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Internal Monochromatic Wave Propagating over Two Bars

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Abstract

Wave propagation in two-fluid system of different density is considered over two bars. The wave is monochromatic, and its dispersion relation is determined, for infinite depth of the upper fluid, which involves the physical quantities for each layers, i.e. the density and the fluid depth. That is then used to observe the effect of the upper fluid to the transmitted wave amplitude after passing the two bars. The optimal width of the bar and the optimal distance between the two bars are determined so that the transmitted amplitude is minimum. In case in the absence of the upper fluid, the result agrees with the surface wave of one fluid.

Mathematics Subject Classification: 35J25, 35Q35, 76B55

Keywords: Internal monochromatic wave, dispersion relation, transmitted wave, reflected wave

1 Introduction

We concern with wave propagating between two fluids with different density. The lighter fluid lies on the heavier one, and the wave is the representation of the interface between those fluids. That wave has been studied by many researchers formulated into a model of equation depending on the layer depth, such as Benjamin [1] (KdV equation), Joseph [8] and Kubota, et. al. [9] (Intermediate Long Wave (ILW) equation), Benjamin [2], Davis and Acrivos [7] and Ono[11] (Benjamin-Ono (BO) equation), Matsuno [10], Choi and Camassa [5], Choi and Camassa [6] and Camassa, et.al. [4] (combination between KdV and ILW equations). Tuck and Wiryanto [13] derived a composite long wave equation as the model of steady internal wave.

In case the bottom topography is involved, the model is more complicated, an extra term should appear in the model as the external forced, similar to the model derived by Shen, at. al. [12]. To reduce the difficulties, we consider a special type of internal wave having one angular frequency, and a bar is submerged in the lower layer. The amplitude of the wave before and after passing the bar is the problem that will be compared. The dimension of the bar is another problem that will be observed to the wave amplitudes, similarly for the wave amplitude after passing two bars. In case the surface wave of one layer, Wiryanto [14] calculated the optimal dimension of the bar that gives minimal transmitted amplitude. This is then extended by Wiryanto and Jamhuri [15] for two bars. Now, in this paper we report our work as an extension of surface wave, by observing the effect of the upper layer for infinite depth to the internal wave.

In formulating the problem, the dispersion relation is determined for the monochromatic internal wave propagating in the two-fluid system with flat boundaries. This relation is used to determine the wave number related to the upper and lower layer depths. The wave amplitude before and after passing the bar is formulated based on wave and flux continuations, as the change of the depth of the lower layer. That formulation gives a system of equations that can be solved analytically or numerically. Therefore, the optimal dimension of the bar can be obtained by observing the minimal amplitude of the wave after passing the bar. The result can be compared to the surface wave of one layer by taking the upper layer density tends to zero.

2 Dispersion relation

We investigate the wave propagation of finite amplitude at the surface of contact between the lower fluid layer with density ρ_1 and the upper fluid layer with density ρ_2. These layers are divided by the contact surface $y = \eta(x, t)$, and they are limited by solid flat boundary, with one or two bars at the bottom. The fluid depth is h_1 for the lower layer and h_2 for the other. As the coordinates, we define Cartesian with the horizontal x-axis chosen along the undisturbed interface and the vertical y-axis is perpendicular to the horizontal axis, as illustrated in Figure 1.

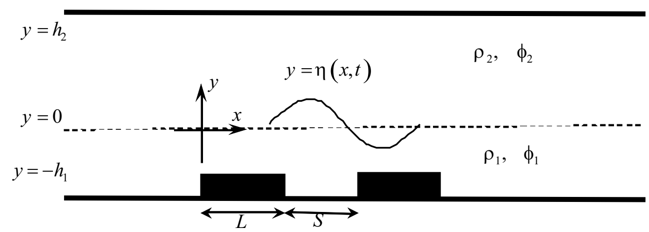

Figure 1: Sketch of the fluid system and the coordinates

We assume the fluids are ideal and the flow is irrotational. The internal wave, in the system of two fluids between two flat boundaries, has amplitude a and it is monochromatic expressed in form of

$$\eta(x,t) = ae^{-i(kx-\omega t)}, \tag{1}$$

where $i = \sqrt{-1}$, k and ω are the wave-number and the angular frequency. The relation between k and ω is our task to be determined in this section, together with the potential function ϕ_1 and ϕ_2, more precisely the result for infinite depth of the upper fluid. To do so, we solve the linearized governing equations as follow. The potential function for each layer and elevation η satisfy

$$\phi_{j_{xx}} + \phi_{j_{yy}} = 0, \tag{2}$$

the index $j = 1$ in (2), and also further equations, is for the lower fluid and $j = 2$ is for the upper fluid, subject to the solid boundaries

$$\phi_{j_y} = 0, \tag{3}$$

at $y = -h_1$ for $j = 1$ and at $y = h_2$ for $j = 2$, and the boundaries at the interface $y = 0$ satisfying

$$\eta_t - \phi_{j_y} = 0 \tag{4}$$

and continuation of the pressure p from upper and lower fluids. The pressure is calculated from Bernoulli's equation, so that we have

$$\rho_1 \left(\phi_{1_t} + g\eta\right) = \rho_2 \left(\phi_{2_t} + g\eta\right). \tag{5}$$

In determining ϕ_j, we follow Budiasih et. al. [3] and similar procedure in Wiryanto [14], using variable-separation method. We express the potential functions as

$$\phi_j(x,y,t) = S_j(x,t)F_j(y). \tag{6}$$

We first substitute (6) to (4), giving $S_j = i\omega\eta/F_j'(0)$ and $F_1'(0)$ is proportional to $F_2'(0)$, namely $F_2'(0) = \lambda F_1'(0)$. Without loss generalization, we choose

$F_1'(0) = 1$, so that we have $F_2'(0) = \lambda$. Beside that, we substitute (6) into (3), giving $F_1'(-h_1) = 0$ and $F_2'(h_2) = 0$. Together with the condition of F_j' at $y = 0$ and using the result of S_j to (6), Laplace equation (2) can be solved for F_j, and we obtain the potential functions

$$\phi_1(x, y, t) = i\omega\eta(x, t)\left[\left(\frac{1}{k} + B_1\right)e^{ky} + B_1 e^{-ky}\right], \tag{7}$$

$$\phi_2(x, y, t) = i\omega\eta(x, t)\left[\left(\frac{1}{k} + B_2\right)e^{ky} + B_2 e^{-ky}\right], \tag{8}$$

where

$$B_1 = \frac{e^{h_1 k}}{2k\sinh(h_1 k)},$$

$$B_2 = \frac{-e^{h_2 k}}{2k\sinh(h_2 k)}.$$

The constant λ does not appear as it is canceled from $S_2(x, t)$ and $F_2(y)$.

Now, we determine the relation between k and ω by substituting (7) and (8) into (5). After doing some algebraic operations, we come to

$$\left(\frac{\omega^2}{k}\right)\left[-\rho_1\left(\frac{e^{-h_1 k}}{\sinh(h_1 k)} + 1\right) + \rho_2\left(\frac{-e^{-h_2 k}}{\sinh(h_2 k)} + 1\right)\right] = (\rho_2 - \rho_1)\,g. \tag{9}$$

In the absence of the upper fluid, we substitute $\rho_2 = 0$ in (9), and the first term of the left hand side of (9) in the square brackets can be written in different form as

$$\left(\frac{e^{-h_1 k}}{\sinh(h_1 k)} + 1\right) = \frac{\cosh(h_1 k)}{\sinh(h_1 k)},$$

so that we can cancel ρ_1 from both sides and we have

$$\omega^2 = gk\tanh(h_1 k),$$

agrees to the dispersion relation for surface wave. We can also observe (9) for very deep of the upper fluid. For $h_2 \to \infty$ the ratio between exponent and sinh in (9) tends to zero, so that the dispersion relation becomes

$$\frac{\omega^2}{gk}\left[-\rho_1\frac{\cosh(h_1 k)}{\sinh(h_1 k)} + \rho_2\right] = \rho_2 - \rho_1. \tag{10}$$

That limiting process is also applied to the potential function (8), where

$$\frac{e^{h_2 k}}{2\sinh(h_2 k)} \to 1,$$

as $h_2 \to \infty$, so that (8) becomes

$$\phi_2(x, y, t) = \frac{-i\omega\eta(x, t)}{k}e^{-ky}. \tag{11}$$

To see the effect of the upper fluid, we make a comparison between the dispersion relation of surface wave and (10). For the lower layer with $h_1 = 1$, $\rho_1 = 1$, we calculate k related to $\omega = 2$ and using $g = 10$, the surface wave gives $k = 0.678$. It is slightly smaller compared to the result of (10), giving $k = 0.686$ corresponding to $\rho_2 = 0.05$, and increasing k by increasing ρ_2, for $\rho_2 = 0.8$ we obtain $k = 1.011$.

3 Optimal width of one bar

In this section we observe the effect of the upper fluid to the transmitted and reflected amplitude of internal monochromatic wave passing over a bar. As illustrated in Fig. 1, the width of the bar is L and bar height is d. We can also say the fluid depth above the bar is $h_{12} = h_{11} - d$. The lower fluid before and after the bar has the same depth h_{11}. Therefore, the wave-number corresponding to the fluid depth h_{11} and h_{12} is k_{11} and k_{12} respectively, calculated from (10). Meanwhile, the x domain is divide into three sub-domains $(-\infty, 0)$, $(0, L)$ and (L, ∞). Since the wave propagates to the right, in the first sub-domain there are incoming and reflected waves with amplitude a and a_t, similar to the second sub-domain with amplitude b_r and b_t, and the third sub-domain there is only transmitted wave with amplitude c_t. This situation is expressed mathematically by

$$\eta(x,t) = \begin{cases} ae^{-i(k_{11}x-\omega t)} + a_r e^{-i(-k_{11}x-\omega t)}, & \text{for } x < 0, \\ b_t e^{-i(k_{12}x-\omega t)} + b_r e^{-i(-k_{12}x-\omega t)}, & \text{for } 0 < x < L, \\ c_t e^{-i(k_{11}x-\omega t)} & \text{for } x > L. \end{cases} \tag{12}$$

Our task is to determine c_t/a for given bar width L and d. In case $L \to 0$ it should be $c_t/a \to 1$, no wave that is reflected. When it can be calculated, our next concern is to determine the optimal value of L so that it gives minimum value for e_t. In answering this problem, the formulation (12) has to be continue. This gives two equations obtained by limiting from both sides at $x \to 0$ and $x \to L$, resulting

$$\left.\begin{aligned} a + a_r &= b_t + b_r, \\ b_t e^{-ik_{12}L} + b_r e^{ik_{12}L} &= c_t e^{-ik_{11}L}. \end{aligned}\right\} \tag{13}$$

The other two equations are obtained by continuing the flux at $x = 0$ and $x = L$. This brings us to formulate the flux.

At $x < 0$, the flux is defined by

$$Q_1 = \int_{-h_{11}}^{\infty} \phi_{j_x} dy.$$

The integral is divided into two following the validity of ϕ , i.e.

$$Q_1 = \int_{-h_{11}}^{0} \phi_{1_x} dy + \int_{0}^{\infty} \phi_{2_x} dy.$$

By applying (7) and (11), the integral gives

$$Q_1 = \frac{-i\omega}{k_{11}^2}\eta_x.$$

Similarly for $0 < x < L$ and $L < x$, we obtain

$$Q_2 = \frac{-i\omega}{k_{12}^2}\eta_x, \qquad Q_3 = \frac{-i\omega}{k_{11}^2}\eta_x,$$

respectively. We obtain the same formula for Q_3 and Q_1, as the lower layer has the same depth. These fluxes are then taken its limit, giving

$$\left.\begin{aligned} k_{12}\left(a - a_r\right) &= k_{11}\left(b_t - b_r\right), \\ k_{11}\left(b_t e^{-ik_{12}L} - b_r e^{ik_{12}L}\right) &= k_{12} c_t e^{-ik_{11}L}. \end{aligned}\right\} \tag{14}$$

Equations (13) and (14) are then solved for determining a_r, b_t, b_r and c_t. These unknowns are scaled with respect to the incoming amplitude a. Since the solution is possible in complex, all amplitudes are calculated as the modulus of the unknown.

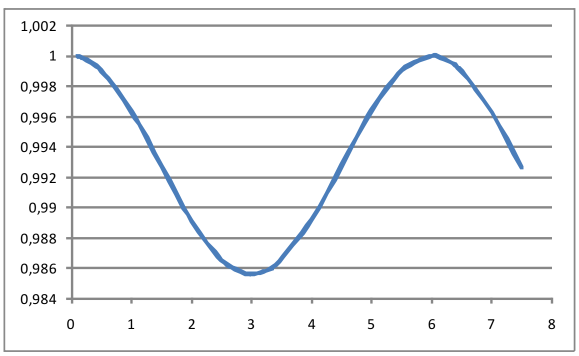

Figure 2: Plot of $|c_t|$ (vertical) versus L (horizontal) for $\rho_2 = 0.1$, obtained $L_{opt} = 3.0$ corresponding to $|c_t| = 0.98562$.

In presenting the result, we use the physical quantities that have been used in [14] for surface wave, so that we then can compare and observe the effect of the upper fluid. The lower fluid in our calculation is $h_{11} = 3.5$, $h_{12} = 2.0$ and $\rho_1 = 1$. The angular frequency is $\omega = 2$ and we use $g = 10$. For $L = 3.04$ Wiryanto [14] obtained the transmitted amplitude $|c_t| = 0.987$. When the upper layer is included with $\rho_2 = 0.1$, our calculation gives $|c_t| = 0.98563$ and smaller transmitted amplitude for larger ρ_2. We obtain $|c_t| = 0.9777$ for $\rho_2 = 0.5$. The result in [14] is the limiting case for $\rho_2 \to 0$. We then collect the transmitted amplitude for various L. Plot of $|c_t|$ versus L using $\rho_2 = 0.1$ is presented in Fig. 2. Our calculations show that the minimum transmitted amplitude can be obtained for $L = 3.0$ corresponding to $|c_t| = 0.98562$. For $L > 3.0$ the minimum value of $|c_t|$ can be repeated obtained with the same value. Therefor we define the optimal of the bar width L_{opt} is the smallest L giving smallest $|c_t|$, in this case $L_{opt} = 3.0$. This is smaller than the result in [14]. For $\rho_2 = 0.5$, we obtain $L_{opt} = 0.73$ corresponding to $|c_t| = 0.97697$.

4 Optimal distance between two bars

In the previous section we described the reduction of wave amplitude after it passes a bar. The reduction can be done for the next bar. Wiryanto and Jamhuri [15] observed the amplitude reduction after the wave passing two bars, but for surface wave. Their formulation shows that the calculation for the amplitude reduction after passing two bars can be larger than if we calculate the reduction twice but using formulation for one bar. That work is extended in this paper for internal wave.

The internal wave propagating over two bars can be formulated by combination between transmitted and reflected waves, similar to (12) but for there are 5 sub-domains. The two bars have the same width L and height d, and the distance between them is S, see Fig. 1. The depth of the lower fluid is h_{11} for the sub-domain without bar and $h_{12} = h_{11} - d$ for sub-domain with a bar. The upper fluid has infinite depth. The wave number corresponding to those sub-domains is k_{11} and k_{12}. Therefore, the wave is expressed as

$$\eta(x,t) = \begin{cases} ae^{-i(k_{11}x-\omega t)} + a_r e^{-i(-k_{11}x-\omega t)}, & \text{for } x < 0, \\ b_t e^{-i(k_{12}x-\omega t)} + b_r e^{-i(-k_{12}x-\omega t)}, & \text{for } 0 < x < L, \\ c_t e^{-i(k_{11}x-\omega t)} + c_r e^{-i(-k_{11}x-\omega t)}, & \text{for } L < x < L+S, \\ d_t e^{-i(k_{12}x-\omega t)} + d_r e^{-i(-k_{12}x-\omega t)}, & \text{for } L+S < x < 2L+S, \\ e_t e^{i(-k_{11}x-\omega t)}, & \text{for } 2L+S < x. \end{cases} \quad (15)$$

Now, our task is to determine all wave amplitudes for given the incoming amplitude, we can choose $a = 1$, especially the transmitted amplitude $|e_t|$ after the wave passing both bars. Moreover, for the bars with L_{opt} we determine the optimal distance of S that gives minimum $|e_t|$, we then denote S_{opt}. Following the previous section, the wave (15) and the flux have to be continue. The limiting proses to (15) is then done and it gives a system of equations

$$\left.\begin{aligned} 1 + a_r &= b_t + b_r, \\ e^{-ik_{12}L} b_t + e^{ik_{12}L} b_r &= e^{-ik_{11}L} c_t + e^{ik_{11}L} c_r, \\ e^{-ik_{11}(L+S)} c_t + e^{ik_{11}(L+S)} c_r &= e^{-ik_{12}(L+S)} d_t + e^{ik_{12}(L+S)} d_r, \\ e^{-ik_{12}(2L+S)} d_t + e^{ik_{12}(2L+S)} d_r &= e^{-ik_{11}(2L+S)} e_t, \\ k_{12}(1 - a_r) &= k_{11}(b_t - b_r), \\ k_{11}\left(e^{-ik_{12}L} b_t - e^{ik_{12}L} b_r\right) &= k_{12}\left(e^{-ik_{11}L} c_t - e^{ik_{11}L} c_r\right), \\ k_{12}\left(e^{-ik_{11}(L+S)} c_t - e^{ik_{11}(L+S)} c_r\right) &= k_{11}\left(e^{-ik_{12}(L+S)} d_t - e^{ik_{12}(L+S)} d_r\right), \\ k_{11}\left(e^{-ik_{12}(2L+S)} d_t - e^{ik_{12}(2L+S)}\right) d_r &= k_{12} e^{-ik_{11}(2L+S)} e_t. \end{aligned}\right\} \quad (16)$$

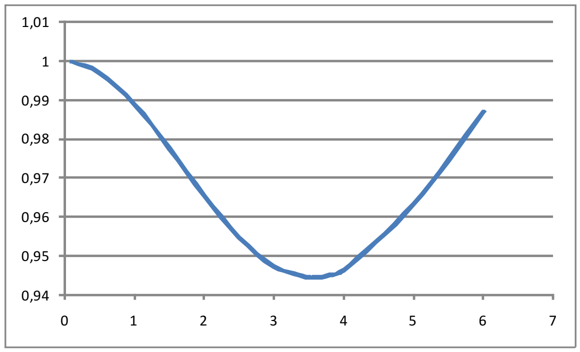

Figure 3: Plot of $|e_t|$ (vertical) versus S (horizontal) for $\rho_2 = 0.1$, obtained $S_{opt} = 3.55$ corresponding to $|c_t| = 0.9445$.

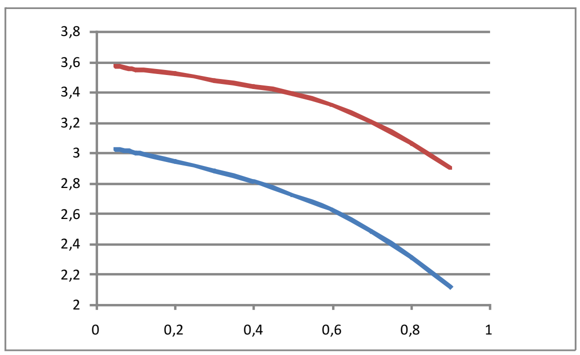

Figure 4: Plot of L_{opt} (lower curve) and S_{opt} (vertical) versus ρ_2 (horizontal).

The first four equations are from the continuation of η, and the others are from the continuation of flux. The wave numbers k_{11} and k_{12} are calculated from (10).

In solving (16), we set some quantities $h_{11} = 3.5$, $h_{12} = 2.0$ and $\rho_1 = 1$ for the lower layer, upper fluid density $\rho_2 = 0.1$, and the bar width $L = 3.0$. The system of equations gives the transmitted amplitude $|e_t| = 0.9890$ for distance $S = 1.0$. For decreasing S we obtain increasing $|e_t|$, and $|e_t| \to 1$ as $S \to 0$. This can be explained that for $S = 0$ we are back to one bar with width $L = 6.0$, and our previous result gives the same transmitted amplitude, see plot in Fig. 2. We then collect some data S and $|e_t|$ from our calculations. They are plotted in Figure 3. The optimal distance is obtained at $S_{opt} = 3.55$ giving transmitted amplitude $|e_t| = 0.9445$.

For different values of ρ_2, the above formulations are used to determine L_{opt}, and then used to determine S_{opt}. Plots of ρ_2 versus L_{opt} and versus S_{opt} are shown in Figure 4. The curve for L_{opt} and S_{opt} has the same profile, decreasing by increasing ρ_2, with lower curve for L_{opt}. We also found that the transmitted amplitude corresponding to those optimal values also decreases. When $\rho_2 \to 0$ we obtain surface wave as obtained in [14] and [15]. On the other hand, $\rho_2 \to 1$ no wave is transmitted, since the fluids become one.

5 Conclusion

We have presented the effect of the upper fluid of infinite depth to the internal monochromatic wave propagating over two bars. The percentage of the wave transmitted and reflected was formulated based on the continuation of the wave elevation and its flux. The optimal width of bar was obtained and it depends on the density of the upper fluid, similarly for the optimal distance between two bars. In case the upper density tends to zero, our results confirm to the surface wave.

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