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OPTIMIZING RANGE NORM OF THE IMAGE SET OF MATRIX OVER INTERVAL MAX-PLUS ALGEBRA

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Abstract

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$, $\varepsilon = -\infty$. Max-plus algebra is the set $\mathbb{R}_\varepsilon$ that is equipped with two operations maximum and addition. It can be formed matrices in the size of $m \times n$,

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whose elements belong to $\mathbb{R}_\varepsilon$, called matrix over max-plus algebra. Let $I(\mathbb{R})_\varepsilon = \{x = [\underline{x}, \overline{x}] | \underline{x}, \overline{x} \in \mathbb{R}, \varepsilon < \underline{x} \leq \overline{x}\} \cup \{\varepsilon\}$ and $\varepsilon = \{\varepsilon, \varepsilon\}$. The set $I(\mathbb{R})_\varepsilon$ which is equipped with two operations maximum and addition is called interval max-plus algebra. It can be formed matrices in the size $m \times n$, whose elements belong to $I(\mathbb{R})_\varepsilon$, called matrices over interval max-plus algebra. Optimizing range norm of the image set of matrix over max-plus algebra has been discussed. In this paper, we discuss optimizing range norm of the image set of matrix over interval max-plus algebra.

I. Introduction

Max-plus algebra is the set $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$, of which $\mathbb{R}$ are all the sets of real numbers and $\varepsilon = -\infty$ that is equipped with two operations $\oplus$ (maximum) and $\otimes$ (addition). Max-plus algebra has been used to model and analyze problems in planning, communication, production system, queueing system with finite capacity, parallel computation, and traffic using algebra [3] and also several other applications, i.e., in flight scheduling and airport problems [2] whereas min-plus algebra is the set $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{\varepsilon'\}$ and $\varepsilon' = \infty$ that is equipped with two operations $\oplus'$ (minimum) and $\otimes'$ (addition) [12]. Tam [12] also discussed complete max-plus algebra, of which the set $\overline{\mathbb{R}}_\varepsilon = \mathbb{R} \cup \{\varepsilon, \varepsilon'\}$ is equipped with operations $\oplus$ and $\otimes$, and complete min-plus algebra, of which the set $\overline{\mathbb{R}}_{\varepsilon'} = \mathbb{R} \cup \{\varepsilon, \varepsilon'\}$ is equipped with two operations $\oplus'$ and $\otimes'$. Furthermore, $\overline{\mathbb{R}}_\varepsilon$ and $\overline{\mathbb{R}}_{\varepsilon'}$ are written as $\overline{\mathbb{R}}$.

It can be formed a set of matrices in the size $m \times n$ of the sets $\mathbb{R}_\varepsilon$, $\mathbb{R}_{\varepsilon'}$, and $\overline{\mathbb{R}}$. The set of matrices with components in $\mathbb{R}_\varepsilon$ is denoted by $\mathbb{R}_\varepsilon^{m \times n}$. The matrix $A \in \mathbb{R}_\varepsilon^{m \times n}$ is called *matrix over max-plus algebra*. For $n = 1$, a set of vectors over max-plus algebra is obtained and written as $\mathbb{R}_\varepsilon^m$, that is $\mathbb{R}_\varepsilon^m = \{(x_1, x_2, ..., x_m)^T \mid x_1, x_2, ..., x_m \in \mathbb{R}_\varepsilon\}$ [5, 6]. Moreover, if $m = n$, then the set $\mathbb{R}_\varepsilon^{n \times n}$ equipped with operations $\oplus$ and $\otimes$ forms idempotent semiring [1, 7, 10].

Tam [12] illustrated an example of application of max-plus algebra in the production system. For example $A = (A_{ij}) \in \mathbb{R}_{\varepsilon}^{n \times n}$ is a production matrix, where A_{ij} shows the period of time of the production process from machine M_j to M_i, while $x(k) = (x_i(k)) \in \mathbb{R}_{\varepsilon}^{n}$ is a vector, where $x_i(k)$ is the starting time of machine M_i at the kth stage. In this production process, we have the equation $x(k+1) = A \otimes x(k)$. One of the criteria used by manufacturer is that the production process is expected to be continue periodically in certain periods, e.g., λ, so that the equation $x(k+1) = \lambda \otimes x(k)$ is obtained. From $x(k+1) = A \otimes x(k)$ and $x(k+1) = \lambda \otimes x(k)$, we get $A \otimes x(k) = \lambda \otimes x(k)$. The eigenvalue and eigenvector problem of matrix A is to determine the eigenvalue λ and the eigenvector $x(k)$ which satisfy the equation $A \otimes x(k) = \lambda \otimes x(k)$.

In the real life situation, there are several ways for the manufacturer to determine the starting time of each machine. One of the ways is by choosing eigenvector as the starting time, so that the system will immediately reach the steady state; that is the process continues periodically with the eigenvalue as its periods. However, in reality, there can exist more than one independent eigenvectors for the manufacturer to choose. In that case, a set of linear combination of the independent eigenvectors is constructed, and as such, the manufacturer needs an additional criterion to choose one element of the set. The additional criterion is by considering the difference between the largest and the smallest of the starting time of each machine. The difference is expressed as range norm of the starting time of each machine. Manufacturer can optimize (either minimize or maximize) the range norm of the starting time of each machine [4, 12]. The readiness of the raw material, the availability of the resources and the distribution of the product become the determining factors for the manufacturer to optimize the range norm of the starting time of each machine. In solving the problems of that production system, Tam [12] has discussed optimizing range norm of the image set of matrix over max-plus algebra.

Kreinovich [8] stated that any measurement is never 100% accurate. Thus, the value of the measurement $\tilde{x}$ in general differs from the real value x. Specifically, in the case of interval uncertainty, after it is being measured and the result of the measurement $\tilde{x}$ is obtained, the acquired information that the real value x is contained in the interval $x = [\tilde{x} - \Delta, \tilde{x} + \Delta]$.

Based on the above statement, the estimation on the period of time of a certain process can be given, for example the period of time of the process in a production system. Therefore, the period of time of a certain production process can be given in an interval of time.

According to the Kreinovich's idea [8] about the probability to give an estimation of the period of time of a certain process and Rudhito [9] about max-plus algebra generalization, i.e., interval max-plus algebra, we try to expand the concept in max-plus algebra, i.e., the optimizing range norm of the image set of matrix over interval max-plus algebra. Siswanto et al. [11] have investigated how to minimize range norm of the image set of matrix over interval max-plus algebra. In this research, we develop the range norm definition in [11]. Using the new range norm definition, we investigate the way to minimize and maximize range norm of the image set of matrix over interval max-plus algebra.

Before discussing the result of this paper, several concepts which support the discussion are observed as follows [12]:

Definition 1.1. Given that $a \in \mathbb{R},$ the conjugate of a is $a^* = a^{-1} = -a$. Suppose that $A \in \overline{\mathbb{R}}^{m \times n}$. Then the conjugate of matrix A is $A^* = (a^*_{ji})$ or $A^* = -A^T$.

Definition 1.2. Given that $A \in \overline{\mathbb{R}}^{m \times n}$ and if matrix A at least has one finite element in each row, then matrix A is called *row* $\mathbb{R}$*-astic*. If matrix A at least has one finite element in each column, then matrix A is called *column* $\mathbb{R}$*-astic*. If matrix A at least has one finite element in each row and each column, then matrix A is called *double* $\mathbb{R}$*-astic*.

Definition 1.3. Suppose that $A \in \mathbb{R}_{\varepsilon}^{m\times n}$, $\text{Im}(A)$ is defined as $\text{Im}(A) = \{A \otimes x \mid x \in \mathbb{R}_{\varepsilon}^{n}\}$, namely the image set of matrix *A*.

Moreover, the definition of the range norm and the problem of optimizing range norm of the image set of matrix over max-plus algebra are given. The notations of $M = \{1, 2, ..., m\}$ and $N = \{1, 2, ..., n\}$ are used to simplify.

Definition 1.4. If $x \in \mathbb{R}^{m}$, then the function $\delta(x) = \sum_{i\in M}^{\oplus} x_i - \sum_{i\in M}^{\oplus'} x_i$ is called the *range norm* of *x*. In other words, the range norm of $x =$ the largest value in *x* - the smallest value in *x*.

In this case, the norm term is a mere identification, in contrast to the norm defined in inner product space in general.

Problem 1.1. Given that matrix $A \in \mathbb{R}_{\varepsilon}^{m\times n}$, solve:

$$\text{minimize } \delta(b)$$

$$\text{subject to } b \in \text{Im}(A).$$

Definition 1.5. If $x \in \mathbb{R}_{\varepsilon}^{m}$, then the function $\tilde{\delta}(x) = \sum_{\substack{i\in M \\ x_i \neq \varepsilon}}^{\oplus} x_i - \sum_{\substack{i\in M \\ x_i \neq \varepsilon}}^{\oplus'} x_i$

is the range norm of *x*, considering only the finite component of *x*.

Problem 1.2. Given that matrix $A \in \mathbb{R}_{\varepsilon}^{m\times n}$, solve:

$$\text{minimize } \tilde{\delta}(b)$$

$$\text{subject to } b \in \text{Im}(A).$$

Problem 1.3. Given that matrix $A \in \mathbb{R}_{\varepsilon}^{m\times n}$, solve:

$$\text{maximize } \delta(b)$$

$$\text{subject to } b \in \text{Im}(A).$$

Other than the above concepts, the concepts on interval max-plus algebra is also needed [9].

Closed interval x in $\mathbb{R}_\varepsilon$ is a subset of $\mathbb{R}_\varepsilon$ in the form $x = [\underline{x}, \overline{x}] = \{x \in \mathbb{R}_\varepsilon \mid \underline{x} \le x \le \overline{x}\}$. The interval x in $\mathbb{R}_\varepsilon$ is known as interval max-plus. A number $x \in \mathbb{R}_\varepsilon$ can be expressed as interval $[x, x]$.

Definition 1.6. $I(\mathbb{R})_\varepsilon = \{x = [\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \mathbb{R}, \varepsilon < \underline{x} \le \overline{x}\} \cup \{\varepsilon\}$ is formed with $\varepsilon = [\varepsilon, \varepsilon]$. On the set $I(\mathbb{R})_\varepsilon$, the operations $\overline{\oplus}$ and $\overline{\otimes}$ with $x \overline{\oplus} y = [\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}]$ and $x \overline{\otimes} y = [\underline{x} \otimes \underline{y}, \overline{x} \otimes \overline{y}]$ for every $x, y \in I(\mathbb{R})_\varepsilon$ are defined. Furthermore they are known as interval max-plus algebra and notated as $I(\Re)_{\max} = (I(\mathbb{R})_\varepsilon; \overline{\oplus}, \overline{\otimes})$.

Definition 1.7. The set of matrices in the size $m \times n$ with the elements in $I(\mathbb{R})_\varepsilon$ is notated as $I(\mathbb{R})_\varepsilon^{m\times n}$, namely:

$$I(\mathbb{R})_\varepsilon^{m\times n} = \{A = [A_{ij}] \mid A_{ij} \in I(\mathbb{R})_\varepsilon; \text{ for } i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n\}.$$

Matrices which belong to $I(\mathbb{R})_\varepsilon^{m\times n}$ are known as matrices over interval max-plus algebra or shortly interval matrices.

Definition 1.8. For $A \in I(\mathbb{R})_\varepsilon^{m\times n}$, matrices $\underline{A} = [\underline{A}_{ij}] \in \mathbb{R}_\varepsilon^{m\times n}$ and $\overline{A} = [\overline{A}_{ij}] \in \mathbb{R}_\varepsilon^{m\times n}$ are defined, each is known as lower bound matrix and upper bound matrix of the interval matrix A.

Definition 1.9. Given that interval matrix $A \in I(\mathbb{R})_\varepsilon^{m\times n}$, of which $\underline{A}$ and $\overline{A}$ each act as lower bound matrix and upper bound matrix of the interval matrix A. The matrix interval of A is defined, namely $[\underline{A}, \overline{A}] = \{A \in \mathbb{R}_\varepsilon^{m\times n} \mid \underline{A} \le A \le \overline{A}\}$ and $I(\mathbb{R}_\varepsilon^{m\times n})_b = \{[\underline{A}, \overline{A}] \mid A \in I(\mathbb{R})_\varepsilon^{m\times n}\}$.

The semimodule $I(\mathbb{R})_\varepsilon^{n\times n}$ over $I(\mathbb{R})_\varepsilon$ is isomorphic to the semimodule $I(\mathbb{R}_\varepsilon^{n\times n})_b$ over $I(\mathbb{R})_\varepsilon$ with the mapping of $f : I(\mathbb{R})_\varepsilon^{n\times n} \to I(\mathbb{R}_\varepsilon^{n\times n})_b$, $f(A) = [\underline{A}, \overline{A}]$, $\forall A \in I(\mathbb{R})_\varepsilon^{n\times n}$. The matrix interval $[\underline{A}, \overline{A}] \in I(\mathbb{R}_\varepsilon^{n\times n})_b$ is known as matrix interval which corresponds to interval matrix $A \in I(\mathbb{R})_{\varepsilon}^{n \times n}$ and which is represented with $A \approx [\underline{A}, \overline{A}]$.

Definition 1.10. It is defined that

$$I(\mathbb{R})_{\varepsilon}^{n} = \{x = [x_1, x_2, \ldots, x_n]^T \mid x_i \in I(\mathbb{R})_{\varepsilon}; i = 1, 2, \ldots, n\}.$$

Set $I(\mathbb{R})_{\varepsilon}^{n}$ can be considered as set $I(\mathbb{R})_{\varepsilon}^{n \times 1}$. The elements of $I(\mathbb{R})_{\varepsilon}^{n}$ are called *interval vector* over $I(\mathbb{R})_{\varepsilon}$. Interval vector x corresponds to vector interval $[\underline{x}, \overline{x}]$, i.e., $x \approx [\underline{x}, \overline{x}]$.

The concept of interval min-plus algebra is defined in the same way with interval max-plus algebra concepts. Following are the definitions of complete interval max-plus algebra and complete interval min-plus algebra [11].

Definition 1.11. Complete interval max-plus algebra is set $I(\overline{\mathbb{R}})_{\varepsilon} = I(\mathbb{R})_{\varepsilon} \cup \{\varepsilon'\}$, $\varepsilon' = [\varepsilon', \varepsilon']$ that is completed with two operations $\overline{\oplus}$ and $\overline{\otimes}$, meanwhile complete interval min-plus algebra is set $I(\overline{\mathbb{R}})_{\varepsilon'} = I(\mathbb{R})_{\varepsilon'} \cup \{\varepsilon\}$ with the operations $\overline{\oplus}'$ and $\overline{\otimes}'$. Furthermore, $I(\overline{\mathbb{R}})_{\varepsilon} = I(\mathbb{R})_{\varepsilon'}$ are written as $I(\overline{\mathbb{R}})$.

In the same way as in the interval max-plus algebra and interval min-plus algebra, in complete interval max-plus algebra and complete interval min-plus algebra, can be defined as the set of matrices in the size $m \times n$ are notated as $I(\overline{\mathbb{R}})^{m \times n}$.

II. Main Results

In this section, the optimizing range norm of the image set of matrix over interval max-plus algebra is presented:

A. Minimizing the range norm of matrix image set over interval max-plus algebra

The following are definitions of the range norm of vector and the image set of matrix over interval max-plus algebra.

Definition 2.1. Suppose that $x \in I(\mathbb{R})^m$ with $x \approx [\underline{x}, \overline{x}]$, $\underline{x}, \overline{x} \in \mathbb{R}^m$. The function $I\delta(x) = [\min(\delta(\underline{x}), \delta(\overline{x})), \delta(\overline{x})]$ is called the *range norm* of *x*. In other words, the range norm of *x* is interval $[\min(\delta(\underline{x}), \delta(\overline{x})), \delta(\overline{x})]$, where $\min(\delta(\underline{x}), \delta(\overline{x}))$ = minimum of $\delta(\underline{x})$ or $\delta(\overline{x})$.

Definition 2.2. Given that matrix $A \in I(\mathbb{R})_\varepsilon^{m\times n}$ with $A \approx [\underline{A}, \overline{A}]$, $\underline{A}, \overline{A} \in \mathbb{R}_\varepsilon^{m\times n}$, it is then defined that $\mathrm{Im}(A) = \{A \overline{\otimes}\, p \mid p \in I(\mathbb{R})_\varepsilon^n\}$.

Based on the above definitions, the problem of minimizing the range norm of matrix image set can be formulated as follows:

Problem 2.1. Given that matrix $A \in I(\mathbb{R})_\varepsilon^{m\times n}$, solve:

$$\text{minimize } I\delta(b)$$

$$\text{subject to } b \in \mathrm{Im}(A).$$

Definition 2.3. Given that matrix $b \in I(\mathbb{R})_\varepsilon^m$ with $b \approx [\underline{b}, \overline{b}]$. The function $I\delta(b)$ is said *minimum* if and only if $\delta(\underline{b})$ and $\delta(\overline{b})$ are minimum.

If the image vector has infinite components, the definition about the range norm and the problem of minimizing the range norm of matrix image set can be formulated as follows:

Definition 2.4. Suppose that $x \in I(\mathbb{R})_\varepsilon^m$ with $x \approx [\underline{x}, \overline{x}]$. The function $\tilde{I}\delta(x) = [\min(\tilde{\delta}(\underline{x}), \tilde{\delta}(\overline{x})), \tilde{\delta}(\overline{x})]$ is the range norm of *x*, after only considering the finite component.

Problem 2.2. Given that matrix $A \in I(\mathbb{R})_\varepsilon^{m\times n}$, solve:

$$\text{minimize } \tilde{I}\delta(b)$$

$$\text{subject to } b \in \mathrm{Im}(A).$$

Definition 2.5. Given that matrix $b \in I(\mathbb{R})_\varepsilon^m$ with $b \approx [\underline{b}, \overline{b}]$. The function $\tilde{I}\delta(b)$ is said *minimum* if and only if $\tilde{\delta}(\underline{b})$ and $\tilde{\delta}(\overline{b})$ are minimum.

Minimizing the range norm if its image vector is finite

The first we investigate minimizing the range norm if its image vector is finite, i.e., Problem 2.1. The next lemmas, definition and theorems are needed to answer Problem 2.1:

Lemma 2.6. *Suppose that* $A \in I(\mathbb{R})_{\varepsilon}^{m\times n}$ *with* $A \approx [\underline{A}, \overline{A}]$. If $A = [A_{ij}]$ *with* $A_{ij} \neq [\varepsilon, \varepsilon]$ *for each* $i \in M$, $j \in N$; $b \approx [\underline{b}, \overline{b}]$ *and* $0 \approx [\underline{0}, \overline{0}]$ *with* $\underline{b} = \underline{A} \otimes (\underline{A}^* \otimes' \underline{0})$ *and* $\overline{b} = \overline{A} \otimes (\overline{A}^* \otimes' \overline{0})$, *then*

(i) $b \leq 0$ *and*

(ii) $b_i = 0$ *for some* $i \in M$.

Proof. Suppose that $b \approx [\underline{b}, \overline{b}]$ with $\underline{b} = \underline{A} \otimes (\underline{A}^* \otimes' \underline{0})$ and $\overline{b} = \overline{A} \otimes (\overline{A}^* \otimes' \overline{0})$. Since $A = [A_{ij}]$ with $A_{ij} \neq [\varepsilon, \varepsilon]$ for each $i \in M$ and $j \in N$, $\underline{A} = [\underline{A}_{ij}]$ with $\underline{A}_{ij} \neq \varepsilon$ for each $i \in M$ and $j \in N$ and $\overline{A} = [\overline{A}_{ij}]$ with $\overline{A}_{ij} \neq \varepsilon$ for each $i \in M$ and $j \in N$. According to concept in max-plus algebra,

(i) $\underline{b} \leq \underline{0}$ and $\overline{b} \leq \overline{0}$,

(ii) $\underline{b}_i = \underline{0}$ and $\overline{b}_i = \overline{0}$ for some $i \in M$.

Since $b \approx [\underline{b}, \overline{b}]$ and $0 \approx [\underline{0}, \overline{0}]$,

(i) $b \leq 0$,

(ii) $b_i = 0$ for some $i \in M$. □

Lemma 2.7. *If* $x, y \in I(\mathbb{R})^m$ *and* $a \in I(\mathbb{R})$ *with* $x \approx [\underline{x}, \overline{x}]$, $y \approx [\underline{y}, \overline{y}]$ *and* $a \approx [\underline{a}, \overline{a}]$, *then*

(i) $I\delta(x \overline{\oplus} y) \leq I\delta(x) \overline{\oplus} I\delta(y)$,

(ii) $I\delta(x) = I\delta(a \overline{\otimes} x)$.

Proof. Since $x \approx [\underline{x}, \overline{x}]$, $y \approx [\underline{y}, \overline{y}]$ and $a \approx [\underline{a}, \overline{a}]$, $x \overline{\oplus} y \approx [\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}]$ and $a \overline{\otimes} x \approx [\underline{a} \otimes \underline{x}, \overline{a} \otimes \overline{x}]$. Next, according to concept in max-plus algebra,

(i) $\delta(\underline{x} \oplus \underline{y}) \le \delta(\underline{x}) \oplus \delta(\underline{y})$ and $\delta(\overline{x} \oplus \overline{y}) \le \delta(\overline{x}) \oplus \delta(\overline{y})$,

(ii) $\delta(\underline{x}) = \delta(\underline{a} \otimes \underline{x})$ and $\delta(\overline{x}) = \delta(\overline{a} \otimes \overline{x})$.

Therefore,

(i)

$$\begin{aligned} I\delta(x \overline{\oplus} y) &= [\min(\delta(\underline{x} \oplus \underline{y}), \delta(\overline{x} \oplus \overline{y})), \delta(\overline{x} \oplus \overline{y})] \\ &\le [\min(\delta(\underline{x}) \oplus \delta(\underline{y}), \delta(\overline{x}) \oplus \delta(\overline{y})), \delta(\overline{x}) \oplus \delta(\overline{y})] \\ &= [\min(\delta(\underline{x}), \delta(\overline{x})), \delta(\overline{x})] \oplus [\min(\delta(\underline{y}), \delta(\overline{y})), \delta(\overline{y})] \\ &= I\delta(x) \overline{\oplus} I\delta(y), \end{aligned}$$

(ii)

$$\begin{aligned} I\delta(x) &= [\min(\delta(\underline{x}), \delta(\overline{x})), \delta(\overline{x})] \\ &= [\min(\delta(\underline{a} \otimes \underline{x}), \delta(\overline{a} \otimes \overline{x})), \delta(\overline{a} \otimes \overline{x})] = I\delta(a \overline{\otimes} x). \end{aligned}$$ □

Definition 2.8. The interval matrix $A \in I(\overline{\mathbb{R}})^{m \times n}$, $A \approx [\underline{A}, \overline{A}]$ is considered as double $I(\mathbb{R})$-astic if A is double $\mathbb{R}$-astic for each $A \in [\underline{A}, \overline{A}]$.

Theorem 2.9. *The matrix* $A \in I(\overline{\mathbb{R}})^{m \times n}$ *with* $A \approx [\underline{A}, \overline{A}]$ *is double* $I(\mathbb{R})$*-astic if and only if* $\underline{A}$ *is double* $\mathbb{R}$*-astic.*

Proof. It is known that $A \in I(\overline{\mathbb{R}})^{m \times n}$ with $A \approx [\underline{A}, \overline{A}]$ is double $I(\mathbb{R})$-astic. Based on the definition, A is double $\mathbb{R}$-astic for each $A \in [\underline{A}, \overline{A}]$. If $A = \underline{A}$, then $\underline{A}$ is double $\mathbb{R}$-astic. It is known that $\underline{A}$ is double $\mathbb{R}$-astic. Since $\underline{A}$ is the lower bound matrix for the matrix interval $[\underline{A}, \overline{A}]$, $\underline{A} \le \overline{A}$. Therefore, since it is a matrix $\underline{A}$ double $\mathbb{R}$-astic, matrix A

double $\mathbb{R}$-astic for each $A \in [\underline{A}, \overline{A}]$, thus $A \in I(\overline{\mathbb{R}})^{m\times n}$ double $I(\mathbb{R})$-astic is obtained. □

The following theorem presents solution to Problem 2.1:

Theorem 2.10. *Suppose that* $A \in I(\mathbb{R})_{\varepsilon}^{m\times n}$ *is double* $I(\mathbb{R})$*-astic and* $v_a \in I(\mathbb{R})^m$ *is a vector whose each component is equal to* $a \in I(\mathbb{R})$. *Then* $b \approx [\underline{b}, \overline{b}]$ *is a solution for Problem* 2.1 *such that* $\underline{b} \leq \underline{A} \otimes (\underline{A}^* \otimes' \underline{v_a})$ *and* $\overline{b} = \overline{A} \otimes (\overline{A}^* \otimes' \overline{v_{\overline{a}}})$.

Proof. Suppose that $A \approx [\underline{A}, \overline{A}]$, $v_a \approx [\underline{v_a}, \overline{v_{\overline{a}}}]$ and $a \approx [\underline{a}, \overline{a}]$. Since $A \in I(\mathbb{R})_{\varepsilon}^{m\times n}$ is double $I(\mathbb{R})$-astic, $v_a \in I(\mathbb{R})^m$ and v_a is a vector whose each component is equal to $a \in I(\mathbb{R})$, which means that:

(i) Matrix $\underline{A} \in \mathbb{R}_{\varepsilon}^{m\times n}$ is double $\mathbb{R}$-astic and $\underline{v_a}$ is a vector whose component is equal to $\underline{a} \in \mathbb{R}$. Based on concept in max-plus algebra, $\underline{A} \otimes (\underline{A}^* \otimes' \underline{v_a})$ is a solution for Problem 1.1.

(ii) Matrix $\overline{A} \in \mathbb{R}_{\varepsilon}^{m\times n}$ is double $\mathbb{R}$-astic and $\overline{v_{\overline{a}}}$ is a vector whose component is equal to $\overline{a} \in \mathbb{R}$. Based on concept in max-plus algebra, $\overline{A} \otimes (\overline{A}^* \otimes' \overline{v_{\overline{a}}})$ is a solution for Problem 1.1.

Therefore, $[\underline{b}, \overline{b}]$ with $\underline{b} \leq \underline{A} \otimes (\underline{A}^* \otimes' \underline{v_a})$ and $\overline{b} = \overline{A} \otimes (\overline{A}^* \otimes' \overline{v_{\overline{a}}})$ is a solution for Problem 2.1. □

Minimizing the range norm if its image vector is not finite

To solve Problem 2.2, the following theorem is used:

Theorem 2.11. *Suppose that* $A \in I(\mathbb{R})_{\varepsilon}^{m\times n}$, *where* $A = [A_{ij}]$ *for each* $i \in M$ *and* $j \in N$ *with* $A_{ij} \neq [\varepsilon, \varepsilon]$, *and* $v_a \in I(\mathbb{R})^m$ *is a vector whose each component is equal to constant interval* a. *If* $x \in I(\mathbb{R})_{\varepsilon}^{n}$ *such that* $b' \leq$

$A \overline{\otimes} x$ *is a solution for Problem* 2.2, *then* $\forall j \in N$ *satisfies*:

a. $x_j = [\varepsilon, \varepsilon]$ *or*

b. $x_j = [\min((\underline{A}^* \otimes' \underline{v}_a)_j, (\overline{A}^* \otimes' \overline{v}_{\overline{a}})_j), (\overline{A}^* \otimes' \overline{v}_{\overline{a}})_j]_j$.

Proof. Suppose that $A \approx [\underline{A}, \overline{A}]$, $v_a \approx [\underline{v}_a, \overline{v}_{\overline{a}}]$ and $a \approx [\underline{a}, \overline{a}]$, Since $A = [A_{ij}]$ for each $i \in M$ and $j \in N$ with $A_{ij} \neq [\varepsilon, \varepsilon]$, and $v_a \in I(\mathbb{R})^m$ is a vector whose each component is equal to constant interval *a*, it means that:

(i) Matrix $\underline{A} = [\underline{A}_{ij}]$ for each $i \in M$ and $j \in N$ with $\underline{A}_{ij} \neq \varepsilon$ and $\underline{v}_a \in \mathbb{R}^m$ is a vector whose each component is equal to constant $\underline{a}$.

(ii) Matrix $\overline{A} = [\overline{A}_{ij}]$ for each $i \in M$ and $j \in N$ with $\overline{A}_{ij} \neq \varepsilon$ and $\overline{v}_{\overline{a}} \in \mathbb{R}^m$ is a vector whose each component is equal to constant $\overline{a}$.

Then, suppose that $x \approx [\underline{x}, \overline{x}]$, $b' \approx [\underline{b}', \overline{b}']$ and $x_j \approx [\underline{x}_j, \overline{x}_j]$. It is known that $x \in I(\mathbb{R})_\varepsilon^n$ such that $b' = A \overline{\otimes} x$ is a solution for Problem 2.2, which according to concept in max-plus algebra means:

(iii) $\underline{x} \in \mathbb{R}_\varepsilon^n$ with $\underline{b}' \leq \underline{A} \otimes \underline{x}$ is the solution for Problem 1.2, so $\forall j \in N$ satisfies:

a′. $\underline{x}_j = \varepsilon$ or

b′. $\underline{x}_j = \min((\underline{A}^* \otimes' \underline{v}_a)_j, (\overline{A}^* \otimes' \overline{v}_{\overline{a}})_j)$.

(iv) $\overline{x} \in \mathbb{R}_\varepsilon^n$ with $\overline{b}' = \overline{A} \otimes \overline{x}$ is the solution for Problem 1.2, so $\forall j \in N$ satisfies:

a′. $\overline{x}_j = \varepsilon$ or

b′. $\overline{x}_j = (\overline{A}^* \otimes' \overline{v}_{\overline{a}})_j$.

Thus, from (i) and (iii) and also from (ii) and (iv), we have:

If $x \in I(\mathbb{R})_\varepsilon^n$ such that $b' = A \overline{\otimes} x$ is the solution for Problem 2.2, then $\forall j \in N$ satisfies:

a. $x_j = [\varepsilon, \varepsilon]$ or

b. $x_j = [\min((\underline{A}^* \otimes' \underline{v}_{\underline{a}})_j, (\overline{A}^* \otimes' \overline{v}_{\overline{a}})_j), (\overline{A}^* \otimes' \overline{v}_{\overline{a}})_j]_j$. □

B. Maximizing the range norm of matrix image set over interval max-plus algebra

The problem of maximizing the range norm of matrix image set can be formulated as follows:

Problem 2.3. Given that matrix $A \in I(\mathbb{R})_\varepsilon^{m\times n}$, solve:

$$\text{maximize } I\delta(b)$$

$$\text{subject to } b \in \mathrm{Im}(A).$$

Definition 2.12. Given that matrix $b \in I(\mathbb{R})_\varepsilon^m$ with $b \approx [\underline{b}, \overline{b}]$. The function $I\delta(b)$ is said to be *maximum* if and only if $\delta(\overline{b})$ and $\delta(\underline{b})$ are maximum.

Maximizing the range norm if its matrix is finite

In this section, we discuss maximizing the range norm if its matrix is finite.

Theorem 2.13. *Given that matrix* $A \in I(\mathbb{R})^{m\times n}$ *with* $A = (A_1\ A_2 \cdots A_n)$ *and* $A_j \approx [\underline{A}_j, \overline{A}_j]$; $j = 1, 2, \ldots, n$; *then for each* $b \in \mathrm{Im}(A)$,

$$I\delta(b) \le [\min(\max_{j=1,2,\ldots,n} \delta(\underline{A}_j), \max_{j=1,2,\ldots,n} \delta(\overline{A}_j)), \max_{j=1,2,\ldots,n} \delta(\overline{A}_j)].$$

Proof. Suppose that $A \in I(\mathbb{R})^{m\times n}$ and $b \in \mathrm{Im}(A)$ with $A \approx [\underline{A}, \overline{A}]$ and $b \approx [\underline{b}, \overline{b}]$, it means that $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$ and $\underline{b} \in \mathrm{Im}(\underline{A})$, $\overline{b} \in \mathrm{Im}(\overline{A})$. Based on concept in max-plus algebra, it is obtained that:

$$\delta(\underline{b}) \leq \max_{j=1,2,\ldots,n} \delta(\underline{A}_j)$$

$$= \max_{j=1,2,\ldots,n}(\max_{i=1,2,\ldots,m} \underline{a}_{ij} - \min_{i=1,2,\ldots,m} \underline{a}_{ij}) \text{ and}$$

$$\delta(\overline{b}) \leq \max_{j=1,2,\ldots,n} \delta(\overline{A}_j)$$

$$= \max_{j=1,2,\ldots,n}(\max_{i=1,2,\ldots,m} \overline{a}_{ij} - \min_{i=1,2,\ldots,m} \overline{a}_{ij}).$$

Therefore,

$$I\delta(b) = [\min(\delta(\underline{b}), \delta(\overline{b})), \delta(\overline{b})]$$

$$\leq [\min(\max_{j=1,2,\ldots,n} \delta(\underline{A}_j), \max_{j=1,2,\ldots,n} \delta(\overline{A}_j)), \max_{j=1,2,\ldots,n} \delta(\overline{A}_j)]$$

$$= [\min(\max_{j=1,2,\ldots,n}(\max_{i=1,2,\ldots,m} \underline{a}_{ij} - \min_{i=1,2,\ldots,m} \underline{a}_{ij}),$$

$$\max_{j=1,2,\ldots,n}(\max_{i=1,2,\ldots,m} \overline{a}_{ij} - \min_{i=1,2,\ldots,m} \overline{a}_{ij})),$$

$$\max_{j=1,2,\ldots,n}(\max_{i=1,2,\ldots,m} \overline{a}_{ij} - \min_{i=1,2,\ldots,m} \overline{a}_{ij})]. \qquad \square$$

Maximizing the range norm if its matrix is not finite

In the last section, we discuss maximizing the range norm if its matrix is not finite.

Theorem 2.14. *Given that* $A \in I(\mathbb{R})_{\varepsilon}^{m\times n}$ *is double* $I(\mathbb{R})$*-astic and not finite, i.e.,* $\exists i \in M,\ j \in N$ *such that* $a_{ij} = [\varepsilon, \varepsilon]$, *then Problem* 2.3 *is unbounded*.

Proof. Suppose that $A \in I(\mathbb{R})_{\varepsilon}^{m\times n}$ with $A \approx [\underline{A}, \overline{A}]$ is double $I(\mathbb{R})$-astic and infinite so $\exists i \in M,\ j \in N$ such that $a_{ij} = [\varepsilon, \varepsilon]$ which means that $\underline{A}, \overline{A} \in \mathbb{R}_{\varepsilon}^{m\times n}$ is double $\mathbb{R}$-astic and $\exists i \in M,\ j \in N$ such that $\underline{a}_{ij} = \varepsilon$ and

$\overline{a}_{ij} = \varepsilon.$ Therefore, according to concept in max-plus algebra, Problem 1.3 is related to matrices $\underline{A}$ and $\overline{A}$ which have unbounded solution, namely:

a. Given that matrix $\underline{A} \in \mathbb{R}_{\varepsilon}^{m \times n},$ solve:

$$\text{maximize } \delta(\underline{b})$$

$$\text{subject to } \underline{b} \in \mathrm{Im}(\underline{A}).$$

b. Given that matrix $\overline{A} \in \mathbb{R}_{\varepsilon}^{m \times n},$ solve:

$$\text{maximize } \delta(\overline{b})$$

$$\text{subject to } \overline{b} \in \mathrm{Im}(\overline{A}).$$

Therefore, Problem 2.3, that is given matrix $A \in I(\mathbb{R})_{\varepsilon}^{m \times n},$ solve:

$$\text{maximize } I\delta(b)$$

$$\text{subject to } b \in \mathrm{Im}(A)$$

has unbounded solution. □

III. Concluding Remarks

The main results are obtained the way to:

a. Minimizing the range norm of matrix image set over interval max-plus algebra, with its finite and not finite image vector.

These results are presented in Theorem 2.10 and Theorem 2.11.

b. Maximizing the range norm of matrix image set over max-plus algebra, with its finite and not finite matrix.

These results are presented in Theorem 2.13 and Theorem 2.14.

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