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A staggered grid Jin–Xin relaxation method used to solve the Burgers equation

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Abstract. Burgers equation is considered. Burgers equation has applications in fluid dynamics, traffic flows, etc. It mimics the Navier–Stokes equations. Therefore, in this paper, an accurate method to solve the Burgers equations is sought. Jin–Xin relaxation method has been an active research area for collocated grids. However, the literature suggests that it has never been tested to solve models on staggered grids. In this paper, we test the performance of the Jin–Xin relaxation method in solving a hyperbolic problem on staggered grids. Our research results show that the Jin–Xin method is successful in solving the Burgers equations on staggered grids.

1. Introduction

Burgers equation is a partial differential equation that possesses the behaviour of conservation laws. Conservation laws have been used to model a number of physics problems, which involve the dynamics of mass, momentum, and energy of a physical system [1-2]. The Burgers equation is nonlinear and hyperbolic. Its solution can be either smooth or nonsmooth. It mimics the characteristics of the Navier–Stokes equations [3].

Conservation laws are governed by

$$u_t + f(u)_x = 0 \tag{1}$$

where the time variable $t \in [0, \infty)$ and the space variable $x \in (-\infty, \infty)$. The initial condition is denoted by

$$u(x, 0) = u_0(x) \tag{2}$$

Here, $u = u(x, t)$ is the variable of the conserved quantity which is dependent on the space and time variables. The function $f(u(x, t))$ represents the flux at the space point x and time t. The flux function for the Burgers equation is $f(u) = 1/2u^2$. The function $u_0(x)$ gives the function satisfies the Burgers equation at time $t = 0$. Some previous studies on the Burgers equation have been conducted, for example, see [4-7].

Staggered grid numerical methods have been active research areas for solving conservation law problems. On the other hand, the Jin–Xin relaxation method [8] has been successful to solve conservation laws on collocated grids [9-10]. The Burgers equation has also been attempted to be solved numerically [11-12]. However, the Jin–Xin relaxation method has never been tested to solve the Burgers equation via relaxation system, such as the Jin–Xin relaxation system. The goal of this paper is to implement a staggered grid Jin–Xin relaxation method for solving the Burgers Equation.

The rest of this paper is structured as follows. We recall the Jin–Xin relaxation method on collocated grids in Section 2. We propose the Jin–Xin relaxation method on staggered grids for solving the Burgers equation in Section 3. Numerical results are presented in Section 4. We conclude the paper with some remarks inferred in Section 5.

2. Problem formulation

Conservation laws (1) can be written in a relaxation form. Jin and Xin proposed that its relaxation system is

$$u_t + v_x = 0 \tag{3}$$

$$v_t + au_x = -\frac{1}{\varepsilon}(v - f(u)) \tag{4}$$

where the initial condition is now given by

$$u(x,0) = u_0(x) \tag{5}$$

$$v(x,0) = f\big(u_0(x)\big). \tag{6}$$

Here, $v = v(x,t)$ is a dummy variable and a is a positive constant satisfying

$$-\sqrt{a} \le f'(u) \le \sqrt{a} \tag{7}$$

for all u. The parameter ε is a small positive constant, which is called the relaxation rate. For smooth solutions, equations (3) and (4) is equivalent to

$$u_t + f(u)_x = \varepsilon(au_{xx} - u_n). \tag{8}$$

For $\varepsilon \to 0^+$, equation (8) approaches equation (1). This means that equations (3) and (4) is about equivalent to equation (1). We note that here for $\varepsilon \to 0^+$, we have approximated equation (1) which is nonlinear using the system of equations (3) and (4) which are linear.

3. Numerical method

In this section, we propose the staggered grid Jin–Xin relaxation method used to solve the Burgers equation. To do so, let us consider $u_i^n \approx u(x_i, t^n)$,

$$u_t|_i \approx \left.\frac{\Delta u}{\Delta t}\right|_i = \frac{u_i^{n+1} - u_i^n}{\Delta t}, \tag{9}$$

$$v_x|_n \approx \left.\frac{\Delta v}{\Delta x}\right|_n = \frac{v_{i+1/2}^n - v_{i-1/2}^n}{\Delta x}. \tag{10}$$

Equation (3) can then be written in a fully discrete form as [8, 11-12]

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+\frac{v_{i+1/2}^n-v_{i-1/2}^n}{\Delta x}=0\,, \tag{11}$$

or

$$u_i^{n+1}=u_i^n-\frac{\Delta t}{\Delta x}\left(v_{i+\frac{1}{2}}^n-v_{i-\frac{1}{2}}^n\right). \tag{12}$$

Equation (4) can be written is a fully discrete form as

$$\frac{v_{i+1/2}^{n+1}-v_{i+1/2}^n}{\Delta t}+a\frac{u_{i+1}^n-u_i^n}{\Delta x}=-\frac{1}{\varepsilon}(v_{i+1/2}^n-f(u_{i+1/2}^n))\,, \tag{13}$$

or

$$v_{i+1/2}^{n+1}=v_{i+1/2}^n-a\frac{\Delta t}{\Delta x}\left[(u_{i+1}^n-u_i^n)-\frac{1}{\varepsilon}(v_{i+1/2}^n-f\left(u_{i+1/2}^n\right)\right]. \tag{14}$$

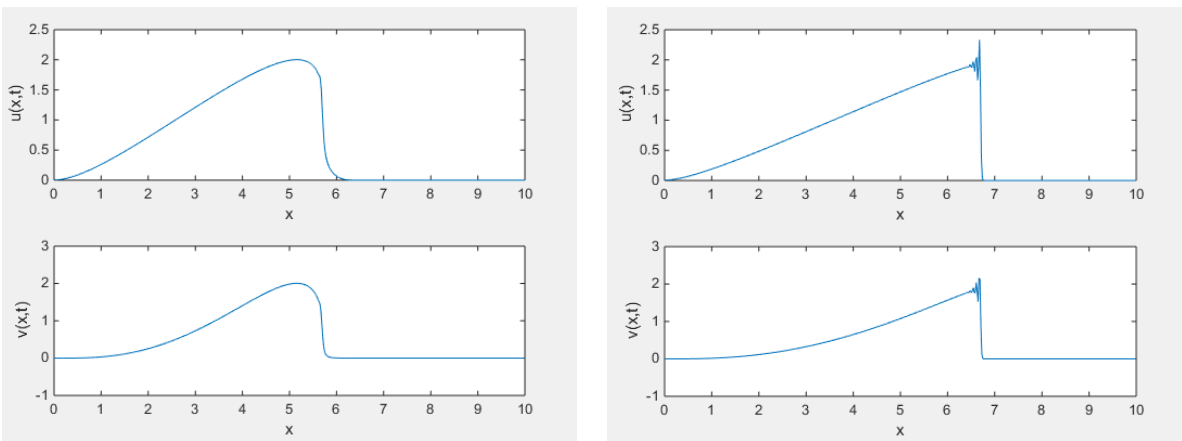
Here x_i denotes the i −th spatial point with $i=1,2,3,\ldots$ and t^n is the n −th temporal point with $0,1,2,\ldots$ In addition, Δt is the time step and Δx is the cell width. Note that the subscript i is written for the index of the space variable and the superscript n is intended for the index of the time variable.

We obtain that the staggered grid numerical scheme to solve the Burgers equation via its relaxation system is given by equations (12) and (14) simultaneously. All values at the right-hand sides of equations (12) and (14) are known, except the value of $f\left(u_{i+1/2}^n\right)$. The value of $f\left(u_{i+1/2}^n\right)$ is approximated using the average of $f(u_{i+1}^n)$ and $f(u_i^n)$.

4. Numerical results

In this section, we present our numerical results. We implement the proposed staggered grid Jin–Xin relaxation method to solve the (inviscid) Burgers equation

$$u_t+\left(\frac{1}{2}u^2\right)_x=0\,. \tag{15}$$



(a). Jin–Xin staggered grid solution at $t=1$. (b). Jin–Xin staggered grid solution at $t=2$.

Figure 1. Jin–Xin solution for the Burgers problem on staggered grids.

Let us consider that the Burgers equation (15) is defined for $0 \leq x \leq 10$, with the initial condition

$$u_0(x) = \begin{cases} 1 - \cos(x) \text{ if } 0 \leq x \leq 2\pi \\ 0 \qquad\qquad \text{if } 2\pi < x \leq 10 \end{cases} \tag{16}$$

and boundary condition

$$u(0,t) = u(10,t) = 0 \tag{17}$$

for all $t > 0$. The cell width is calculated as $\Delta x = L/(N-1)$ where N is the number of spatial points x, and L is the length of the space domain. In our simulation, we take $N = 1000$ and $L = 10$. We choose to take $\Delta t = 0.1\,\Delta x$ and $\varepsilon = 10^{-3}$ for stability.

Figure 1 shows the numerical results of the Burgers problem. For an initially given hyperbolic bump, as time evolves we obtain that the solution moves to the right based on its characteristics until a discontinuity is reached. Then this solution keeps moving to the right with its amplitude gets smaller as time evolves. These results are consistent with those in [11-12].

5. Conclusion

We have proposed a staggered grid Jin–Xin relaxation method for solving the Burgers equation. The Burgers equation is indeed solved numerically in a stable behaviour. As the Burgers equation is a nonlinear hyperbolic conservation law, so our proposed method should be able to be extended for other types of hyperbolic conservation laws. This could be a future research direction regarding this numerical method.

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References

[1] LeVeque R J 1992 *Numerical Methods for Conservation Laws* (Basel: Springer)
[2] Banda M K and Herty M 2012 Adjoint IMEX-based schemes for control problems governed by hyperbolic conservation laws *Computation Optimization and Applications* **51** 909
[3] Landajuela M 2011 Burgers equation *BCAM Internship-Summer 2011* (Bilbao: Basque Center for Applied Mathematics)
[4] Basto M, Semiano V and Calheiros F L 2007 Numerical study of modified Adomian's method applied to Burgers equation *Journal of Computational and Applied Mathematics* **206** 927
[5] Saka B 2008 A numerical study of the Burgers' equation *Journal of the Franklin Institute* **345** 328
[6] Joseph K T and Sachdev P L 2010 Initial boundary value problems for scalar and vector Burgers equations *Studies in Applied Mathematics* **106** 481
[7] Mungkasi S 2015 Some advantages of implementing an adaptive moving mesh for the solution to the Burgers equation *IOP Conference Series: Materials Science and Engineering* **78** 012031
[8] Jin S and Xin Z 1995 The relaxation schemes for systems of conservation laws in arbitrary space dimensions *Communication on Pure and Applied Mathematics* **48** 235
[9] Murthy A S V 2007 An alternate relaxation approximation to conservation laws *Journal of Computational and Applied Mathematics* **203** 437
[10] Zou Q, Zhao H and Wang T 2012 The Jin–Xin relaxation approximation of scalar conservation

laws in several dimensions with large initial perturbation *Journal of Differential Equations* **253** 563

[11] Mungkasi S and Laras F 2017 Jin–Xin relaxation solution to the spatially-varying Burgers equation in one dimension *Journal of Physics: Conference Series* **795** 012043

[12] Mungkasi S, Harini B H and Wiryanto L H 2017 Jin–Xin relaxation method used to solve the one-dimensional inviscid Burgers equation *Journal of Physics: Conference Series* **856** 012007