

Source details

[Feedback](#)
[Compare sources](#)

Journal of Physics: Conference Series

[Open Access](#)

Scopus coverage years: from 2005 to Present

Publisher: IOP Publishing Ltd.

ISSN: 1742-6588

Subject area: Physics and Astronomy

[Set document alert](#)
[Journal Homepage](#)
[Visit Scopus Journal Metrics](#)

 CiteScore 2016
0.45

 SJR 2015
0.211

 SNIP 2015
0.247
[CiteScore](#)
[CiteScore rank & trend](#)
[Scopus content coverage](#)

CiteScore 2016

Calculated on 23 May, 2017

0.45 = $\frac{\text{Citation Count 2016}}{\text{Documents 2013 - 2015}^*}$
[Citation Count 2016](#)
[6924 Citations](#)
[Documents 2013 - 2015*](#)
[15502 Documents](#)

*CiteScore includes all available document types

[View CiteScore methodology](#)
[CiteScore FAQ](#)

CiteScore rank

In category: Physics and Astronomy

Percentile: 25th

Rank: #147/198

[View CiteScore trends](#)

Jin–Xin relaxation method used to solve the one-dimensional inviscid Burgers equation

This content has been downloaded from IOPscience. Please scroll down to see the full text.

2017 J. Phys.: Conf. Ser. 856 012007

(<http://iopscience.iop.org/1742-6596/856/1/012007>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 202.94.83.93

This content was downloaded on 06/06/2017 at 09:44

Please note that [terms and conditions apply](#).

You may also be interested in:

[Jin–Xin relaxation method for solving a traffic flow problem in one dimension](#)

Bernadetta Ambar Sulistiyawati and Sudi Mungkasi

[The Hirota bilinear method for the coupled Burgers equation and the high-order Boussinesq–Burgers equation](#)

Zuo Jin-Ming and Zhang Yao-Ming

[Moving boundary problems for the Burgers equation](#)

S De Lillo

[Darboux Transformations and Soliton Solutions for Classical Boussinesq–Burgers Equation](#)

Xu Rui

[An Invariance for \(2+1\)-Extension of Burgers Equation and Formulae to Obtain Solutions of KP Equation](#)

Tian Yong-Bo, Tian Chou and Shao Nan

[Traffic Current Fluctuation and the Burgers Equation](#)

Toshimitsu Musha and Hideyo Higuchi

[Burgers equation on the semiline](#)

F Calogero and S De Lillo

[The Burgers equation on the semi-infinite and finite intervals](#)

F Calogero and S de Lillo

[Perturbative Influence on the Shockwave of Burgers Equation](#)

Yi Tang, Jia-Ren Yan and Kai-Wang Zhang

Jin–Xin relaxation method used to solve the one-dimensional inviscid Burgers equation

Sudi Mungkasi¹, Bernadeta Wuri Harini² and Leo Hari Wiryanto³

¹ Department of Mathematics, Faculty of Science and Technology, Sanata Dharma University, Mrican, Tromol Pos 29, Yogyakarta 55002, Indonesia

² Department of Electrical Engineering, Faculty of Science and Technology, Sanata Dharma University, Mrican, Tromol Pos 29, Yogyakarta 55002, Indonesia

³ Department of Mathematics, Faculty of Mathematics and Natural Sciences, Bandung Institute of Technology, Jalan Ganesha 10, Bandung 40132, Indonesia

E-mail: sudi@usd.ac.id, wuribernard@usd.ac.id, leo@math.itb.ac.id

Abstract. We consider the Burgers equation and seek for its numerical solutions. The relaxation method of Jin and Xin, called the Jin–Xin relaxation method, is tested to solve the Burgers equation. We find that the Jin–Xin relaxation method solves the Burgers equation successfully. In addition, the Jin–Xin relaxation method holds the analytical properties of the Burgers equation better than the standard Lax–Friedrichs finite-volume method does.

1. Introduction

Conservation laws govern a number of physical and mathematical problems. In physics, they relate to transport of conserved quantities, such as mass, momentum and energy. In mathematics, they may occur as a constraint in optimal control problems and other optimization problems, for example, minimizing costs, maximizing revenue, etc.

Some previous works related to conservation laws are available in the literature. Conservation laws were successfully applied to simulate real world problems, such as floods and tsunamis as presented by Mungkasi and Darmawan [1]. Investigation on Adjoint IMEX-based schemes for control problems was conducted by Banda and Herty [2] using the relaxation method of Jin and Xin [3] (the Jin–Xin relaxation method). Yohana [4], as well as Yohana and Banda [5], extends the investigation to the Euler equations of gas dynamics employing the work of Armijo [6]. Even though the Jin–Xin relaxation method has been implemented successfully by Jin and Xin [3] themselves, as well as by the authors of [2, 4, 5], to solve some problems, to our knowledge, there is no publication testing the method to solve the one-dimensional inviscid Burgers equation.

In this paper we solve the one-dimensional inviscid Burgers equation using the Jin–Xin relaxation method. Note that the Burgers equation is a member of the class of conservation laws. Our work complements the aforementioned literature. We also solve the Burgers equation using the standard Lax–Friedrichs finite-volume method, as a benchmark, to compare the results with.

This paper is organized as follows. We write the problem formulation of the Jin–Xin relaxation system in Section 2. The numerical scheme to solve the problem is given in Section 3. Numerical results and discussion are presented in Section 4. Finally, we draw some concluding remarks in Section 5.



2. Problem formulation

In this section, we describe the problem that we want to deal with. We write the mathematical model under consideration first and the approach to solve the model afterward. We assume that the model is one-dimensional and scalar.

First, let us recall the general mathematical model that we want to solve. The general mathematical model governing the one-dimensional scalar conservation law is

$$u_t + f(u)_x = 0, \tag{1}$$

for the time variable $t \in [0, \infty)$ and the space variable $x \in (-\infty, \infty)$ with the initial condition

$$u(x, t = 0) = u_0(x). \tag{2}$$

Here $u = u(x, t)$ denotes the conserved quantity and $f(u)$ represents the flux function. The function u_0 gives the initial condition of the conserved quantity. The conserved quantity can be mass, momentum or energy. Equation (1) is generally nonlinear. Note that, taking $f(u) = \frac{1}{2}u^2$, (1) becomes the one-dimensional inviscid Burgers equation [7].

Now, to solve (1) with the initial condition (2) let us recall the relaxation approach of Jin and Xin [3], that is, the Jin–Xin relaxation method. Jin and Xin [3] proposed that (1) and (2) be approximated by a relaxation system, which is a linear system of partial differential equations. The Jin–Xin relaxation system for the mathematical model (1) is [3]

$$u_t + v_x = 0, \tag{3}$$

$$v_t + av_x = -\frac{1}{\epsilon}(v - f(u)), \tag{4}$$

where the initial conditions relating to (2) are

$$u(x, 0) = u_0(x), \tag{5}$$

$$v(x, 0) = f(u_0(x)). \tag{6}$$

Here $v = v(x, t)$ is a dummy variable and a is a positive constant satisfying

$$-\sqrt{a} \leq f'(u) \leq \sqrt{a}, \tag{7}$$

for all u . The parameter ϵ is positive and called the relaxation rate. When the solution is smooth, we eliminate variable v from (3) and (4) to obtain

$$u_t + f(u)_x = \epsilon(au_{xx} - u_{tt}). \tag{8}$$

For $\epsilon \rightarrow 0^+$, we observe that (8) tends to (1). That is, for $\epsilon \rightarrow 0^+$, (3) and (4) simultaneously tend to (1). Here we have approximated the nonlinear equation (1) using the system of linear equations (3) and (4) for $\epsilon \rightarrow 0^+$. However, if we take $\epsilon = 0$, then (3) and (4) are not defined. Ideally ϵ is a small positive number, which is close to zero. More details about the Jin–Xin relaxation system can be found in the work of Jin and Xin [3].

3. Numerical method

We write the numerical scheme to solve (3) and (4), that is, the relaxation system of (1).

Let us assume that the space and time domains are discretized in the finite volume framework. The space domain is discretized uniformly with the cell width h . (Readers interested in nonuniform space discretization can consult the work of Mungkasi [8].) The time domain is discretized uniformly with the time step k . The notation u_j^n is for pointing the approximate value of u in the j -th cell at the n -th time step.

In this paper we use the upwind numerical technique as follows [3]. Discretizing (3) and (4) with respect to space and time domains, we obtain the one-step conservative explicit scheme:

$$\frac{u_j^{n+1} - u_j^n}{k} + \frac{v_{j+1/2}^n - v_{j-1/2}^n}{h} = 0, \tag{9}$$

$$\frac{v_j^{n+1} - v_j^n}{k} + a \frac{u_{j+1/2}^n - u_{j-1/2}^n}{h} = -\frac{1}{\epsilon} (v_j^n - f(u_j^n)). \tag{10}$$

Once again, a is a positive constant satisfying (7), so we can take $a = f'(u)^2$. Jin and Xin [3] derived the first-order upwind relaxing fluxes, which are given by

$$u_{j+1/2}^n = \frac{1}{2} (u_j^n + u_{j+1}^n) - \frac{1}{2\sqrt{a}} (v_{j+1}^n - v_j^n), \tag{11}$$

$$v_{j+1/2}^n = \frac{1}{2} (v_j^n + v_{j+1}^n) - \frac{\sqrt{a}}{2} (u_{j+1}^n - u_j^n). \tag{12}$$

Using the upwind fluxes (11) and (12) in the one-step conservative explicit schemes (9) and (10) we obtain the upwind numerical method for (3) and (4) as

$$\frac{u_j^{n+1} - u_j^n}{k} + \frac{1}{2h} (v_{j+1}^n - v_{j-1}^n) - \frac{\sqrt{a}}{2h} (u_{j+1}^n - 2u_j^n + u_{j-1}^n) = 0, \tag{13}$$

$$\frac{v_j^{n+1} - v_j^n}{k} + \frac{a}{2h} (u_{j+1}^n - u_{j-1}^n) - \frac{\sqrt{a}}{2h} (v_{j+1}^n - 2v_j^n + v_{j-1}^n) = -\frac{1}{\epsilon} (v_j^n - f(u_j^n)). \tag{14}$$

Therefore, given that we know the quantities u_j^n and v_j^n at the n -th time step for all j cells, using (13) and (14) we can compute the quantities and at the $(n + 1)$ -th time step for all j cells.

This can be done by iterations. Starting from the initial condition (2), we have (5), so we obtain u_j^0 . We also have (6), so we obtain v_j^0 . Then the values of $u_j^1, v_j^1, u_j^2, v_j^2, u_j^3, v_j^3, \dots$ are also obtained for all j using (13) and (14).

4. Numerical results

In this section we present results of our research. The Jin–Xin relaxation method is applied to solve the inviscid Burgers equation. The results of this method are analyzed if they hold some analytical properties of the Burgers equation. We also compare the results of the Jin–Xin relaxation method to those of the standard Lax-Friedrichs finite-volume method to show that the Jin–Xin relaxation method performs better. All quantities are assumed to be in the SI units, so we do not need to write their units.

As we have mentioned in Section 2, let us take $f(u) = \frac{1}{2}u^2$. Equation (1) becomes the nonlinear inviscid Burgers equation

$$u_t + \left(\frac{1}{2}u^2 \right)_x = 0. \tag{15}$$

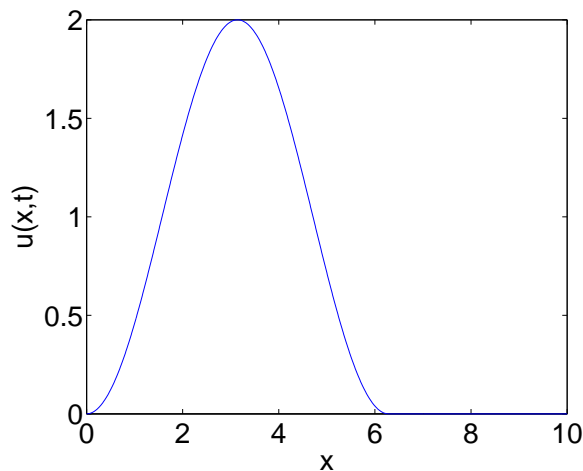


Figure 1. Initial condition for the Burgers equation at time $t = 0$.

For simulations, we set up the followings. We consider the space domain $0 \leq x \leq 10$. The initial condition is given by

$$u_0(x) = \begin{cases} 1 - \cos x, & \text{if } 0 < x \leq 2\pi, \\ 0, & \text{if } 2\pi < x \leq 10. \end{cases} \quad (16)$$

The boundary condition is

$$u(0, t) = u(10, t) = 0 \quad (17)$$

for all $t > 0$. For all simulations, we take the cell width to be $h = 10^{-2}$ and the time step to be $k = 10^{-3}$. This time-step value is chosen so, in order that methods are stable. The Jin–Xin relaxation rate is taken as $\epsilon = 10^{-3}$.

Let us collect some analytical properties of our simulation setting. With the initial condition given by (16), the maximum height of the wave is 2, as shown in figure 1. Due to the characteristics of the Burgers equation, the wave propagates from the left to the right direction. Based on the formulation of LeVeque [9], the wave will form a shock discontinuity starting at time $t_s = -1/\min\{u'_0(x)\}$, that is, $t_s = 1$. Therefore, a shock wave will propagate from the left to the right at time $t > 1$. When the shock occurs, the wave loses some energy, so the wave height gets smaller over time. Using these properties, we shall analyze our numerical results.

Representations of Jin–Xin relaxation results are shown in figure 2. From our simulation, we observe that the wave height is kept at the value of 2 until time $t_s = 1$, as given in figure 2(a). Then for time $t > 1$, a shock occurs and the wave height gets smaller than 2. We also notice that the shock is sharply resolved by the Jin–Xin relaxation method, as illustrated in figure 2(b).

In this section we also record numerical results of the standard Lax–Friedrichs finite-volume method applied directly to the Burgers equation (15). Readers interested in this finite-volume method are referred to the work of LeVeque [10]. In the Lax–Friedrichs finite volume simulation, we use the same initial and boundary conditions as well as the same values of the cell width and time step, as mentioned before. Representations of Lax–Friedrichs finite volume results are shown in figure 3. The Lax–Friedrichs finite-volume method cannot preserve the wave height at the value of 2 at time $t_s = 1$, as given in figure 3(a). In addition, the Lax–Friedrichs finite-volume method cannot sharply resolve the shock wave for time $t > 1$, as shown in figure 3(b).

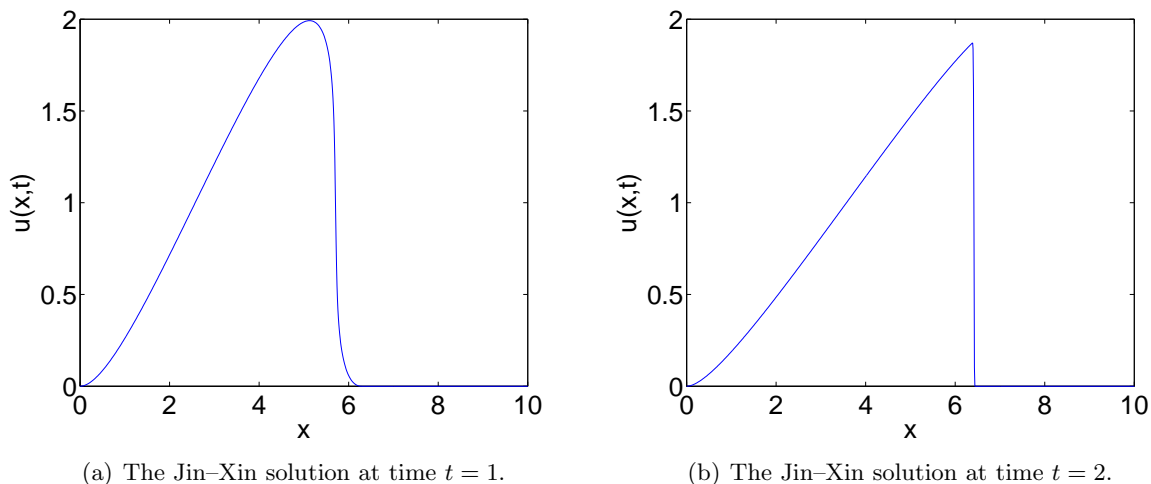


Figure 2. Solutions to the Burgers equation using the Jin-Xin relaxation method.

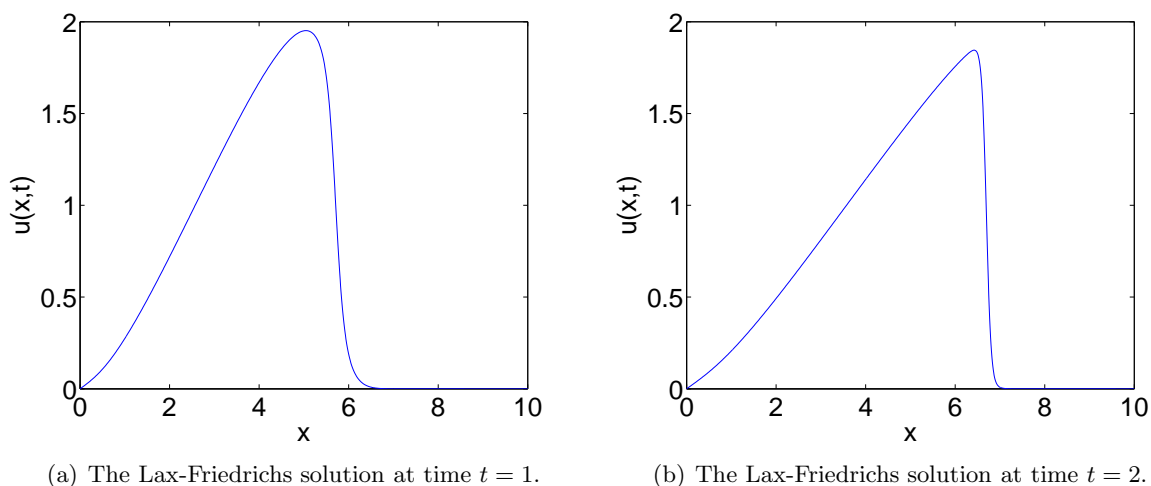


Figure 3. Solutions to the Burgers equation using the Lax-Friedrichs finite-volume method.

We can now compare the results of the Jin-Xin relaxation method and those of the Lax-Friedrichs finite-volume method. Comparing figure 2(a) and figure 3(a), we obtain that the Jin-Xin relaxation method holds the analytical property of the wave height better than the Lax-Friedrichs finite-volume method does. Furthermore, comparing figure 2(b) and figure 3(b), we notice that the Jin-Xin relaxation method holds the analytical property of the shock discontinuity better than the Lax-Friedrichs finite-volume method does.

5. Conclusion

We have solved the inviscid Burgers equation using the Jin-Xin relaxation method successfully. Our results in this paper complement the test cases already available in the literature regarding the method. The method is able to resolve smooth and nonsmooth solutions. In addition, the Jin-Xin relaxation method has better performance than the standard Lax-Friedrichs finite-volume method does.

Acknowledgments

This work was financially supported by Sanata Dharma University and a research grant from *Direktorat Riset dan Pengabdian Masyarakat* (DRPM) of Ministry of Research, Technology and Higher Education of the Republic of Indonesia year 2016.

References

- [1] Mungkasi S and Darmawan J B B 2015 *Proc. 4th Int. Conf. on Soft Computing, Intelligent System and Information Technology (Bali)* vol 516 ed Intan R *et al* (Springer-Verlag) p 469
- [2] Banda M K and Herty M 2012 *Comput. Optim. Appl.* **51** 909
- [3] Jin S and Xin Z 1995 *Commun. Pure Appl. Math.* **48** 235
- [4] Yohana E 2012 MSc Thesis University of the Witwatersrand at Johannesburg
- [5] Yohana E M and Banda M K 2016 *J. Numer. Math.* **24** 45
- [6] Armijo L 1966 *Pac. J. Math.* **16** 1
- [7] Mungkasi S 2015 *IOP Conf. Ser.: Mater. Sci. Eng.* **78** 012031
- [8] Mungkasi S 2016 *Adv. Math. Phys.* **2016** 7528625
- [9] LeVeque R J 1992 *Numerical Methods for Conservation Laws* (Basel: Springer)
- [10] LeVeque R J 2002 *Finite Volume Methods for Hyperbolic Problems* (Cambridge: Cambridge University Press)