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Reconstruction of initial conditions of the Burgers equation in conservation law problems

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Abstract. In this paper, we solve the Burgers equation, which is a representation of nonlinear conservation laws. The research question is that after we solve the Burgers equation for the final time, how can we reconstruct the initial condition if we need the initial condition, whereas the initial condition has evolved to the final solution. To answer this research question, we propose the use of the Lax–Friedrichs finite volume method to reconstruct the initial condition of the Burgers equation in one dimension. To demonstrate our proposal, we solve the Burgers equation first using the Jin–Xin relaxation method until we accomplish the final solution. After the final solution is obtained, we reconstruct the initial condition backward in time and compare it with the exact initial condition which is used earlier. We find that the proposed numerical method is successful in reconstructing the initial condition accurately.

1. Introduction

Burgers equation contains some characteristics of the Navier–Stokes equation for fluid dynamics [1]. The Burgers equation is a nonlinear hyperbolic partial differential equation [2-3]. Literature has been available largely in correspondence with solving the Burgers equations, for example, see References [4-8]. Nevertheless, how to track the history of a solution to the Burgers equation is still questionable. In this paper, we concern on reconstructing the initial condition of the Burgers equation in one dimension.

First, the Jin–Xin relaxation method [9-11] is used to solve the Burgers equation until we discover the final numerical solution. After that, using the final numerical solution, we calculate backward in time the estimation of initial condition using the Lax–Friedrichs finite volume method. We choose the Jin–Xin relaxation method because of its linearity in the models, so we do not need too much efforts to calculate the solution, while the Lax–Friedrichs finite volume method presents the simplicity in the models.

To see the accuracy of our reconstruction, we compare the reconstructed initial condition with the exact initial condition. We examine the error between them and observe the effectiveness of the Lax–Friedrichs finite volume method to reconstruct the initial condition. This research can be used to estimate the history of wave if all we have is present data without any background from the past.

The paper is organised as follows. Section 2 recalls the defined model and the implemented methods. Results and discussion are presented in Section 3. Conclusion is expressed in Section 4.



2. Model and methods

We consider conservation laws in one dimension

$$u_t + f(x, u)_x = 0 \quad (1)$$

where $f(x, u)$ is a flux function. Here, the flux for the Burgers equation is

$$f(u) = \frac{u^2}{2}. \quad (2)$$

We assume that the initial condition for u when $1 < x < 1 + 2\pi$ is

$$u(x, 0) = 1 + \sin\left(x - \frac{\pi}{2} - 1\right) + \epsilon, \quad (3)$$

otherwise the value is ϵ , which is a sufficiently small positive number. Here x is a spatial variable, t is the time variable and u is a conserved quantity depending on x and t .

There are two methods which are used in this paper. They are the Jin–Xin relaxation method and the Lax–Friedrichs finite volume method. The Jin–Xin relaxation method is implemented to obtain the final numerical solution of the Burgers equation. After we obtain the final numerical solution, the research question is:

If the initial condition has gone and unknown, can we reconstruct the initial condition?

In this paper, we propose the Lax–Friedrichs finite volume method backward in time to reconstruct the initial conditions. Note that the equation solved using the Lax–Friedrichs finite volume method is different from those solved using the Jin–Xin relaxation method.

2.1 Jin–Xin relaxation method

Equation (1) can be written in a linear system [9]

$$\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} = 0 \quad (4)$$

and

$$\frac{\partial v}{\partial t} + a \frac{\partial u}{\partial x} = -\frac{1}{\epsilon}(v - f(u)) \quad (5)$$

where v is defined by $v = f(u)$, while ϵ is a small positive constant and a is a constant satisfying $a = \max (f'(u))^2$.

We can write equation (4) and (5) in the discretised space of the conservative form

$$\frac{\partial}{\partial t} u_i + \frac{1}{\Delta x} (v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}) = 0 \quad (6)$$

and

$$\frac{\partial}{\partial t} v_i + \frac{1}{\Delta x} a (u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}) = -\frac{1}{\epsilon} (v_i - f(u_i)). \quad (7)$$

The fully explicit schemes of the Jin–Xin relaxation method in one dimension are

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x} \left(v_{i+1}^n - v_i^n - a^{\frac{1}{2}}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \right), \quad (8)$$

and

$$v_i^{n+1} = v_i^n - \frac{\Delta t}{2\Delta x} \left(a(u_{i+1}^n - u_i^n) - a^{\frac{1}{2}}(v_{i+1}^n - 2v_i^n + v_{i-1}^n) \right) - \frac{1}{\epsilon} (v_i^n - f(u_i^n)). \quad (9)$$

2.2 Lax–Friedrichs finite volume method

The Lax–Friedrichs finite volume method is used for the reconstruction of initial condition. First, we are given the final solution $u(x, t^M)$, where t^M is assumed to be the final time. We use this final solution as the initial value of p , where p is the variable for the reconstruction of the history of the solution:

$$u(x, t^M) = p(x, t^M). \quad (10)$$

We can write equation (4) and (5) in adjoint equations [11]

$$-q_t - p_x = -\frac{q}{\epsilon}, \quad (11)$$

and

$$-p_t - aq_x = \frac{q}{\epsilon} f'(u). \quad (12)$$

Based on equations (11) and (12), we obtain [12]

$$-p_t - f'(u)p_x = \epsilon ap_{xx} \quad (13)$$

and solve equation (13) using the Lax–Friedrichs finite volume method backward in time. The explicit numerical scheme of the finite volume method backward in time is [3]

$$P_i^n = P_i^{n+1} - \frac{\Delta t}{\Delta x} \left(F_{i+\frac{1}{2}}^{n+1} - F_{i-\frac{1}{2}}^{n+1} \right) \quad (14)$$

where

$$\begin{aligned} F_{i+\frac{1}{2}}^{n+1} &= \frac{1}{2} (F_{i+1}^{n+1} + F_i^{n+1}) - \frac{\Delta x}{2\Delta t} (P_{i+1}^{n+1} - P_i^{n+1}) \\ &= \frac{1}{2} \left(\frac{1}{2} (P_{i+1}^{n+1})^2 + \frac{1}{2} (P_i^{n+1})^2 \right) - \frac{\Delta x}{2\Delta t} (P_{i+1}^n - P_i^n), \end{aligned} \quad (15)$$

and

$$\begin{aligned} F_{i-\frac{1}{2}}^{n+1} &= \frac{1}{2} (F_i^{n+1} + F_{i-1}^{n+1}) - \frac{\Delta x}{2\Delta t} (P_i^{n+1} - P_{i-1}^{n+1}) \\ &= \frac{1}{2} \left(\frac{1}{2} (P_i^{n+1})^2 + \frac{1}{2} (P_{i-1}^{n+1})^2 \right) - \frac{\Delta x}{2\Delta t} (P_i^n - P_{i-1}^n). \end{aligned} \quad (16)$$

Here $F_{i+1/2}^{n+1} \approx f(p(x_{i+1/2}, t^{n+1}))$ is the flux, and $P_i^{n+1} \approx p(x_i, t^{n+1})$ is an approximation of the conserved quantity, Δx is the cell-width, Δt is the time step, i represents the index of space, and n denotes the index of time.

To see the accuracy of our reconstructed initial conditions, numerical error and convergence rate should be calculated. We define N as the number of cells based on how large the cell-width is and E_i as the error of the solution based on the cell-width Δx_i . Therefore, the formulas for the numerical error (E) and the convergence rate (CR) are as follows [13]:

$$E = \frac{1}{N} \sum_{i=1}^N |u(x, 0) - p(x, 0)| \quad (17)$$

and

$$CR = \frac{1}{N-1} \sum_i^{N-1} R_i \quad (18)$$

where

$$R_i = \frac{\log\left(\frac{E_i}{E_{i+1}}\right)}{\log\left(\frac{\Delta x_i}{\Delta x_{i+1}}\right)}. \quad (19)$$

3. Results and discussion

In this section, we present our results and give a discussion.

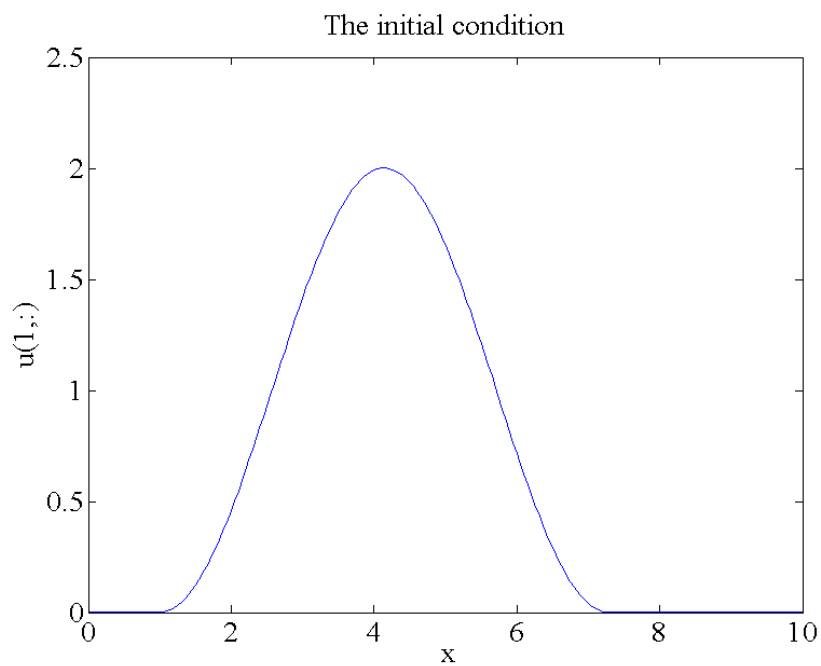


Figure 1. The initial condition that is used as a test case in the Burgers equation. It is evolved for the final solution of the Burgers equation.

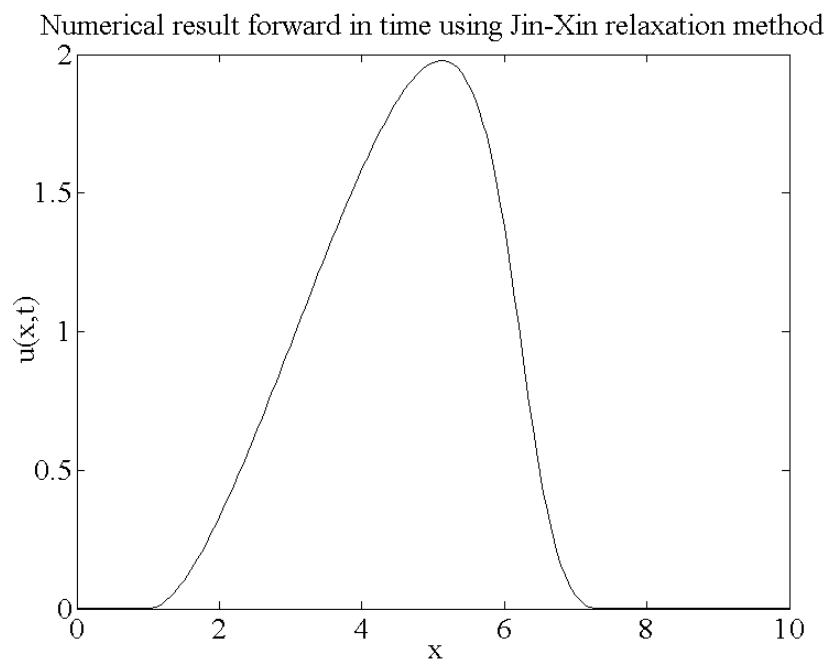


Figure 2. Numerical solution of Burgers equation using Jin–Xin relaxation method forward in time for case 1.

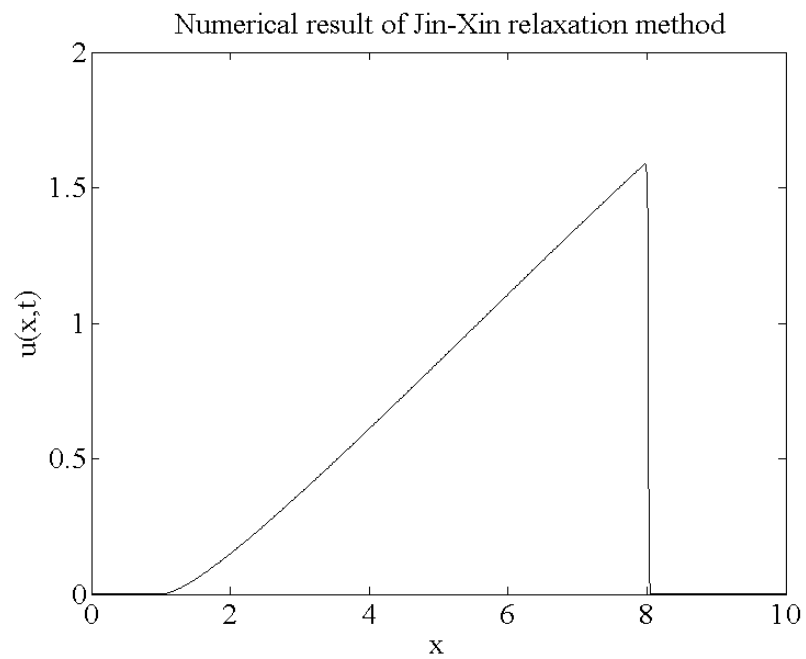


Figure 3. Numerical solution of Burgers equation using Jin–Xin relaxation method forward in time for case 2.

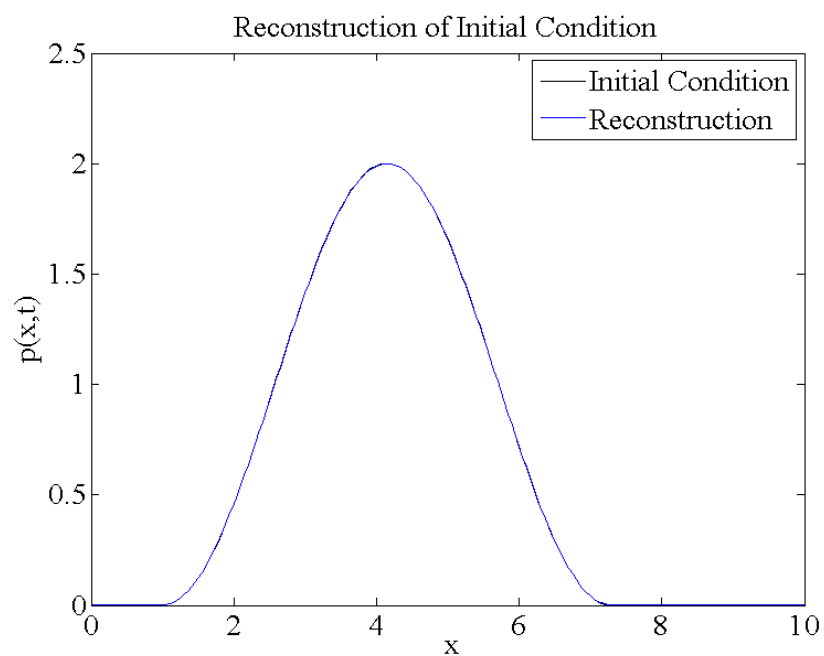


Figure 4. Comparison between the exact initial condition and the reconstruction for case 1.

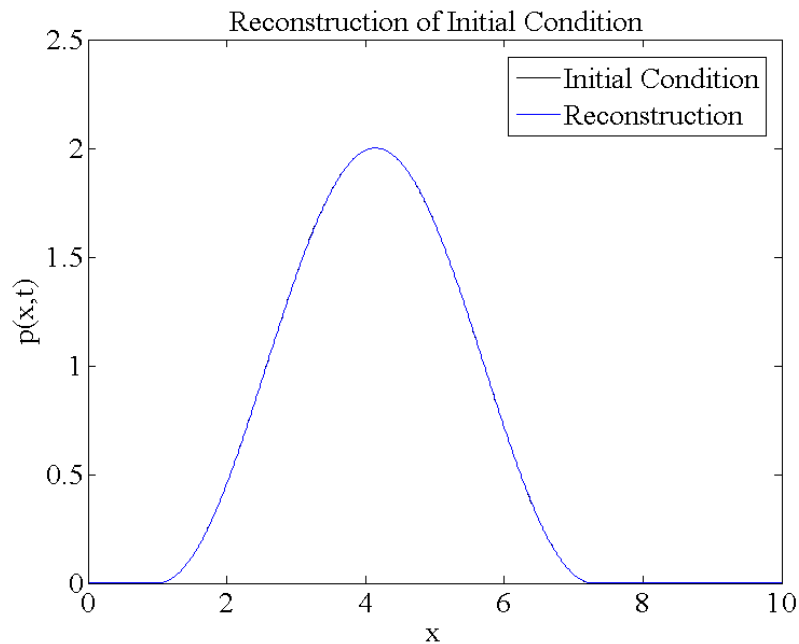


Figure 5. Comparison between the exact initial condition and the reconstruction for case 2.

3.1. Initial condition of Burgers equation

For computation in MATLAB, we take $x \in [0,10]$ with cell-width $\Delta x = 0.05$ and $t \in [0,0.5]$ with time step $\Delta t = 0.025 \Delta x$. The initial condition has been mentioned in equation (3). Figure 1 illustrates the exact initial condition that we will reconstruct later.

3.2. Results using Jin–Xin relaxation method forward in time

The Jin–Xin relaxation method is used to solve the Burgers equation with the boundary condition $u(0,t)$ and $u(10,t)$ are set to be transmissive and $\epsilon = 10^{-3}$.

Case 1: We take spatial domain $0 \leq x \leq 10$ and time domain $0 \leq t \leq 0.5$ with $\Delta x = 0.05$ and time step $\Delta t = 0.025 \Delta x$. Figure 2 shows the results at the final time.

Case 2: We take spatial domain $0 \leq x \leq 10$ and time domain $0 \leq t \leq 3$ with $\Delta x = 0.0125$ and time step $\Delta t = 0.025 \Delta x$. Figure 3 draws the results at the final time.

We notice that as time evolves the final solution becomes steeper to form (almost) a discontinuity.

3.3. Reconstruction of initial conditions using Lax–Friedrichs finite volume method backward in time

To reconstruct the initial condition, we set the final solution which was obtained earlier as the initial condition in the Lax–Friedrichs finite volume method backward in time. The boundary conditions are set to be transmissive and $\epsilon = 10^{-3}$.

Case 1: We use the same Δx and Δt as in the previous calculation in case 1 on Subsection 3.2. Figure 4 shows the reconstruction results.

Case 2: We select numerical settings the same as the previous simulation in case 2 on Subsection 3.2. Figure 5 illustrates the reconstruction results.

As we observe in Figures 4 and 5, the error between the initial condition and its reconstruction is insignificant.

3.4. Results of convergence rate

We take variety of Δx which are 0.4, 0.2, 0.1, 0.05, 0.025 and 0.0125, and $x \in [0,10]$, $t \in [0,0.5]$, and Δt are the same as the previous simulation. Table 1 contains the errors and convergence rate for problems of either case 1 or case 2 in the reconstruction of the initial condition.

Table 1. The numerical error and convergence rate of either case 1 or case 2.

Δx	Error	CR
0.4	0.0319	-
0.2	0.0090	1.825560
0.1	0.0030	1.584963
0.05	0.0012	1.321928
0.025	5.3692×10^{-4}	1.160255
0.0125	2.5736×10^{-4}	1.060919
	Rate	1.738406

From Table 1, we notice that the smaller Δx gives the smaller error. The convergence rate of our proposed method is one as Δx approaches to zero.

4. Conclusion

We have proposed a finite volume approach to reconstruct the initial condition for problems of conservation laws. The conservation law under investigation is the Burgers equation. The numerical method being proposed is the Lax–Friedrichs finite volume method. Our convergence rate of the numerical method is one, as the cell width is taken smaller approaching to zero.

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