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A simple but accurate explicit finite difference method for the advection-diffusion equation

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Abstract. We consider the advection-diffusion equation in one dimension. The equation is solved using an explicit finite difference method. As the method is explicit, it is simple to implement. We investigate the performance of the method. We obtain that the method is indeed accurate.

1. Introduction
The advection-diffusion equation is a model that can be used for simulation of the spreading of pollutant. It governs the process of advection and diffusion concurrently. A large number of authors (to mention some of them, see [1-15]) have attempted to solve and use this equation in their simulations.

In this paper, we investigate the performance of a finite difference method due to Karahan [4] for solving the advection-diffusion equation of pollutant transports. The finite difference approach is appropriate because the equation is a partial differential equation having the parabolic type. We use the MATLAB software in the implementation of the finite difference method. The iterations are started from a given initial condition using an arbitrary space width and time step.

This paper is organised simply as follows. Sections 2-5 contains the mathematical model, numerical method, numerical results, and conclusion, respectively.

2. Advection-diffusion equation
The mathematical model describing the pollutant transport is the one-dimensional advection-diffusion equation [4]

$$\frac{\partial c}{\partial t}+u\frac{\partial c}{\partial x}=D\frac{\partial^2 c}{\partial x^2},\ 0<x<L,\ 0<t\leq T \tag{1}$$

with the initial condition

$$c(x,0)=f(x),\quad 0\leq x<L \tag{2}$$

and boundary conditions

$$c(0,t)=g(t),\ \ 0<t\leq T \tag{3}$$

$$c(L,t)=h(t),\ \ 0<t\leq T \tag{4}$$

in a finite space domain of x and a finite time domain of t. The variable $c = c(x,t)$ denotes the concentration of a pollutant. In addition, f, g and h are known functions; u, x, D respectively represent the velocity, the direction, and the dispersion coefficient. Here u and D are assumed to be positive constants. As advection-diffusion equation contains advection and diffusion phenomena, the solution should reveal those two phenomena.

3. Finite difference method

We use the finite difference method due to Karahan [4] to solve the problem. Note that the equation is a partial differential equation of the parabolic type, so finite difference methods should be able to solve the problem. This is because finite difference methods are based on the direct-discretisation of differential equations. Solutions of parabolic partial differential equations are always continuous even when the initial condition is discontinuous.

We discretise the problem by choosing a stepsize Δx as the spatial step in x and stepsize Δt as the temporal step in t. We also approximate the concentration c on a grid of point in the xt-plane. Moreover,

$$x_i = i\Delta x,\ i = 1, 2, 3, ..., M,\ \Delta x = \frac{L}{M} \tag{5}$$

$$t_{\mathrm{n}} = n\Delta t,\ t = 1,2,3,\ldots,N,\ \Delta t = \frac{T}{N} \tag{6}$$

Furthermore, we take the following discretisations

$$\frac{\partial c}{\partial t} = \frac{c(i,n+1) - c(i,n)}{\Delta t} \tag{7}$$

$$u\frac{\partial c}{\partial x} = u\left(\frac{1}{2\Delta x}\right)\left(c(i,n+1) - c(i-1,n+1) + c(i+1,n) - c(i,n)\right) \tag{8}$$

$$D\frac{\partial^2 c}{\partial x^2} = D\left[\frac{\theta}{(\Delta x)^2}\right]\left[c(i-1,n+1) - c(i,n+1) + c(i+1,n) - c(i,n)\right] + D\left[\frac{1-\theta}{(\Delta x)^2}\right][c(i-1,n) - 2c(i,n) + c(i+1,n)] \tag{9}$$

where θ is the weighting factor. Here the time integration (7) is forward difference, so it has the first order of accuracy.

Substituting the equations above into equation (1), we have

$$\frac{c(i,n+1)-c(i,n)}{\Delta t} + u\left(\frac{1}{2\Delta x}\right)[c(i,n+1) - c(i-1,n+1) + c(i+1,n) - c(i,n)] = D\left[\frac{\theta}{(\Delta x)^2}\right] [c(i-1,n+1) - c(i,n+1) + c(i+1,n) - c(i,n)] + D\left[\frac{1-\theta}{(\Delta x)^2}\right][c(i-1,n) - 2c(i,n) + c(i+1,n)] \tag{10}$$

for $1 \le i \le M$ and $1 \le n \le N$. Equation (10) is equivalent to

$$\left[1 + \frac{Cr}{2} + \theta\left(\frac{Cr}{Pe}\right)\right]c(i,n+1) - \left[\frac{Cr}{2} + \theta\left(\frac{Cr}{Pe}\right)\right]c(i-1,n+1) = \left[\left(\frac{Cr}{Pe}\right) - \theta\left(\frac{Cr}{Pe}\right)\right]c(i-1,n) + \left[1 + \left(\frac{Cr}{2}\right) - 2\left(\frac{Cr}{Pe}\right) + \theta\left(\frac{Cr}{Pe}\right)\right]c(i,n) + \left[\left(\frac{Cr}{Pe}\right) - \left(\frac{Cr}{2}\right)\right]c(i+1,n) \tag{11}$$

where the Courant number $Cr = u\Delta t / \Delta x$, and the Peclet number $Pe = u\Delta x / D$. The boundary condition, equation (3), makes the equation is explicit, so we obtain

$$c(i,n+1)=\left\{\left[\left(\frac{Cr}{Pe}\right)-\theta\left(\frac{Cr}{Pe}\right)\right]c(i-1,n)+\left[1+\left(\frac{Cr}{2}\right)-2\left(\frac{Cr}{Pe}\right)+\theta\left(\frac{Cr}{Pe}\right)\right]c(i,n)+\left[\left(\frac{Cr}{Pe}\right)-\left(\frac{Cr}{2}\right)\right]\right.$$
$$\left.c(i+1,n)+\left[\frac{Cr}{2}+\theta\left(\frac{Cr}{Pe}\right)\right]c(i-1,n+1)\right\}\left[1+\frac{Cr}{2}+\theta\left(\frac{Cr}{Pe}\right)\right]^{-1} \quad (12)$$

Equation (12) is the explicit finite difference method due to Karahan [4] for solving the advection-diffusion equation (1).

4. Results and discussion

In this section, we present our numerical results and discuss them. We assume that all quantities are in SI units with the MKS system.

As the numerical test, we consider the following settings. The initial condition is

$$c(x,0)=\exp\left(-\frac{(x-1)^2}{D}\right). \quad (13)$$

This initial condition is illustrated in Figure 1. Parameters being used are $D=0.005$ and $u=0.8$. The spatial step is $\Delta x=0.025$, and the temporal step is $\Delta t=0.0125$. We consider the spatial domain $0\le x\le 9$. We seek the solution at time $t=2.5$ and $t=5$.

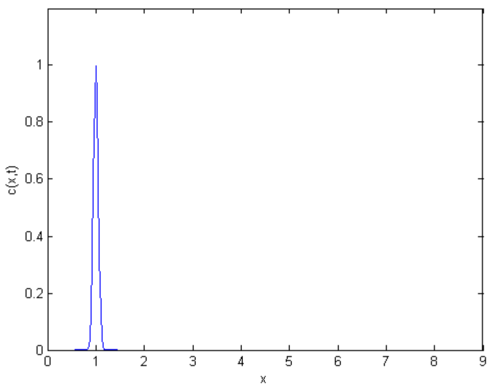

Figure 1. Initial condition ($t=0$) of an advection-diffusion problem.

The analytical solution for this problem is [4]

$$c(x,t)=\frac{1}{\sqrt{4t+1}}\exp\left(-\frac{(x-1-ut)^2}{D\,(4t+1)}\right). \quad (14)$$

Note that this problem is chosen, because the analytical solution has been found (see Sankaranarayanan et al. [11]). In general, the analytical solution to the advection-diffusion equation is not available. Therefore, we do need numerical methods to solve the advection-diffusion equation.

Numerical results show that the method is simple to implement, yet gives accurate solutions. This is shown in Figure 2 as representatives of our numerical results. This figure shows the solution at time $t=2.5$ and $t=5$, respectively. Notice that our solutions reveal advection and diffusion phenomena at the same time. That is, as time evolves the solution translates to the right (advection phenomenon) and it diffuses from the centre of the concentration position (diffusion phenomenon).

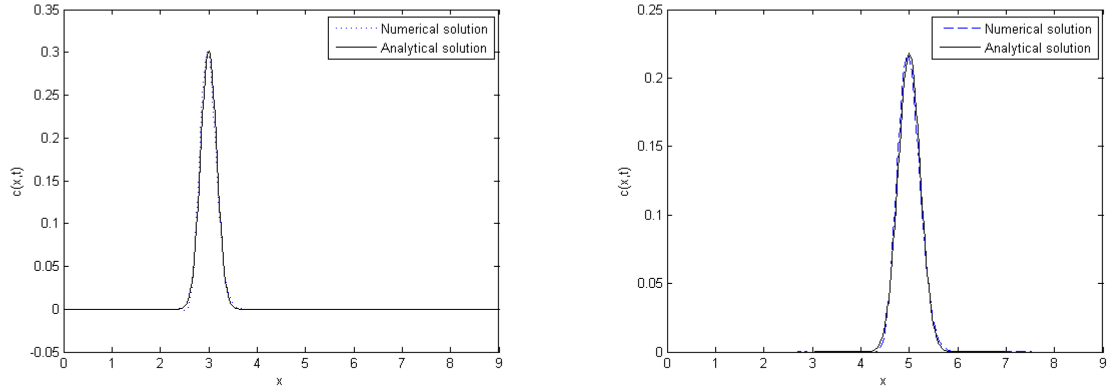

Figure 2. Numerical and analytical solutions using $\Delta x = 0.025$ and $\Delta t = 0.0125$ at time $t = 2.5$ (left figure) and $t = 5.0$ (right figure) respectively. We observe that the numerical solution is very accurate, as the error is quite small with respect to the analytical solution.

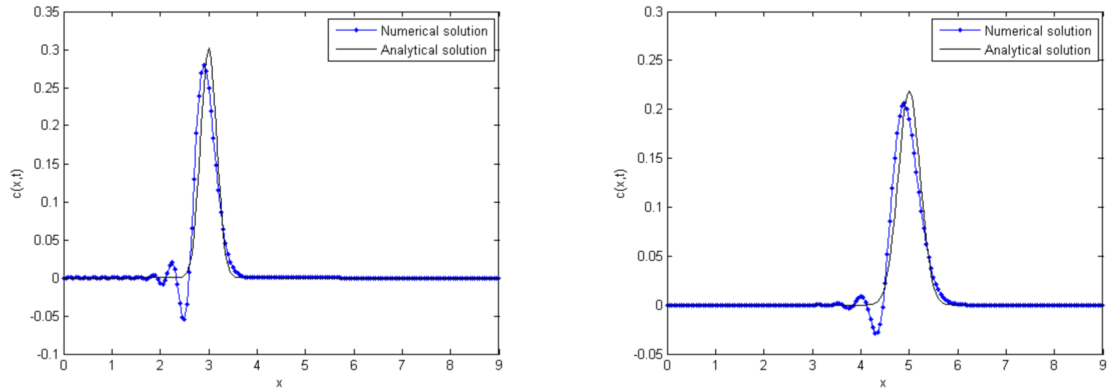

Figure 3. Numerical and analytical solutions using $\Delta x = 0.05$ and $\Delta t = 0.0125$ at time $t = 2.5$ (left figure) and $t = 5.0$ (right figure) respectively. We observe that artificial oscillations occur if the spatial step is taken too large.

We should not take the spatial step Δx too large, as it may not be able to cover the accuracy of our numerical discretisation. Taking the spatial step Δx too large may lead to unphysical solutions. That is, artificial oscillations may occur in the numerical solutions if the spatial step Δx is too large. This is shown in Figure 3. This figure shows the solution at time $t = 2.5$ and $t = 5$, respectively using the same numerical settings as before, except the spatial step here is $\Delta x = 0.05$.

5. Conclusion

We have investigated the performance of an explicit finite difference method for solving the advection-diffusion equation. As we use the forward difference for the time derivative, numerical solutions should have the first order of accuracy at best. The general analytical solution to the advection-diffusion equation is not available, so we need the numerical method to solve the advection-diffusion equation for the general case. The proposed method can be extended for solving the advection-diffusion equation with a source term.

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