A smoothness indicator for numerical solutions to the Ripa model

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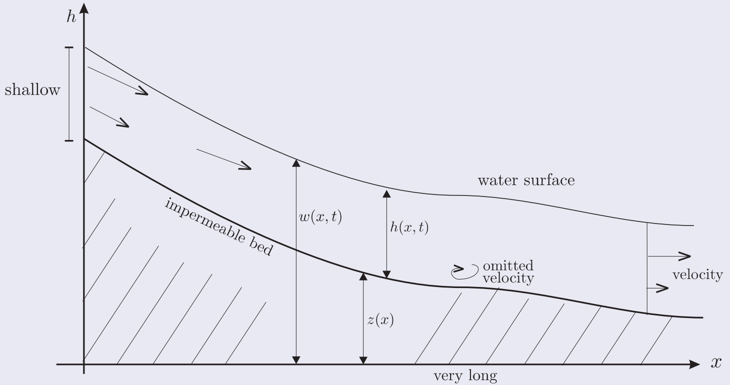
Outline

- Introduction: governing equations
- Finite Volume Method (FVM)
- Numerical entropy production (NEP)
- Numerical tests
- Conclusions

Illustration for shallow water flows



The Ripa model

The shallow water wave equations involving the water temperature fluctuations are

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0, \tag{1}$$

$$\frac{\partial (hu)}{\partial t} + \frac{\partial (hu^2 + \frac{1}{2}gh^2\theta)}{\partial x} = -gh\theta\frac{dz}{dx}, \tag{2}$$

$$\frac{\partial (h\theta)}{\partial t} + \frac{\partial (h\theta u)}{\partial x} = 0. \tag{3}$$

A balance law

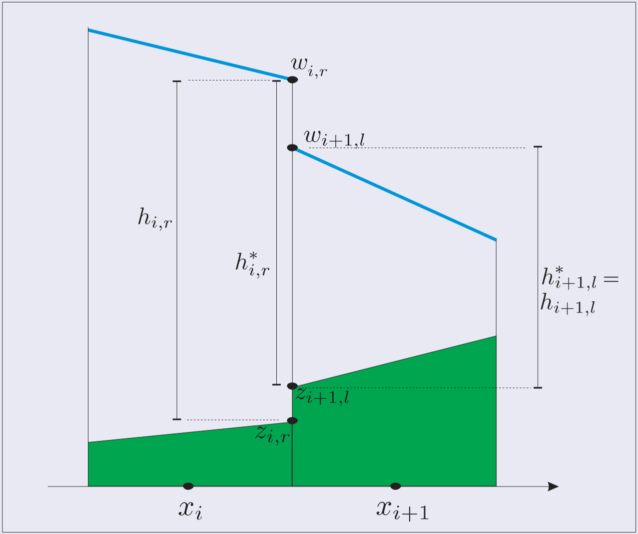
The Ripa model can be rewritten as a balance law

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial x} = \mathbf{s}(\mathbf{q})\frac{dz}{dx} \tag{4}$$

where the vectors of conserved quantities, fluxes, and sources are respectively given by

$$\mathbf{q} = \begin{bmatrix} h \\ hu \\ h\theta \end{bmatrix}, \quad \mathbf{f}(\mathbf{q}) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}g\theta h^2 \\ h\theta u \end{bmatrix}, \quad \mathbf{s}(\mathbf{q}) = \begin{bmatrix} 0 \\ -g\theta h \\ 0 \end{bmatrix}. \tag{5}$$

Illustration for reconstruction



Finite volume scheme

A semi-discrete finite volume scheme for the homogeneous Ripa model is

$$\Delta x_j \frac{d}{dt}\mathbf{Q}_j + \mathcal{F}(\mathbf{Q}_j, \mathbf{Q}_{j+1}) - \mathcal{F}(\mathbf{Q}_{j-1}, \mathbf{Q}_j) = \mathbf{0} \tag{6}$$

where $\mathcal{F}$ is a numerical flux function consistent with the homogeneous Ripa model. Here Δx_j is the cell-width of the jth cell.

We continue discretising the semi-discrete scheme (6) using the first order Euler method for ordinary differential equations. We obtain the fully-discrete scheme

$$\mathbf{Q}_j^{n+1} = \mathbf{Q}_j^n - \lambda_j^n \left(\mathbf{F}_{j+\frac{1}{2}}^n - \mathbf{F}_{j-\frac{1}{2}}^n \right) . \tag{7}$$

Entropy, entropy flux, and entropy inequality

Entropy solutions of the Ripa model must satisfy the entropy inequality

$$\frac{\partial \eta}{\partial t} + \frac{\partial \psi}{\partial x} \leq 0 \tag{8}$$

in the weak sense for all entropies. We consider the entropy pair

$$\eta(\mathbf{q}) = h\frac{u^2}{2} + \frac{g}{2}h\theta(h+z), \tag{9}$$

$$\psi(\mathbf{q}) = hu(\frac{u^2}{2} + g\theta(h+z)) \tag{10}$$

as the entropy function and the entropy flux function.

Semi discrete scheme for entropy

We take a semi discrete scheme

$$\Delta x_i \frac{d}{dt}\Theta_i + \Psi^r(\mathbf{Q}_i, \mathbf{Q}_{i+1}, z_{i,r}, z_{i+1,l}) - \Psi^l(\mathbf{Q}_{i-1}, \mathbf{Q}_i, z_{i-1,r}, z_{i,l}) = 0 \tag{11}$$

to get the value of Θ_i^n , where

$$\Psi^r(\mathbf{Q}_i, \mathbf{Q}_{i+1}, z_{i,r}, z_{i+1,l}) := \Psi(\mathbf{Q}^*_{i,r}, \mathbf{Q}^*_{i+1,l}, z^*_{i+1/2}) \tag{12}$$

and

$$\Psi^l(\mathbf{Q}_{i-1}, \mathbf{Q}_i, z_{i-1,l}, z_{i,r}) := \Psi(\mathbf{Q}^*_{i-1,r}, \mathbf{Q}^*_{i,l}, z^*_{i-1/2}) \tag{13}$$

are the right and left numerical entropy fluxes of the ith cell calculated at $x_{i+1/2}$ and $x_{i-1/2}$ respectively.

Definition

The numerical entropy production is

$$E_i^n = \frac{1}{\Delta t}\left(\eta\left(\mathbf{Q}_i^n\right) - \Theta_i^n\right) , \tag{14}$$

which is the local truncation error of the entropy.

We compare the results of NEP (Numerical Entropy Production) to

Karni, Kurganov, and Petrova's (KKP) local truncation error [2]

The KKP indicator is defined at $x = x_i$, $t = t^n$:

$$E_i^n = \frac{1}{12}\left\{\Delta x\left[h_{i+1}^{n+1} - h_{i+1}^{n-1} + 4\left(h_i^{n+1} - h_i^{n-1}\right) + h_{i-1}^{n+1} - h_{i-1}^{n-1}\right]\right.$$
$$+\Delta t\left[h_{i+1}^{n+1}u_{i+1}^{n+1} - h_{i-1}^{n+1}u_{i-1}^{n+1} + 4\left(h_{i+1}^{n}u_{i+1}^{n} - h_{i-1}^{n}u_{i-1}^{n}\right)\right.$$
$$\left.\left. + h_{i+1}^{n-1}u_{i+1}^{n-1} - h_{i-1}^{n-1}u_{i-1}^{n-1}\right]\right\} .$$

Constantin and Kurganov's (CK) local truncation error [1]

The CK indicator is defined at $x = x_{i+1/2}$, $t = t^{n-1/2}$:

$$E_{i+1/2}^{n-1/2} = \frac{1}{2}\left\{\Delta x\left[h_i^n - h_i^{n-1} + h_{i+1}^n - h_{i+1}^{n-1}\right]\right.$$
$$\left.+\Delta t\left[h_{i+1}^{n-1}u_{i+1}^{n-1} - h_i^{n-1}u_i^{n-1} + h_{i+1}^n u_{i+1}^n - h_i^n u_i^n\right]\right\} .$$

Moving shock in a dam break problem

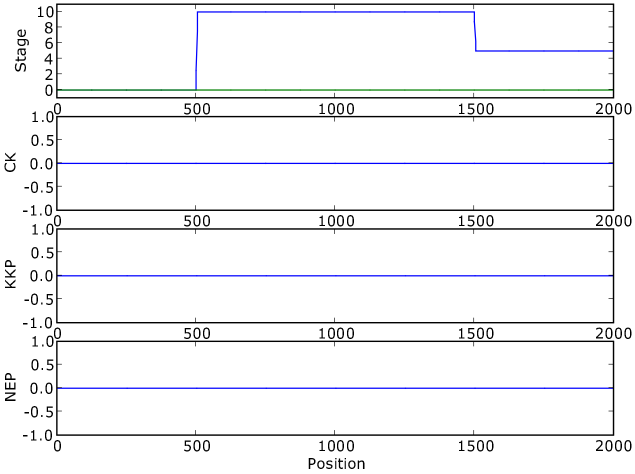
We consider a reservoir with horizontal topography

$$z(x) = 0\,, \quad 0 < x < 2000\,, \tag{15}$$

and an initial condition

$$u(x,0) = 0\,, \quad w(x,0) = \begin{cases} 0 & \text{if} \quad 0 < x < 500\,, \\ 10 & \text{if} \quad 500 < x < 1500\,, \\ 5 & \text{if} \quad 1500 < x < 2000\,. \end{cases} \tag{16}$$

Moving shock in a dam break problem, 20 s:



Stationary shock on a parabolic obstruction

We consider a channel of length 25 with topography

$$z(x) = \begin{cases} 0.2 - 0.05\,(x-10)^2 & \text{if} \quad 8 \le x \le 12\,, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

The initial condition

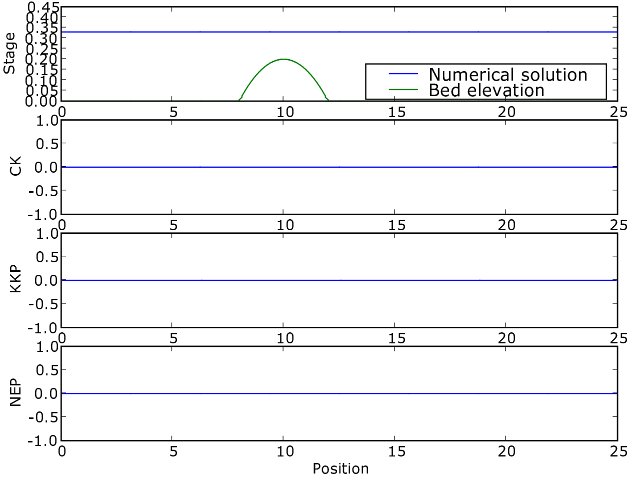
$$u(x,0) = 0\,, \quad w(x,0) = 0.33 \tag{18}$$

together with the Dirichlet boundary conditions

$$[w, m, z, h, u] = \left[0.42, 0.18, 0.0, 0.42, \frac{0.18}{0.42}\right] \quad \text{at} \quad x = 0^-\,, \tag{19}$$

$$[w, m, z, h, u] = \left[0.33, 0.18, 0.0, 0.33, \frac{0.18}{0.33}\right] \quad \text{at} \quad x = 25^+\,. \tag{20}$$

Stationary shock on a parabolic obstruction, 50 s:



Shock-like detection

We consider the initial condition

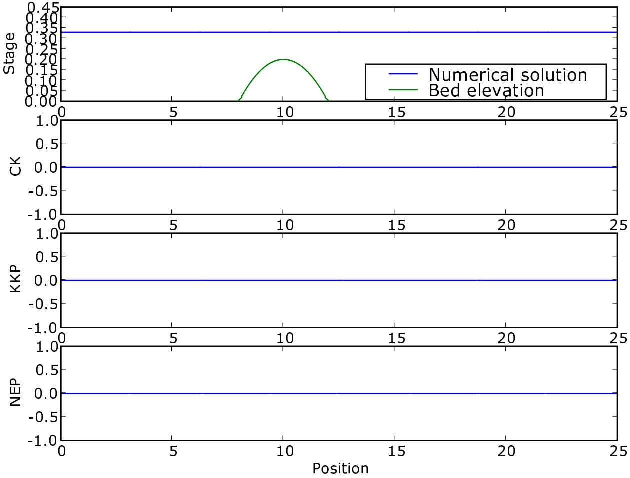
$$u(x,0) = 0\,, \quad w(x,0) = 0.33 \tag{21}$$

together with the Dirichlet boundary conditions

$$[w,m,z,h,u] = \left[0.42, 0.18, 0.0, 0.42, \frac{0.18}{0.42}\right] \quad \text{at} \quad x = 0^-\,, \tag{22}$$

$$[w,m,z,h,u] = \left[0.1, 0.18, 0.0, 0.1, \frac{0.18}{0.1}\right] \quad \text{at} \quad x = 25^+\,. \tag{23}$$

Shock-like detection, 100 s:

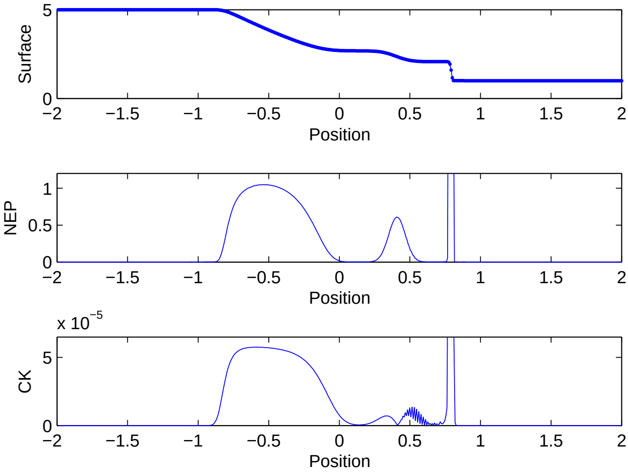


The Ripa problem

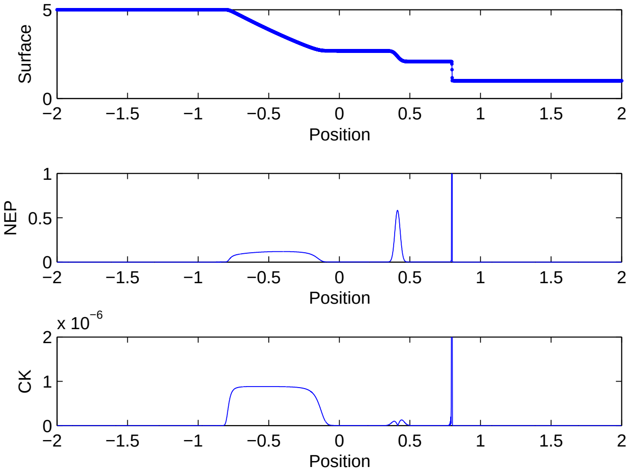
We consider the initial condition

$$\mathbf{q}(x, t=0) = \begin{cases} (5, 0, 15)^t & \text{if} \quad x < 0, \\ (1, 0, 5)^t & \text{if} \quad x > 0. \end{cases} \tag{24}$$

The Ripa problem at time $t = 0.2$s:



The Ripa problem at time $t = 0.2$s:



Conclusion and future direction

- The numerical entropy production detects the location of a shock nicely.
- Future research will implement the numerical entropy production as a smoothness indicator for an adaptive FVM.

References

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Thank you.