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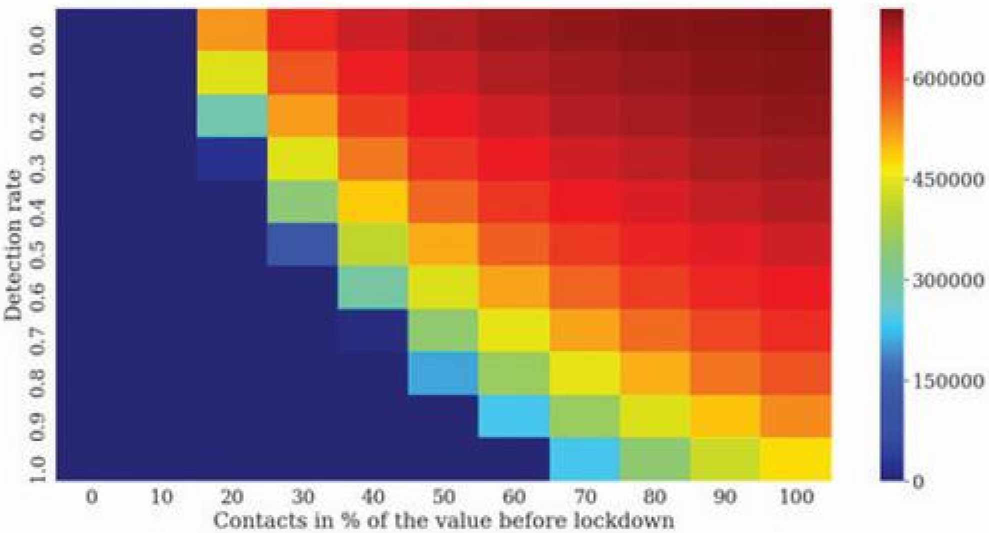

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PREFACE: 9th Jagna International Workshop: Stochastic Analysis – Mathematical Methods and Real-World Models

Stochastic fluctuations are ubiquitous in complex biological, physical, and social systems. Literature abounds in studies applying commonly encountered stochastic processes: Wiener, Lévy, Ornstein-Uhlenbeck, Poisson, noise in various forms – white, grey and colored, and fractional stochastic processes for anomalous diffusion. However, rapid advances in technology continue to spew large volumes of data. Maintaining a big picture perspective and decoding information from such rapid data outflow require an expanded mathematical toolbox that minimizes complicated estimation and approximation methods. To address this issue, the "*9th Jagna International Workshop*: *Stochastic Analysis – Mathematical Methods and Real-World Models*" held on 8 – 18 January 2020 at the Research Center for Theoretical Physics, Central Visayan Institute Foundation, Jagna, Bohol, Philippines, served as a forum for discussion of stochastic analytical methods for real world models of complex systems such as those exhibiting memory or dissipation.

The informal nature of the Jagna Workshop series is meant to foster active interaction among speakers and participants, and to encourage graduate students and postdoctoral researchers to get insights from invited lecturers so that viable research targets could be clearly defined. There were 63 participants from Japan, USA, Portugal, Germany, Indonesia, Poland, and the Philippines. This volume contains the Proceedings of the Workshop so that others may benefit from the talks presented. Our gratitude to the speakers for submitting the author-prepared PDFs for this Proceedings volume.

We thank the participants, speakers, and sponsors for their active participation which made the 9th Jagna International Workshop successful. For research and teaching capacity building in local institutes, we wish to thank Prof. Ludwig Streit, Prof. Tyll Krüger, and Prof. José Luís da Silva for conducting introductory lectures on stochastic analysis one week prior to the research conference.

The publication of this volume was made possible by support from SMART Communications, Inc. and the Philippine Council for Industry, Energy and Emerging Technology Research and Development, Department of Science and Technology (Philippines). We are grateful for their continuing support for the Jagna Workshops.

C. C. Bernido, M. V. Carpio-Bernido, J. B. Bornales and L. Streit
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Herry Pribawanto Suryawan

A White Noise Approach to Subfractional Brownian Motion

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Abstract. In this paper we study the subfractional Brownian motion by using white noise analysis. First we recall a representation of subfractional Brownian motion on the white noise space and prove the existence of the Donsker delta function of subfractional Brownian motion. We also solve the Langevin equation and a Wick-type linear stochastic differential equation driven by subfractional Brownian motion by using the S-transform method.

INTRODUCTION

The fractional Gaussian processes have been intensively studied due to the fact that the theory is interesting in itself and at the same time it has a broad range of applications. Fractional Brownian motion (fBm) seems to be the simplest fractional Gaussian process. It was introduced by Kolmogorov in [1] with the definition as follows. The fBm with Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B^H = \left(B^H(t)\right)_{t\geq 0}$ with $B^H(0) = 0$ *a.s.* and covariance function

$$\mathrm{Cov}\left(B^H(t), B^H(s)\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s,t \geq 0.$$

The case $H = \frac{1}{2}$ corresponds to the standard Brownian motion (Bm). Mandelbrot and Van Ness in [2] further studied this process and, in particular, obtained a representation of fBm as a stochastic integral with respect to Bm. It is well-known that fBm satisfies self-similarity, long/short-range dependence, Hölder continuity of the sample paths, and stationarity of increments. Furthermore, fBm is the only continuous Gaussian process which is self-similar and has stationary increments. Due to these properties fBm has been used as an important tool for stochastic modeling in hydrology, telecommunication, turbulence, image processing and segmentation, medical image analysis, network traffic analysis, financial mathematics, etc. A comprehensive study of fBm can be found in [3, 4, 5, 6, 7] and references therein.

Another generalization of the Brownian motion is the so-called subfractional Brownian motion (sfBm). It was introduced by Bojdecki *et al* in [8] in connection with the study of occupation time fluctuations of branching particle systems with Poisson initial condition. The sfBm with parameter $H \in (0,1)$ is a centered Gaussian process $S^H = \left(S^H(t)\right)_{t\geq 0}$ with $S^H(0) = 0$ *a.s.* and covariance function

$$\mathrm{Cov}\left(S^H(t), S^H(s)\right) = s^{2H} + t^{2H} - \frac{1}{2}\left((s+t)^{2H} + |t-s|^{2H}\right), \quad s,t \geq 0. \tag{1}$$

When $H = \frac{1}{2}$ one recovers the standard Bm. One way to establish the existence of sfBm is by considering the process $\xi^H = \left(\xi^H(t)\right)_{t\geq 0}$ defined by

$$\xi^H(t) := \frac{B^H(t) + B^H(-t)}{\sqrt{2}}, \quad t \geq 0,$$

where $\left(B^H(t)\right)_{t\in\mathbb{R}}$ is a two-sided fBm. One can easily check that the covariance function of ξ^H coincides with (1). For details we refer to [6, 8].

The sfBm is intermediate between Bm and fBm in the sense that it has properties analogous to those of fBm, but the increments on non-overlapping intervals are more weakly correlated, and their covariance decays polynomially at a higher rate. In some applications, such as turbulence phenomena in hydromechanics, fBm is an adequate model for small increments, but it seems to be inadequate for large increments. For this reason, sfBm may be an alternative to fBm in some stochastic models.

The sfBm is neither a semimartingale nor a Markov process unless $H = \frac{1}{2}$. This implies that the powerful techniques from classical stochastic calculus are not available when dealing with S^H. The sfBm shares many properties to those of fBm such as self-similarity and long-range dependence. On the contrary the increments of sfBm are not stationary.

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It is known that sfBm is a quasi-helix in the sense of Kahane, that is it satisfies the inequalities, for any $s,t \geq 0$ with $t > s$,

$$\left(\left(2-2^{2H-1}\right)\wedge 1\right)(t-s)^{2H} \leq \mathbb{E}\left(S^H(t)-S^H(s)\right)^2 \leq \left(\left(2-2^{2H-1}\right)\vee 1\right)(t-s)^{2H}.$$

As a consequence, sample paths of sfBm is Hölder continuous of order γ for any $\gamma < H$. As a conclusion sfBm is suitable for modeling of random phenomenon which posseses self-similarity, long/short dependence, continuous sample paths but non-stationary increments. Some works on sfBm, including basic and advanced properties, stochastic calculus, local times, can be found in [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20], among others.

In this paper we recall the realization of sfBm on the white noise probability space and prove that the Donsker delta function of sfBm is a white noise distribution in the Hida sense. We also construct solutions to a subfractional version of the Ornstein-Uhlenbeck and a linear stochastic differential with Wick-multiplicative subfractional white noise in a suitable white noise distribution space.

BASICS ON WHITE NOISE ANALYSIS

In this section we give background on the white noise theory used throughout this paper. For a more comprehensive discussions including various applications of white noise theory we refer to [21, 22, 23] and references therein. We start with the Gelfand triple

$$\mathscr{S}_d(\mathbb{R}) \hookrightarrow L^2_d(\mathbb{R}) \hookrightarrow \mathscr{S}'_d(\mathbb{R}),$$

where $\mathscr{S}_d(\mathbb{R})$ is the space of $\mathbb{R}^d$-valued Schwartz test function, $\mathscr{S}'_d(\mathbb{R})$ is the space of $\mathbb{R}^d$-valued tempered distributions, and $L^2_d(\mathbb{R})$ is the real Hilbert space of all $\mathbb{R}^d$-valued Lebesgue square-integrable functions. Next, we construct a probability space $(\mathscr{S}'_d(\mathbb{R}), \mathscr{C}, \mu)$ where $\mathscr{C}$ is the Borel σ-algebra generated by weak topology on $\mathscr{S}'_d(\mathbb{R})$ and the probability measure μ is uniquely determined through the Bochner-Minlos theorem by fixing the characteristic function

$$C(\vec{f}) := \int_{\mathscr{S}'_d(\mathbb{R})} \exp\left(i\langle\vec{\omega},\vec{f}\rangle\right) d\mu(\vec{\omega}) = \exp\left(-\frac{1}{2}|\vec{f}|_0^2\right)$$

for all $\vec{f} \in \mathscr{S}_d(\mathbb{R})$. Here $|\cdot|_0$ denotes the usual norm in the $L^2_d(\mathbb{R})$, and $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $\mathscr{S}'_d(\mathbb{R})$ and $\mathscr{S}_d(\mathbb{R})$. The dual pairing is considered as the bilinear extension of the inner product on $L^2_d(\mathbb{R})$, i.e.

$$\left\langle\vec{g},\vec{f}\right\rangle = \sum_{j=1}^{d}\int_{\mathbb{R}} g_j(x) f_j(x)\, dx,$$

for all $\vec{g} = (g_1,\ldots,g_d) \in L^2_d(\mathbb{R})$ and $\vec{f} = (f_1,\ldots,f_d) \in \mathscr{S}_d(\mathbb{R})$. This probability space is known as the $\mathbb{R}^d$-valued white noise space since it contains the sample paths of the d-dimensional Gaussian white noise. In this setting a d-dimensional Brownian motion can be represented by a continuous modification of the stochastic process $B = (B_t)_{t\geq 0}$ with

$$B(t) := \left(\left\langle\cdot,\mathbf{1}_{[0,t]}\right\rangle,\ldots,\left\langle\cdot,\mathbf{1}_{[0,t]}\right\rangle\right),$$

such that for independent d-tuples of Gaussian white noise $\vec{\omega} = (\omega_1,\ldots,\omega_d) \in \mathscr{S}'_d(\mathbb{R})$

$$B(t,\vec{\omega}) = \left(\left\langle\omega_1,\mathbf{1}_{[0,t]}\right\rangle,\ldots,\left\langle\omega_d,\mathbf{1}_{[0,t]}\right\rangle\right),$$

where $\mathbf{1}_A$ denotes the indicator function of a set $A \subset \mathbb{R}$.

In the sequel we will use the Gel'fand triple

$$(\mathscr{S}) \hookrightarrow L^2(\mu) := L^2\left(\mathscr{S}'_d(\mathbb{R}),\mathscr{C},\mu\right) \hookrightarrow (\mathscr{S})^*$$

where $(\mathscr{S})$ is the space of white noise test functions obtained by taking the intersection of a family of Hilbert subspaces of $L^2(\mu)$. The space of white noise distributions $(\mathscr{S})^*$ is defined as the topological dual space of $(\mathscr{S})$.

Elements of $(\mathscr{S})$ and $(\mathscr{S})^*$ are known as *Hida test functions* and *Hida distributions*, respectively. An important tool in white noise analysis is the S-transform which can be considered as the Laplace transform with respect to the Gaussian measure. The S-transform of $\Phi \in (\mathscr{S})^*$ is defined as

$$(\mathrm{S}\Phi)(\vec{f}) := \left\langle\left\langle \Phi, :\exp\left(\left\langle \cdot, \vec{f}\right\rangle\right): \right\rangle\right\rangle, \quad \vec{f} \in \mathscr{S}_d(\mathbb{R}),$$

where

$$:\exp\left(\left\langle \cdot, \vec{f}\right\rangle\right) ::= C(\vec{f})\exp\left(\left\langle \cdot, \vec{f}\right\rangle\right),$$

is the so-called Wick exponential and $\langle\langle \cdot, \cdot\rangle\rangle$ denotes the dual pairing between $(\mathscr{S})^*$ and $(\mathscr{S})$. We define this dual pairing as the bilinear extension of the sesquilinear inner product on $L^2(\mu)$. The S-transform provides a convenient way to identify a Hida distribution $\Phi \in (\mathscr{S})^*$, in particular, when it is hard to find the explicit form for the Wiener-Itô chaos decomposition of Φ. We now state some properties of the S-transform.

Theorem 1 *[22]*

1. *The S-transform is injective, i.e. if $S\Phi(\varphi) = S\Psi(\varphi)$ for all $\varphi \in \mathscr{S}_1(\mathbb{R})$, then $\Phi = \Psi$.*
2. *Let a stochastic distribution process $X : I \to (\mathscr{S})^*$ is differentiable. Then, $S\frac{d}{dt}X(t)(\varphi) = \frac{d}{dt}SX(t)(\varphi)$ for all $\varphi \in \mathscr{S}_1(\mathbb{R})$.*

In white noise analysis there is a way to define multiplication of two distributions using the S-transform. Let $\Phi, \Psi \in (\mathscr{S})^*$. The Wick product of Φ and Ψ is defined as

$$\Phi \diamond \Psi := \mathrm{S}^{-1}(\mathrm{S}\Phi \cdot \mathrm{S}\Psi).$$

We will need also the Wick exponential function $\exp^\diamond$ which is defined for $\Phi \in (\mathscr{S})^*$ by

$$S\exp^\diamond(\Phi)(\varphi) = \exp^\diamond(S\Phi(\varphi)), \quad \varphi \in \mathscr{S}_1(\mathbb{R}).$$

See [24] for details on Wick product and calculus based on it.

Finally in this section we state a sufficient condition on the Bochner integrability of a family of Hida distributions which depend on an additional parameter.

Theorem 2 *[25] Let $(\Omega, \mathscr{A}, \nu)$ be a measure space and $\lambda \mapsto \Phi_\lambda$ be a mapping from Ω to $(\mathscr{S})^*$. If*

(1) the mapping $\lambda \mapsto \mathrm{S}(\Phi_\lambda)(\vec{f})$ is measurable for all $\vec{f} \in \mathscr{S}_d(\mathbb{R})$, and

(2) there exist $C_1(\lambda) \in L^1(\Omega, \mathscr{A}, \nu)$, $C_2(\lambda) \in L^\infty(\Omega, \mathscr{A}, \nu)$ and a continuous seminorm $\|\cdot\|$ on $\mathscr{S}_d(\mathbb{R})$ such that for all $z \in \mathbb{C}$, $\vec{f} \in \mathscr{S}_d(\mathbb{R})$

$$\left|\mathrm{S}(\Phi_\lambda)(z\vec{f})\right| \leq C_1(\lambda)\exp\left(C_2(\lambda)|z|^2\left\|\vec{f}\right\|^2\right),$$

then Φ_λ is Bochner integrable with respect to some Hilbertian norm which topologizing $(\mathscr{S})^*$. Hence $\int_\Omega \Phi_\lambda \, d\nu(\lambda) \in (\mathscr{S})^*$, and furthermore

$$\mathrm{S}\left(\int_\Omega \Phi_\lambda \, d\nu(\lambda)\right) = \int_\Omega \mathrm{S}(\Phi_\lambda)\, d\nu(\lambda).$$

WHITE NOISE ANALYSIS OF SFBM

In this section we recall a representation of sfBm on the white noise space and show the existence of the Donsker delta function of sfBm. In order to represent sub-fBm on the white noise space, we use the following operator

$$M_-^H f := \begin{cases} C_H D_-^{-(H-\frac{1}{2})} f & \text{if } 0 < H < \frac{1}{2} \\ f & \text{if } H = \frac{1}{2} \\ C_H I_-^{H-\frac{1}{2}} f & \text{if } \frac{1}{2} < H < 1, \end{cases}$$

where $C_H = \sqrt{2H\sin(\pi H)\Gamma(2H)}$ dan Γ denotes the gamma function. Here $I_-^\beta f$, $0<\beta<1$, is the Weyl type fractional integral operator defined by

$$\left(I_-^\beta f\right)(x) := \frac{1}{\Gamma(\beta)}\int_x^\infty f(t)\,(t-x)^{\beta-1}\,dt$$

if the integrals exist for almost all $x\in\mathbb{R}$ and $D_-^\beta f$, $0<\beta<1$ is the Marchaud type fractional derivative operator defined by

$$\left(D_-^\beta f\right)(x) := \lim_{h\to 0^+}\frac{\beta}{\Gamma(1-\beta)}\int_h^\infty \frac{f(x)-f(x+t)}{t^{1+\beta}}\,dt,$$

if the limit exists in $L^p(\mathbb{R})$ for some $p>1$. For any Borel function f on $[0,\infty)$ we define its odd extension f° by

$$f^\circ(x) = \begin{cases} f(x) & \text{if } x\geq 0 \\ -f(-x) & \text{if } x<0. \end{cases}$$

Note that sfBm S^H can be written as a Volterra process with the following moving average representation (see e.g. [8]):

$$S^H(t) = K_H\int_{\mathbb{R}}\left((t-s)_+^{H-\frac{1}{2}} + (t+s)_-^{H-\frac{1}{2}} - 2(-s)_+^{H-\frac{1}{2}}\right)dB(s), \tag{2}$$

where

$$K_H = \frac{1}{\sqrt{2}}\left(\int_0^\infty\left((1+x)^{H-\frac{1}{2}} - x^{H-\frac{1}{2}}\right)^2 dx + \frac{1}{2H}\right)^{-1/2} = \frac{\sqrt{H\sin(\pi H)\Gamma(2H)}}{\Gamma\left(H+\frac{1}{2}\right)},$$

$x_+ = \max\{x,0\}$, $x_- = \max\{-x,0\}$, and $(B(t))_{t\in\mathbb{R}}$ is a two-sided Brownian motion. For more information on a white noise approach to Volterra process see [26]. Based on the moving average representation (2) one can show the following.

Proposition 1 *[17] It holds that $M_-^H\mathbf{1}_{[0,t]}^\circ \in L^2(\mathbb{R})$ and*

$$S^H(t) = \frac{1}{\sqrt{2}}\int_{\mathbb{R}}\left(M_-^H\mathbf{1}_{[0,t]}^\circ\right)(s)\,dB_s = \left\langle \cdot, \frac{1}{\sqrt{2}}M_-^H\mathbf{1}_{[0,t]}^\circ\right\rangle,$$

where $B=(B(t))_{t\in\mathbb{R}}$ is a one-dimensional Brownian motion on the real line and $\mathbf{1}_A^\circ$ denotes the odd extension of the indicator function of a set $A\subset\mathbb{R}$..

The generalized stochastic process $W^H = \left(W(t)^H\right)_{t\geq 0}$ defined by

$$W^H(t) := \frac{d}{dt}\left(S^H(t)\right) = \lim_{h\to 0}\frac{S^H(t+h)-S^H(t)}{h},$$

where the convergence takes place in the Hida distribution space $(\mathscr{S})^*$, is called the one-dimensional subfractional white noise. Note that for any $t\geq 0$ $S^H(t)\in L^2(\mu)$ and $W^H(t)\in(\mathscr{S})^*$. The d-dimensional sfBm can be represented on the white noise space by a continuous modification of the stochastic process $S^H = \left(S^H(t)\right)_{t\geq 0}$ with

$$S^H(t) := \left(\left\langle \cdot, \frac{1}{\sqrt{2}}M_-^H\mathbf{1}_{[0,t]}^\circ\right\rangle, \ldots, \left\langle \cdot, \frac{1}{\sqrt{2}}M_-^H\mathbf{1}_{[0,t]}^\circ\right\rangle\right),$$

such that for independent d-tuples of Gaussian white noise $\vec{\omega} = (\omega_1,\ldots,\omega_d)\in\mathscr{S}_d'(\mathbb{R})$

$$S^H(t,\vec{\omega}) = \left(\left\langle \omega_1, \frac{1}{\sqrt{2}}M_-^H\mathbf{1}_{[0,t]}^\circ\right\rangle, \ldots, \left\langle \omega_d, \frac{1}{\sqrt{2}}M_-^H\mathbf{1}_{[0,t]}^\circ\right\rangle\right).$$

Apart from the operator M_-^H above we will need the following operator:

$$M_+^H f := \begin{cases} C_H D_+^{-(H-\frac{1}{2})} f & \text{if } 0 < H < \frac{1}{2} \\ f & \text{if } H = \frac{1}{2} \\ C_H I_+^{H-\frac{1}{2}} f & \text{if } \frac{1}{2} < H < 1, \end{cases}$$

where, for $0 < \beta < 1$, $I_+^\beta f$ is the Weyl fractional integral operator defined by

$$\left(I_+^\beta f\right)(x) := \frac{1}{\Gamma(\beta)} \int_{-\infty}^{x} f(t)\,(x-t)^{\beta-1}\,dt$$

if the integrals exist for almost all $x \in \mathbb{R}$ and $D_+^\beta f$ is the Marchaud fractional derivative operator defined by

$$\left(D_+^\beta f\right)(x) := \lim_{h\to 0^+} \frac{\beta}{\Gamma(1-\beta)} \int_h^\infty \frac{f(x)-f(x-t)}{t^{1+\beta}}\,dt,$$

if the limit exists in $L^p(\mathbb{R})$ for some $p > 1$. One can show that M_-^H and M_+^H are dual operators in the sense that

$$\left(f, M_-^H g\right)_0 = \left(M_+^H f, g\right)_0$$

for any function f and g satisfying some regularity condition, for example indicator function or Schwartz test function. For details we refer to [27]. The following result shows that the operator $M_\pm^H$ interchanges with the S-transform.

Lemma 1 *[28] Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. If $M_\pm^H X$ exist for some $X : \mathbb{R} \to L^2(\Omega, \mathscr{F}, \mathbb{P})$, then*

$$\mathbb{E}\left(\left(M_\pm^H X\right)_t \Phi\right) = M_\pm^H\left(\mathbb{E}\left(X_t \Phi\right)\right)$$

for all $\Phi \in L^2(\Omega, \mathscr{F}, \mathbb{P})$. For $H < \frac{1}{2}$ the convergence of the fractional derivative on the right-hand side is in the $L^p(\mathbb{R})$ sense, if $M_\pm^{-(H-\frac{1}{2})} X \in L^p(\mathbb{R}; \Omega, \mathscr{F}, \mathbb{P})$. In particular, the operator $M_\pm^H$ interchange with the S-transform.

The S-transform of sfBm and subfractional white noise can be computed explicitely, see [17, 28] for proofs and details.

Lemma 2 *Let $(S^H(t))_{t\geq 0}$ dan $(W^H(t))_{t\geq 0}$ be one-dimensional sfBm and subfractional white noise, respectively. Then, for any $\varphi \in \mathscr{S}_1(\mathbb{R})$ it holds*

1.

$$\mathrm{S}S^H(t)(\varphi) = \left\langle \varphi, \frac{1}{\sqrt{2}} M_-^H \mathbf{1}_{[0,t]}^\circ \right\rangle = \frac{1}{\sqrt{2}} \left(\int_0^t M_+^H \varphi(s)\,ds - \int_{-t}^0 M_+^H \varphi(s)\,ds \right).$$

2.

$$\mathrm{S}W^H(t)(\varphi) = \frac{1}{\sqrt{2}} \left(\left(M_+^H \varphi\right)(t) - \left(M_+^H \varphi\right)(-t) \right).$$

Now we show that the Donsker delta function of a sfBm is a well-defined element in the space of Hida distributions.

Theorem 3 *Let $S^H = \left(S_1^H, \ldots, S_d^H\right)$ be a d-dimensional sfBm and $\vec{x} \in \mathbb{R}^d$. The Bochner integral*

$$\delta\left(S^H(t) - \vec{x}\right) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\left(i\langle \vec{\lambda}, S^H(t) - \vec{x}\rangle\right) d\vec{\lambda}$$

is a Hida distribution with the S-transform given by

$$\mathrm{S}\delta\left(S^H(t) - \vec{x}\right)(\vec{\varphi}) = \left(\frac{1}{2\pi\left(2-2^{2H-1}\right)t^{2H}}\right)^{\frac{d}{2}} \exp\left(-\frac{1}{2\left(2-2^{2H-1}\right)t^{2H}} \sum_{j=1}^d \left(x_j - \left\langle \varphi_j, \frac{1}{\sqrt{2}} M_-^H \mathbf{1}_{[0,t]}^\circ \right\rangle\right)^2\right),$$

for any $\vec{\varphi} \in \mathscr{S}_d(\mathbb{R})$.

Proof. For any $\vec{\varphi} \in \mathscr{S}_d(\mathbb{R})$ we have

$$\begin{aligned}
\mathrm{S}\exp\left(i\langle\vec{\lambda}, S^H(t)-\vec{x}\rangle\right)(\vec{\varphi}) &= \left\langle\left\langle \exp\left(i\langle\vec{\lambda}, S^H(t)-\vec{x}\rangle\right), C(\vec{\varphi})\exp(\langle\cdot,\vec{\varphi}\rangle)\right\rangle\right\rangle \\
&= \exp\left(i\left(\vec{\lambda},\vec{x}\right)-\frac{1}{2}|\vec{\varphi}|_0^2\right)\int_{\mathscr{S}_d'(\mathbb{R})}\exp\left(\left\langle \vec{\omega}, \vec{\varphi}-\frac{i\vec{\lambda}}{\sqrt{2}}M_-^H\mathbf{1}^{\circ}_{[0,t]}\right\rangle\right)d\mu(\vec{\omega}) \\
&= \exp\left(i\left(\vec{\lambda},\vec{x}\right)-\frac{1}{2}|\vec{\varphi}|_0^2\right)\exp\left(\frac{1}{2}\left|\vec{\varphi}-\frac{i\vec{\lambda}}{\sqrt{2}}M_-^H\mathbf{1}^{\circ}_{[0,t]}\right|_0^2\right) \\
&= \exp\left(i\left(\vec{\lambda},\vec{x}-\left\langle\vec{\varphi},\frac{1}{\sqrt{2}}M_-^H\mathbf{1}^{\circ}_{[0,t]}\right\rangle\right)-\frac{1}{2}\left|\vec{\lambda}\right|^2\left|\frac{1}{\sqrt{2}}M_-^H\mathbf{1}^{\circ}_{[0,t]}\right|_0^2\right),
\end{aligned}$$

a measurable function of $\vec{\lambda}$. Moreover, for $\vec{\varphi} \in \mathscr{S}_d(\mathbb{R})$ and $z \in \mathbb{C}$ we obtain

$$\begin{aligned}
\left|\mathrm{S}\exp\left(i\langle\vec{\lambda}, S^H(t)-\vec{x}\rangle\right)(z\vec{\varphi})\right| &\le \prod_{j=1}^{d}\exp\left(-\frac{1}{2}\lambda_j^2\mathrm{Var}\left(S^H(t)\right)\right)\exp\left(|z|\left|\lambda_j\right|\left|\left\langle\varphi_j,\frac{1}{\sqrt{2}}M_-^H\mathbf{1}^{\circ}_{[0,t]}\right\rangle\right|\right) \\
&\le \prod_{j=1}^{d}\exp\left(-\frac{1}{2}\lambda_j^2\mathrm{Var}\left(S^H(t)\right)\right)\exp\left(\frac{1}{4}\lambda_j^2\mathrm{Var}\left(S^H(t)\right)\right)\exp\left(|z|^2|\varphi_j|^2\right) \\
&= \exp\left(-\frac{2-2^{2H-1}}{4}t^{2H}\left|\vec{\lambda}\right|^2\right)\exp\left(|z|^2|\vec{\varphi}|_0^2\right).
\end{aligned}$$

Note that the first factor is an integrable function of $\vec{\lambda}$ and the second factor is independent of $\vec{\lambda}$. Thus, Theorem 2 implies that $\delta\left(S^H(t)-\vec{x}\right) \in (\mathscr{S})^*$. Finally, by calculating the Gaussian integral we get

$$\begin{aligned}
\mathrm{S}\delta\left(S^H(t)-\vec{x}\right)(\vec{\varphi}) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{S}\exp\left(i\langle\vec{\lambda}, S^H(t)-\vec{x}\rangle\right)(\vec{\varphi})\,d\vec{\lambda} \\
&= \prod_{j=1}^{d}\frac{1}{2\pi}\int_{\mathbb{R}}\exp\left(i\lambda_j\left(x_j-\left\langle\varphi_j,\frac{1}{\sqrt{2}}M_-^H\mathbf{1}^{\circ}_{[0,t]}\right\rangle\right)-\frac{1}{2}\lambda_j^2\mathrm{Var}\left(S^H(t)\right)\right)d\lambda_j \\
&= \prod_{j=1}^{d}\frac{1}{\sqrt{2\pi\mathrm{Var}\left(S^H(t)\right)}}\exp\left(-\frac{1}{2\left(2-2^{2H-1}\right)t^{2H}}\left(x_j-\left\langle\varphi_j,\frac{1}{\sqrt{2}}M_-^H\mathbf{1}^{\circ}_{[0,t]}\right\rangle\right)^2\right) \\
&= \left(\frac{1}{2\pi\left(2-2^{2H-1}\right)t^{2H}}\right)^{\frac{d}{2}}\exp\left(-\frac{1}{2\left(2-2^{2H-1}\right)t^{2H}}\sum_{j=1}^{d}\left(x_j-\left\langle\varphi_j,\frac{1}{\sqrt{2}}M_-^H\mathbf{1}^{\circ}_{[0,t]}\right\rangle\right)^2\right).
\end{aligned}$$

Corollary 1 *The transition probability function of a particle in a system described by a d-dimensional sfBm $S^H = (S^H(t))_{t\ge 0}$ to move from $\vec{x}_0 \in \mathbb{R}^d$ to an endpoint $\vec{x}_T \in \mathbb{R}^d$ at a later time $t = T$ is given by*

$$p\left(\vec{x}_0, 0; \vec{x}_T, T\right) = \left(\frac{1}{2\pi\left(2-2^{2H-1}\right)t^{2H}}\right)^{\frac{d}{2}}\exp\left(-\frac{1}{2\left(2-2^{2H-1}\right)t^{2H}}\left|\vec{x}_0-\vec{x}_T\right|_{\mathbb{R}^d}^2\right).$$

Proof. The generalized expectation of the Donsker delta function can be obtained from Theorem 3 by evaluating the value of the S-transform at $\vec{\varphi} = \vec{0}$:

$$\mathbb{E}_\mu\left(\delta(S^H(t)-\vec{c})\right) = \mathrm{S}\delta(S^H(t)-\vec{c})(\vec{0}) = \left(\frac{1}{2\pi\left(2-2^{2H-1}\right)t^{2H}}\right)^{\frac{d}{2}}\exp\left(-\frac{1}{2\left(2-2^{2H-1}\right)t^{2H}}\sum_{j=1}^{d}\left(x_0^j-c^j\right)^2\right).$$

The transition probability function $p\left(\vec{x}_0, 0; \vec{x}_T, T\right)$ follows immediately from the last expression by fixing the endtime $t = T$ with endpoint $\vec{c} = \vec{x}_T$.

For a discussion on the Donsker delta function of general class of stochastic processes with memory see [29].

WICK-TYPE LINEAR SDES DRIVEN BY SFBM

In this section we will solve the Langevin equation driven by sfBm and a linear stochastic differential equation (SDE) of Wick-type driven by sfBm by using the S-transform method. First, let us consider the Langevin equation driven by sfBm

$$dX^H(t) = -\lambda X^H(t)\,dt + \kappa\,dS^H(t), \quad X^H(0) = x_0, \tag{3}$$

where $\kappa \in \mathbb{R}$ dan $\lambda > 0$. The solution $X^H(t)$ of equation (3) represents the velocity at time t of a free particle which performs a subfractional Brownian-type motion but different from $S^H(t)$. The SDE (3) is interpreted in the integral form by

$$X^H(t) = x_0 - \lambda \int_0^t X^H(s)\,ds + \kappa S^H(t).$$

Applying the S-transform to $X^H(t)$ we obtain

$$SX^H(t)(\varphi) = x_0 - \lambda \int_0^t SX^H(s)(\varphi)\,ds + \kappa \left\langle \varphi, \frac{1}{\sqrt{2}} M_-^H \mathbf{1}^\circ_{[0,t]} \right\rangle, \tag{4}$$

for all $\varphi \in \mathscr{S}_1(\mathbb{R})$. Let us denote $z(t) := SX^H(t)(\varphi)$, then the equation (4) is equivalent to

$$z'(t) = -\lambda z(t) + \kappa \frac{d}{dt} \left\langle \varphi, \frac{1}{\sqrt{2}} M_-^H \mathbf{1}^\circ_{[0,t]} \right\rangle. \tag{5}$$

The solution of ODE (5) is given by

$$z(t) = x_0 \exp(-\lambda t) + \frac{\kappa}{\sqrt{2}} \left(\left\langle \varphi, M_-^H \mathbf{1}^\circ_{[0,t]} \right\rangle - \lambda \int_0^t \exp(\lambda(s-t)) \left\langle \varphi, M_-^H \mathbf{1}^\circ_{[0,s]} \right\rangle ds \right).$$

Hence, we have

$$SX^H(t)(\varphi) = x_0 \exp(-\lambda t) + \frac{\kappa}{\sqrt{2}} \left\langle \varphi, M_-^H \mathbf{1}^\circ_{[0,t]} - \lambda \int_0^t \exp(\lambda(s-t))\, M_-^H \mathbf{1}^\circ_{[0,s]}\,ds \right\rangle.$$

Denoting

$$f_H(t) := \frac{1}{\sqrt{2}} \left(M_-^H \mathbf{1}^\circ_{[0,t]} - \lambda \int_0^t \exp(\lambda(s-t))\, M_-^H \mathbf{1}^\circ_{[0,s]} \right)$$

the S-transform of $X^H(t)$ can be rewritten as

$$SX^H(t)(\varphi) = x_0 \exp(-\lambda t) + \kappa \langle \varphi, f_H(t) \rangle.$$

Taking the inverse of the S-transform yields

$$X^H(t) = x_0 \exp(-\lambda t) + \kappa \langle \cdot, f_H(t) \rangle. \tag{6}$$

Definition 1 *The solution $X^H(t)$, $t \geq 0$, given by equation (6) of the Langevin equation (3) is called a subfractional Ornstein-Uhlenbeck process. Its expectation and characteristic function are given by*

$$\begin{aligned} \mathbb{E}\left(X^H(t)\right) &= x_0 \exp(-\lambda t), \\ \mathbb{E}\left(\exp\left(ikX^H(t)\right)\right) &= \exp\left(ikx_0 \exp(-\lambda t) - \frac{1}{2} k^2 \left| f_H(t) \right|_0^2\right). \end{aligned}$$

Next, we will consider linear SDEs of Wick-type of the form

$$dY^H(t) = -\lambda Y^H(t)\,dt + Y^H(t) \diamond dS^H(t), \quad Y^H(0) = \Phi_0 \in (\mathscr{S})^*. \tag{7}$$

In white noise analysis the connection between Wick product and Skorokhod integration can be expressed as

$$\int_{\mathbb{R}} Y(t)\,\delta B(t) = \int_{\mathbb{R}} Y(t) \diamond W(t)\,dt,$$

where the left-hand side denotes the Skorokhod integral of the stochastic process $(Y(t))_{t\geq 0}$ while the right-hand side corresponds to a $(\mathscr{S})^*$-valued Pettis integral. Here, $(B(t))_{t\geq 0}$ is a standard Brownian motion and $(W(t))_{t\geq 0}$ is the corresponding white noise.

In order to give sense and solve equation (7) we recall the notion of integration of a stochastic distribution process in terms of the S-transform.

Theorem 4 *[30] Let $(T,\mathscr{F},m)$ be a measure space and $\Phi(t) \in (\mathscr{S})^*$ for all $t \in T$. If*

1. *$S\Phi(\cdot)(\varphi)$ is measurable for any $\varphi \in \mathscr{S}_1(\mathbb{R})$,*
2. *$S\Phi(\cdot)(\varphi) \in L^1(T)$ for all $\varphi \in \mathscr{S}_1(\mathbb{R})$,*
3. *for any measurable set $C \subseteq T$ the function $\int_C S\Phi(u)(\cdot)\,dm(u)$ is the S-transform of some distribution in $(\mathscr{S})^*$,*

then there exists a unique distribution $\Psi \in (\mathscr{S})^*$ such that

$$S\Psi(\varphi) = \int_T S\Phi(u)(\varphi)\,dm(u), \quad \varphi \in \mathscr{S}_1(\mathbb{R}).$$

The distribution Ψ is denoted by $\int_T \Phi(t)\,dm(t)$ and it is called the Pettis integral of Φ.

Now, equation (7) is interpreted in the integral form by

$$Y^H(t) = \Phi_0 - \lambda \int_0^t Y^H(s)\,ds + \int_0^t Y^H(s) \diamond W^H(t)\,dt, \tag{8}$$

where the integrals are to be understood in the Pettis sense as in Theorem 4. Next, we apply the S-transform to equation (8) and use Lemma 2 to obtain

$$SY^H(t)(\varphi) = S\Phi_0(\varphi) - \lambda \int_0^t SY^H(s)(\varphi)\,ds + \frac{1}{\sqrt{2}} \int_0^t SY^H(s)\left(\left(M_+^H\varphi\right)(s) - \left(M_+^H\varphi\right)(-s)\right)\,ds.$$

Denote $y(t) := SY^H(t)(\varphi)$, then

$$y'(t) = -\lambda y(t) + \frac{1}{\sqrt{2}} y(t)\left(\left(M_+^H\varphi\right)(t) - \left(M_+^H\varphi\right)(-t)\right)$$

and the solution of this ODE is given by

$$y(t) = y(0)\exp(-\lambda t)\exp\left(\frac{1}{\sqrt{2}}\left(\left(M_+^H\varphi\right)(t) - \left(M_+^H\varphi\right)(-t)\right)\right).$$

Rewriting the last expression using S-transform we get

$$SY^H(t)(\varphi) = \exp(-\lambda t)\,S\Phi_0(\varphi)\exp\left(SW^H(t)(\varphi)\right).$$

Finally, inverting the S-transform gives

$$Y^H(t) = \exp(-\lambda t)\,\Phi_0 \diamond \exp^{\diamond}\left(W^H(t)\right), \quad t \geq 0. \tag{9}$$

From the previous considerations we obtain the following theorem.

Theorem 5 *The solution $Y^H(t)$, $t \geq 0$, given by equation (9) of the linear SDE with Wick-multiplicative subfractional white noise is an element of $(\mathscr{S})^*$. Its generalized expectation is given by*

$$\mathbb{E}_\mu\left(Y^H(t)\right) = \mathbb{E}_\mu\left(\Phi_0\right)\exp(-\lambda t).$$

CONCLUSION

We have discussed sfBm in the framework of white noise theory. The sfBm can be represented as functional of white noise by using some fractional integral/differential operator. We proved that Donsker's delta function of sfBm is a Hida distribution. We constructed the subfractional Ornstein-Uhlenbeck process as the solution of the Langevin equation. Finally, by using S-transform method, we also solved a linear SDE with Wick-multiplicative subfractional white noise.

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REFERENCES

1. A. N. Kolmogorov, Doklady Acad. Sci. URSS **26**, 115–118 (1940).
2. B. B. Mandelbrot and J. W. V. Ness, SIAM Review **10**, 422–437 (1968).
3. F. Biagini, Y. Hu, B. Oksendal, and T. Zhang, *Stochastic Calculus for Fractional Brownian Motion and Applications* (Springer-Verlag, 2008).
4. Y. S. Mishura, *Stochastic Calculus for Fractional Brownian Motion and Related Processes* (Springer-Verlag, 2008).
5. I. Nourdin, *Selected Aspects of Fractional Brownian Motion* (Springer, 2012).
6. Y. S. Mishura and M. Zili, *Stochastic Analysis of Mixed Fractional Gaussian Processes* (ISTE Press, 2018).
7. G. Samorodnitsky and M. Taqqu, *Stable Non-Gaussian Random Processes* (CRC Press, 1994).
8. T. Bojdecki, L. G. Gorostiza, and A. Talarczyk, Statist. Probab. Lett. **69**, 717–739 (2004).
9. T. Bojdecki, L. G. Gorostiza, and A. Talarczyk, J. Theoret. Probab. **69**, 405–419 (2004).
10. T. Bojdecki, L. G. Gorostiza, and A. Talarczyk, Elect. Comm. in Probab. **12**, 161–172 (2007).
11. C. El-Nouty, Stochastic Differential Equations and Processes **7**, 179–196 (2012).
12. I. Mendy, Annales Mathematiques Blaise Pascal **17**, 357–374 (2010).
13. C. Tudor, Stochastics **79**, 431–448 (2007).
14. C. Tudor, Statist. Probab. Lett. **78**, 2201–2209 (2008).
15. C. Tudor, Ann. Univ. Bucuresti Mathematica **57**, 199–230 (2008).
16. C. Tudor, J. Math. Anal. Appl. **351**, 456–468 (2009).
17. Z. Wang and L. Yan, Adv. Math. Phys. **2013**, 827192:1–11 (2013).
18. L. Yan and G. Shen, Statist. Probab. Lett. **80**, 296–308 (2010).
19. L. Yan, G. Shen, and K. He, Comm. Stoch. Anal. **5**, 135–159 (2011).
20. H. P. Suryawan, Jurnal Matematika Integratif **15**, 81–87 (2019).
21. T. Hida, H.-H. Kuo, J. Potthoff, and L. Streit, *White Noise. An Infinite Dimensional Calculus* (Kluwer Academic Publishers, 1993).
22. H.-H. Kuo, *White Noise Distribution Theory* (CRC Press, 1996).
23. N. Obata, *White Noise Calculus and Fock Space* (Springer Verlag, 1996).
24. Y. G. Kondratievi, P. Leukert, and L. Streit, Acta Appl. Math. **44**, 269–294 (1996).
25. Y. G. Kondratiev and et al, J. Funct. Anal. **141**, 301–318 (1996).
26. H. P. Suryawan, Int. J. Mod. Phys. Conf. Ser. **36**, 1560005: 1–11 (2015).
27. C. Bender, Stoch. Proc. Appl. **104**, 81–106 (2003).
28. C. Bender, Bernoulli **9**, 955–983 (2003).
29. H. P. Suryawan, J. Math. Fundam. Sci. **51**, 265–277 (2019).
30. H.-H. Kuo, Nonlinear Anal. Theory Methods Appl. **30**, 317–328 (1997).