A staggered grid finite difference method for solving the elastic wave equations

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Abstract. We consider elasticity (elastic wave) equations for solving acoustics problems. We use the finite difference method on staggered grids to solve the elastic wave equations. On staggered grids, the pressure is computed at a set of spatial points, and the velocity is computed at another set of spatial points. An advantage of using staggered grids is that fluxes do not need to be approximated in the solving process, as the flux values are known. Numerical results confirm that our numerical method is reliable.

1. Introduction

Elastic wave is a wave that its propagation is dependent on the nature of the wave propagation medium itself. It has been an attention for a number of researchers (for example, see [1-9]) due to its importance in physics and engineering. One special example of elastic waves that occurs in everyday life is the propagation of sound, called an acoustic wave [10].

Staggered grid numerical methods provide a way to solve wave equations. These methods have a high accuracy for a large number of problems [11-14]. In this paper, we use the finite difference method with staggered grids to solve the elastic wave equation for acoustic problems. The finite difference method is chosen with staggered grids, because the forms of the equations match with the nature of staggered grids. An advantage of using staggered grids is that we do not need to compute an approximate flux at a certain interface at the domain. In particular, in this paper, we propose a numerical scheme to deal with the elastic wave equations when we use the staggered grid method to solve acoustic problems.

The paper is organised simply as follows. We present the mathematical model in Section 2. The numerical method is explained in Section 3. Numerical results are presented in Section 4. We provide some concluding remarks in Section 5.

2. Mathematical model

The general one-dimensional elastic wave equations are [1, 9]

$$\varepsilon_t(x,t) - u_x(x,t) = 0, \tag{1}$$

$$(\rho(x)u(x,t))_t - \sigma(\varepsilon(x,t),x)_x = 0. \tag{2}$$

Here x represents the space domain and t denotes the time. In addition, ε is the strain, u the velocity, ρ the density, and σ the stress. Note that $\varepsilon_t(x,t) = \frac{\partial \varepsilon(x,t)}{\partial t}$. Other derivative notations are analogous.

In equations (1) and (2), the stress σ and the strain ε are assumed to have the relation

$$\sigma(\varepsilon, x) = K(x)\varepsilon. \tag{3}$$

Here $K(x)$ is the bulk compressibility modulus. The pressure $p(x,t)$ is given by $p = -\sigma$, so equations (1) and (2) can be expressed in linear partial differential equations [15]

$$p_t + K(x)u_x = 0, \tag{4}$$

$$\rho(x)u_t + p_x = 0. \tag{5}$$

Assuming that $\rho(x) = 1$ dan $K(x) = 1$, we rewrite equations (4) and (5) as

$$p_t + u_x = 0, \tag{6}$$

$$u_t + p_x = 0. \tag{7}$$

The system of equations (6) and (7) is a hyperbolic system of partial differential equations. Without loss of generality, in the next sections, we shall focus on solving the system of equations (6) and (7), which is a model for acoustic problems.

3. Numerical method

We note that

$$p_t = \frac{\partial p}{\partial t} \quad \text{and} \quad u_x = \frac{\partial u}{\partial x}, \tag{8}$$

$$u_t = \frac{\partial u}{\partial t} \quad \text{and} \quad p_x = \frac{\partial p}{\partial x}. \tag{9}$$

We take the following discretisations of the derivatives

$$\frac{\partial p}{\partial t}\Big|_{t=t^n}^{x=x_j} \approx \frac{p_j^{n+1} - p_j^n}{\Delta t}, \tag{10}$$

$$\frac{\partial u}{\partial x}\Big|_{t=t^n}^{x=x_j} \approx \frac{u_{j+1/2}^n - u_{j-1/2}^n}{\Delta x}. \tag{11}$$

Using approximations (10) and (11), equation (6) can be written in a fully-discrete form as

$$p_j^{n+1} = p_j^n - \frac{\Delta t}{\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right). \tag{12}$$

Furthermore, we take the following discretisations of the derivatives

$$\frac{\partial u}{\partial t}\Big|_{t=t^n}^{x=x_{j+1/2}} \approx \frac{u_{j+1/2}^{n+1} - u_{j+1/2}^n}{\Delta t}, \tag{13}$$

$$\frac{\partial p}{\partial x}\Big|_{t=t^n}^{x=x_{j+1/2}} \approx \frac{p_{j+1}^n - p_j^n}{\Delta x}. \tag{14}$$

Using approximations (13) and (14), the fully-discrete form of equation (7) is

$$u_{j+1/2}^{n+1} = u_{j+1/2}^n - \frac{\Delta t}{\Delta x}\left(p_{j+1}^n - p_j^n\right). \tag{15}$$

The system of equations (12) and (15) is the numerical scheme for solving acoustics problems governed by equations (6) and (7) on staggered grids.

4. Numerical results

This section is devoted to present our numerical results.

We assume that, initially, an acoustics problem has the pressure function

$$p(x,0) = \begin{cases} 1 - \cos(x+\pi), & \text{if} \quad |x| \le \pi, \\ 0, & \text{if} \quad \pi < |x| \le 10, \end{cases} \tag{16}$$

for $-10 \le x \le 10$. This initial pressure is shown in Figure 1. The velocity is $u(x,0) = 0$ for the whole space domain.

Using the given data, the space domain is $[-L, L]$, where $L = 10$. We shall compute the pressure and velocity in the system at every time step. We take that the number of space points is $N = 1000$, the space step is $\Delta x = 2L/(N-1)$, and the time step is $\Delta t = 0.05\Delta x$.

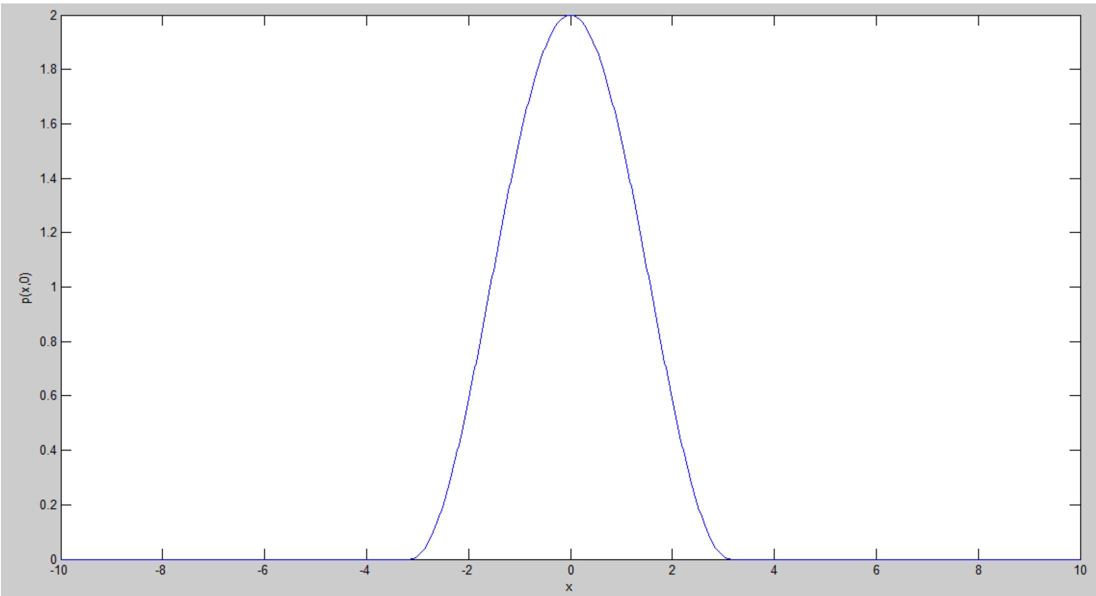

Figure 1. Pressure at time $t = 0$.

As initially, the pressure is centred around $x = 0$, this pressure separates into two waves propagating to the left and right directions. This is illustrated in Figures 2-3. Figure 2 shows the pressure and velocity at time $t = 2.5$, whereas Figure 3 shows the pressure and velocity at time $t = 5$.

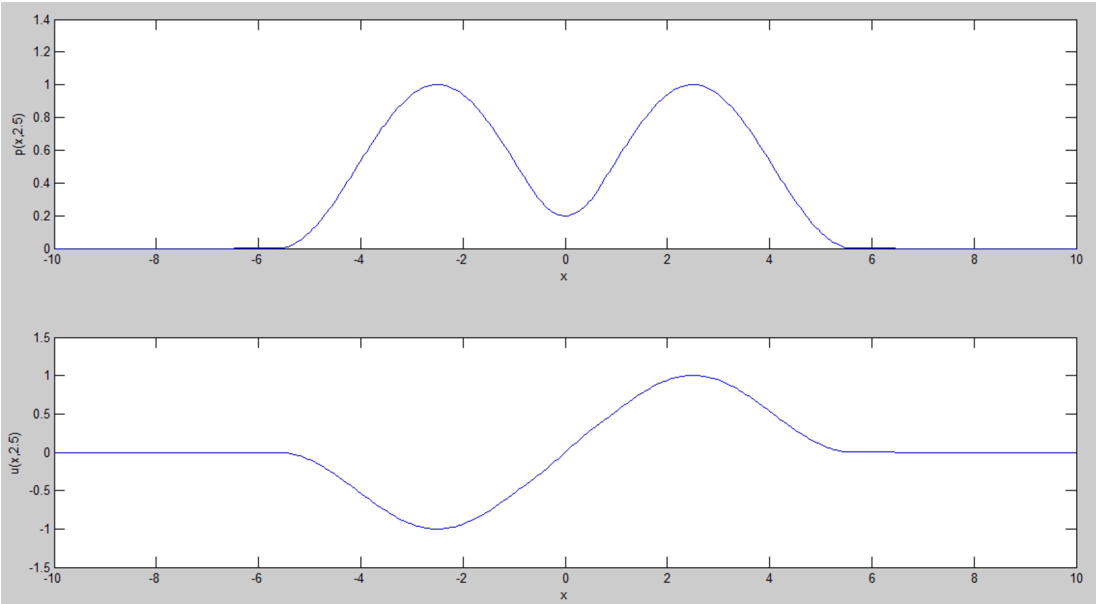

Figure 2. Pressure $p(x, 2.5)$ at time $t = 2.5$.

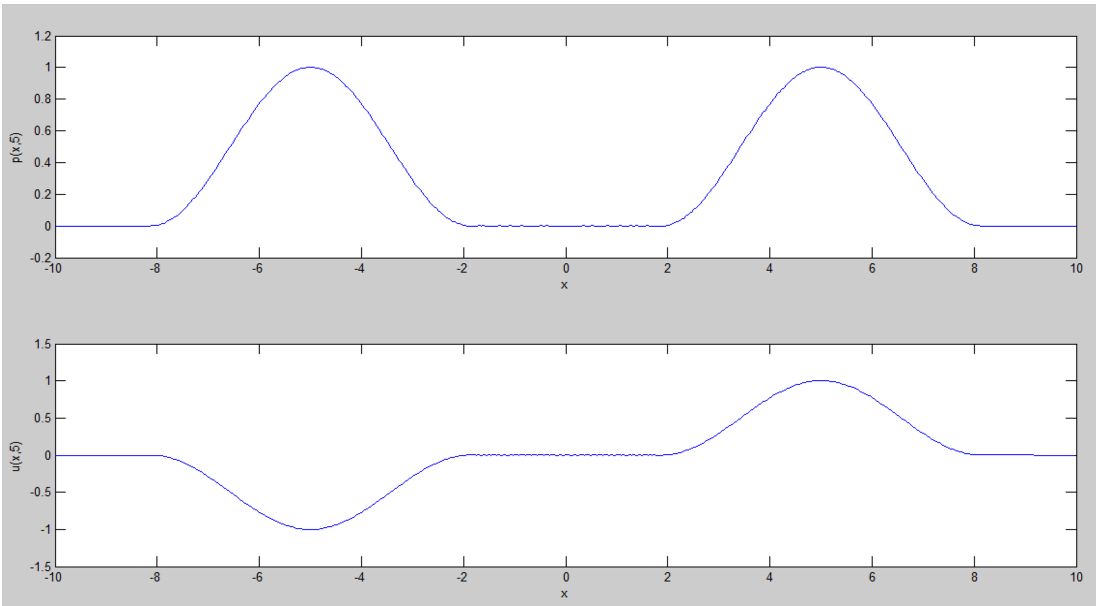

Figure 3. Pressure $p(x, 5)$ at time $t = 5$.

5. Conclusion

The staggered grid finite difference method has been derived and used to solve the elastic wave equations relating to acoustics problems. In this paper, the test uses smooth initial values. We obtain the correct physical behaviour in the numerical solution, as there is a single bump of pressure with zero velocity, the pressure separates into two moving in opposite directions. Future work shall investigate the performance of the numerical method when it is used to solve nonsmooth problems.

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