Numerical Simulation of Blood Flow in Human Artery Using (*A*, *Q*) and (*A*, *u*) Systems

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Numerical Simulation of Blood Flow in Human Artery Using (A, Q) and (A, u) Systems

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Abstract. In this paper, we model blood flow in human artery in the form of (A, Q) and (A, u) systems, then we use the Lax-Friedrichs finite volume method to find the numerical solution of each model. Here A represents the cross sectional area of the artery, Q denotes the discharge of the blood flow, and u is the velocity of the blood flow. We simulate the numerical scheme of each model and investigate how the blood pressure pulse propagates in human artery. Particularly, we use the residual of A to determine which system is better numerically. We obtain that the (A, Q) system is better numerically than the (A, u) system, because the absolute of the residual of A using the (A, Q) system is smaller than the absolute of the residual of A using the (A, u) system.

1. Introduction

Applications of Mathematics have been growing in various fields. Mathematics can be applied in various problems [3, 4], such as blood flow problems [1]. Modelling and simulation of blood flow are useful for human life.

There are a number of mathematical models for blood flow [2, 6]. Two of them are blood flow models in the form of (A, Q) and (A, u) systems [6]. Here A represents the cross sectional area of the artery, Q denotes the discharge of the blood flow, and u is the velocity of the blood flow. These (A, Q) and (A, u) systems can be used to explain phenomena of blood flow in human artery.

In this paper, we focus on modelling blood flow in human artery in the form of (A, Q) and (A, u) systems, solving both systems numerically, simulating both numerical solutions, and determining which system is better numerically. We use a finite volume method, because it is able to solve both continuous and discontinuous problems. Note that blood flow model has either continuous or discontinuous solution even though the initial condition is continuous. Then, we use the residual of A of each system to determine which system is better numerically.

This paper is organised as follows. We provide mathematical models and numerical methods for blood flow in Section 2. Research results and discussion are given in Section 3. We conclude the paper with some remarks in Section 4.

2. Models and methods

In this section, we model blood flow in human artery in the form of (A, Q) and (A, u) systems with several assumptions, and find the numerical method for each model. Blood flow models are derived from mass and momentum conservation laws [6].

We illustrate the shape of human artery in Figure 1. We denote the time variable t and the one-dimensional space variable z. We define that at an arbitrary cross section $S(z, t)$ of the artery

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$$A(z,t) = \int_S d\sigma, \tag{2.1}$$

$$u(z,t) = \frac{1}{A}\int_S \hat{u}\, d\sigma, \tag{2.2}$$

and

$$p(z,t) = \frac{1}{A}\int_S \hat{p}\, d\sigma, \tag{2.3}$$

where A is the cross section area, u is the average velocity, and p is the average pressure, $\hat{u}(z,\sigma,t)$ is the velocity, $\hat{p}(z,\sigma,t)$ is the pressure. We define $Q(z,t) = A(z,t)u(z,t)$ as the volume flux at $S(z,t)$. We assume that blood is incompressible fluid, so the viscosity and the density of blood are constant.

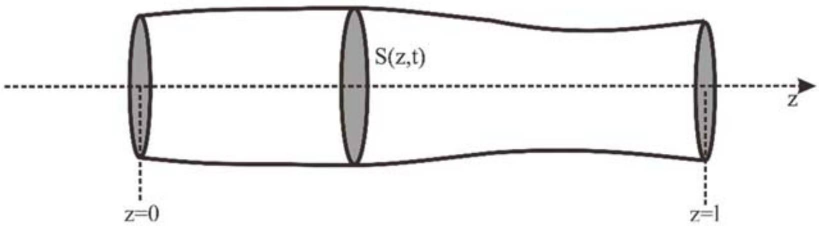

Figure 1. Illustration of human artery.

We have assumed that the shape of human artery is a cylindrical tube with circular cross section $S(z,t)$, and the circular cross section area does not depend on space variables x and y, but depends on the space variable z as well as the time variable t.

2.1. Mass conservation

Mass conservation law states that mass is neither created nor destroyed, so the total of the nett mass flux flowing out of a control volume and the rate of change of mass within a control volume equals to zero. We can write this statement as an equation

$$\rho\frac{dV}{dt} + \rho Q(l,t) - \rho Q(0,t) = 0, \tag{2.1.1}$$

where ρ is the density of blood and V is defined as follows

$$V(t) = \int_0^l A(z,t)\, dz. \tag{2.1.2}$$

Then, we can rewrite equation (2.1.1) as follows

$$\rho\frac{d}{dt}\int_0^l A(z,t)\, dz + \rho\int_0^l \frac{\partial Q}{\partial z}\, dz = 0. \tag{2.1.3}$$

Because the length l of the control volume is constant, we can rewrite equation (2.1.3) as follows

$$\rho\int_0^l \left(\frac{\partial A}{\partial z} + \frac{\partial Q}{\partial z}\right) dz = 0. \tag{2.1.4}$$

Equation (2.1.4) is satisfied for any value of l, so we obtain the mass conservation equation

$$\frac{\partial A}{\partial z} + \frac{\partial Q}{\partial z} = 0. \tag{2.1.5}$$

2.2. Momentum conservation

The Newton's second law states that the change of momentum within a system equals to the total of the applied forces. We assume that there is no flux through the artery walls, so the total of the nett momentum flux flowing out of a control volume and the rate of change of momentum within the control volume equals to the total of applied forces. We can write this statement as an equation

$$\frac{d}{dt}\int_0^l \rho Q(z,t)\,dz + \alpha\rho Q(l,t)u(l,t) - \alpha\rho Q(0,t)u(0,t) = F, \tag{2.2.1}$$

where $\rho Q(z,t)$ is the momentum, and α is the correction factor of the momentum flux. Then, the total of applied forces is defined as follows

$$F = p(0,t)A(0,t) - p(l,t)A(l,t) + \int_0^l p\frac{\partial A}{\partial z}\,dz + \int_0^l f\ dx, \tag{2.2.2}$$

where f is the blood friction force per unit length. If equation (2.2.2) is substituted to equation (2.2.1), then we obtain

$$\frac{d}{dt}\int_0^l \rho Q(z,t)\,dz + \int_0^l \frac{\partial(\alpha\rho Qu)}{\partial z}\,dz = -\int_0^l \frac{\partial(pA)}{\partial z}\,dz + \int_0^l p\frac{\partial A}{\partial z}\,dz + \int_0^l f\ dz. \tag{2.2.3}$$

Because ρ and l are constant, we can rewrite equation (2.2.3) as follows

$$\int_0^l \left(\rho\frac{\partial Q}{\partial t} + \rho\frac{\partial(\alpha Qu)}{\partial z}\right)dz = \int_0^l \left(-\frac{\partial(pA)}{\partial z} + p\frac{\partial A}{\partial z} + f\right)dz. \tag{2.2.4}$$

Equation (2.2.4) is satisfied for any value of l, so we have

$$\rho\frac{\partial Q}{\partial t} + \rho\frac{\partial(\alpha Qu)}{\partial z} = -\frac{\partial(pA)}{\partial z} + p\frac{\partial A}{\partial z} + f. \tag{2.2.5}$$

Noticing that

$$-\frac{\partial(pA)}{\partial z} = -\mathrm{p}\frac{\partial A}{\partial z} - \mathrm{A}\frac{\partial p}{\partial z}, \tag{2.2.6}$$

we obtain the momentum conservation equation

$$\rho\frac{\partial Q}{\partial t} + \rho\frac{\partial(\alpha Qu)}{\partial z} + \mathrm{A}\frac{\partial p}{\partial z} - f = 0. \tag{2.2.7}$$

2.3. *Blood flow models*

Based on the mass and momentum conservation laws, we find a one-dimensional blood flow model in human artery

$$\begin{cases} \dfrac{\partial A}{\partial t} + \dfrac{\partial Q}{\partial z} = 0, \\ \dfrac{\partial Q}{\partial t} + \dfrac{\partial}{\partial z}\left(\dfrac{Q^2}{A}\right) + \dfrac{A}{\rho}\dfrac{\partial p}{\partial z} = 0, \end{cases} \tag{2.3.1}$$

where $\alpha = 1$ and $f = 0$. This model (2.3.1) is called the (A,Q) system of blood flow model.

We have $Q = Au$, so the second equation (2.3.1) can be rewritten as follows

$$u\frac{\partial A}{\partial t} + A\frac{\partial u}{\partial t} + 2uA\frac{\partial u}{\partial z} + u^2\frac{\partial A}{\partial z} + \frac{A}{\rho}\frac{\partial p}{\partial z} = 0. \tag{2.3.2}$$

Thus, we get another one-dimensional blood flow model in human artery

$$\begin{cases} \dfrac{\partial A}{\partial z} + \dfrac{\partial(Au)}{\partial z} = 0 \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial z}\left(\dfrac{u^2}{2}\right) + \dfrac{1}{\rho}\dfrac{\partial p}{\partial z} = 0. \end{cases} \tag{2.3.3}$$

This model (2.3.3) is called the (A,u) system of blood flow model.

2.4. *Finite volume method*

Finding the exact solution of a model is not easy, so we shall solve the considered models numerically using a finite volume method with the formulation of the Lax-Friedrichs flux [4].

Notice that the (A,Q) system has two equations and three dependent variables (that is, A, Q, and p), so we define a relation for the blood pressure and the artery cross section area to get two equations with two dependent variables [2, 6]

$$p = p_{\mathrm{ext}} + \beta\left(\sqrt{A} - \sqrt{A_0}\right), \tag{2.4.1}$$

where A_0 is the artery cross section area when $t = 0$, and p_{ext} is the external pressure. Here, p_{ext} equals to zero, and A_0 is constant. Then, β is a parameter which relates to the wall elastic properties, and it is defined as follows [2]

$$\beta(z) = \frac{4\sqrt{\pi}h_0 E(z)}{3A_0}, \tag{2.4.2}$$

where $E(z)$ is the Young's modulus of elasticity.

For numerical solutions, let us consider the space domain discretisation (as shown in Figure 2) where $\Delta z = z_{i+1/2} - z_{i-1/2}$ or $\Delta z = z_i - z_{i-1}$, and the time domain discretisation $t^n = n\Delta t$ for non-negative integers i and n.

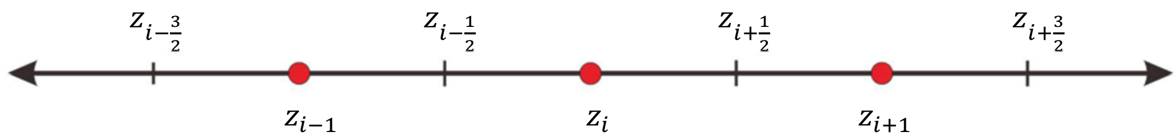

Figure 2. Spatial domain discretisation.

Using algebraic operations [2], we know that

$$\frac{A}{\rho}\frac{\partial p}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\beta A^{\frac{3}{2}}}{3\rho}\right) + \frac{A}{\rho}\frac{d\beta}{dz}\left(\frac{2}{3}\sqrt{A} - \sqrt{A_0}\right), \tag{2.4.3}$$

so the (A, Q) system can be written as a balance law

$$\bar{v}_t + \bar{f}\,(\bar{v})_z = \bar{s}(\bar{v}), \tag{2.4.4}$$

where the quantity, the flux, and the source term are respectively given by

$$\bar{v} = \begin{bmatrix} A \\ Q \end{bmatrix},\ \bar{f}(\bar{v}) = \begin{bmatrix} Q \\ \dfrac{Q^2}{A} + \dfrac{\beta}{3\rho}A^{\frac{3}{2}} \end{bmatrix},\ \text{and}\ \bar{s}(\bar{v}) = \begin{bmatrix} 0 \\ \dfrac{A}{\rho}\dfrac{d\beta}{dz}\left(\sqrt{A_0} - \dfrac{2}{3}\sqrt{A}\right) \end{bmatrix}. \tag{2.4.5}$$

Using the finite volume framework, we assume that $\bar{V}_i^n$, $\bar{f}(\bar{V}_i^n)$, and $\bar{S}_i^n$ are approximate values of $\bar{v}(z_i, t^n)$, $\bar{f}\big(\bar{v}(z_i, t^n)\big)$, and $\bar{s}\big(\bar{v}(z_i, t^n)\big)$, respectively. Then, we have the following vectors

$$\bar{V}_i^n = \begin{bmatrix} A_i^n \\ Q_i^n \end{bmatrix},\ \bar{f}(\bar{V}_i^n) = \begin{bmatrix} Q_i^n \\ \dfrac{(Q_i^n)^2}{A_i^n} + \dfrac{\beta}{3\rho}(A_i^n)^{\frac{3}{2}} \end{bmatrix},\ \text{and}\ \bar{S}_i^n = \begin{bmatrix} 0 \\ \dfrac{A_i^n}{\rho}\dfrac{d\beta}{dz}\left(\sqrt{A_0} - \dfrac{2}{3}\sqrt{A_i^n}\right) \end{bmatrix}. \tag{2.4.6}$$

Thus, the fully-discrete finite volume method [4] for the numerical solution to balance law (2.4.4) is

$$\bar{V}_i^{n+1} = \bar{V}_i^n - \frac{\Delta t}{\Delta z}\left(\bar{F}_{i+\frac{1}{2}}^n - \bar{F}_{i-\frac{1}{2}}^n\right) + \Delta t \bar{S}_i^n \tag{2.4.7}$$

with the formulation of the Lax-Friedrichs flux

$$\bar{F}_{i+\frac{1}{2}}^n = \frac{\bar{f}(\bar{V}_{i+1}^n) + \bar{f}(\bar{V}_i^n)}{2} - \frac{\Delta z}{2\Delta t}(\bar{V}_{i+1}^n - \bar{V}_i^n) \tag{2.4.8}$$

and

$$\bar{F}_{i-\frac{1}{2}}^n = \frac{\bar{f}(\bar{V}_i^n) + \bar{f}(\bar{V}_{i-1}^n)}{2} - \frac{\Delta z}{2\Delta t}(\bar{V}_i^n - \bar{V}_{i-1}^n). \tag{2.4.9}$$

From equations (2.4.7)-(2.4.9), we get the numerical scheme for (A, Q) system (2.3.1) as follows

$$A_i^{n+1} = A_i^n - \frac{\Delta t}{\Delta z}\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right) \tag{2.4.10}$$

with the formulation of the Lax-Friedrichs flux

$$F_{i+\frac{1}{2}}^n = \frac{1}{2}(Q_{i+1}^n + Q_i^n) - \frac{\Delta z}{2\Delta t}(A_{i+1}^n - A_i^n), \tag{2.4.11}$$

$$F_{i-\frac{1}{2}}^n = \frac{1}{2}(Q_i^n + Q_{i-1}^n) - \frac{\Delta z}{2\Delta t}(A_i^n - A_{i-1}^n), \tag{2.4.12}$$

and

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta z}\left(\mathcal{F}_{i+\frac{1}{2}}^n - \mathcal{F}_{i-\frac{1}{2}}^n\right) + \Delta t\left(\frac{A_i^n}{\rho}\frac{d\beta}{dz}\left(\sqrt{A_0} - \frac{2}{3}\sqrt{A_i^n}\right)\right) \tag{2.4.13}$$

with the formulation of the Lax-Friedrichs flux

$$\mathcal{F}_{i+\frac{1}{2}}^n = \frac{1}{2}\left[\frac{(Q_{i+1}^n)^2}{A_{i+1}^n} + \frac{\beta}{3\rho}(A_{i+1}^n)^{\frac{3}{2}} + \frac{(Q_i^n)^2}{A_i^n} + \frac{\beta}{3\rho}(A_i^n)^{\frac{3}{2}}\right] - \frac{\Delta z}{2\Delta t}(Q_{i+1}^n - Q_i^n), \tag{2.4.14}$$

$$\mathcal{F}_{i-\frac{1}{2}}^n = \frac{1}{2}\left[\frac{(Q_i^n)^2}{A_i^n} + \frac{\beta}{3\rho}(A_i^n)^{\frac{3}{2}} + \frac{(Q_{i-1}^n)^2}{A_{i-1}^n} + \frac{\beta}{3\rho}(A_{i-1}^n)^{\frac{3}{2}}\right] - \frac{\Delta z}{2\Delta t}(Q_i^n - Q_{i-1}^n). \tag{2.4.15}$$

Then, let us consider the (A, u) system in equation (2.3.3) as a conservation law

$$\bar{\mathcal{v}}_t + \bar{\mathcal{f}}\,(\bar{\mathcal{v}})_z = \bar{0} \tag{2.4.16}$$

where the quantity and the flux are

$$\bar{\mathcal{v}} = \begin{bmatrix} A \\ u \end{bmatrix} \text{ and } \bar{\mathcal{f}}(\bar{\mathcal{v}}) = \begin{bmatrix} Au \\ \dfrac{u^2}{2} + \dfrac{p}{\rho} \end{bmatrix}. \tag{2.4.17}$$

Using the finite volume framework, we assume that $\bar{\mathcal{V}}_i^n$ and $\bar{\mathcal{f}}(\bar{\mathcal{V}}_i^n)$ are approximate values of $\bar{\mathcal{v}}(z_i, t^n)$ and $\bar{\mathcal{f}}\big(\bar{\mathcal{v}}(z_i, t^n)\big)$, respectively. Then, with the same spatial and time discretisation, we have the following vectors

$$\bar{\mathcal{V}}_i^n = \begin{bmatrix} A_i^n \\ u_i^n \end{bmatrix} \text{ and } \bar{\mathcal{f}}(\bar{\mathcal{V}}_i^n) = \begin{bmatrix} A_i^n u_i^n \\ \dfrac{(u_i^n)^2}{2} + \dfrac{p_i^n}{\rho} \end{bmatrix}. \tag{2.4.18}$$

Thus, the fully-discrete finite volume method [4] for the numerical solution to conservation law (2.4.16) is

$$\bar{\mathcal{V}}_i^{n+1} = \bar{\mathcal{V}}_i^n - \frac{\Delta t}{\Delta z}\left(\bar{\mathrm{F}}_{i+\frac{1}{2}}^n - \bar{\mathrm{F}}_{i-\frac{1}{2}}^n\right) \tag{2.4.19}$$

with the formulation of the Lax-Friedrichs flux

$$\bar{\mathrm{F}}_{i+\frac{1}{2}}^n = \frac{\bar{\mathcal{f}}(\bar{\mathcal{V}}_{i+1}^n) + \bar{\mathcal{f}}(\bar{\mathcal{V}}_i^n)}{2} - \frac{\Delta z}{2\Delta t}(\bar{\mathcal{V}}_{i+1}^n - \bar{\mathcal{V}}_i^n) \tag{2.4.20}$$

and

$$\bar{\mathrm{F}}_{i-\frac{1}{2}}^n = \frac{\bar{\mathcal{f}}(\bar{\mathcal{V}}_i^n) + \bar{\mathcal{f}}(\bar{\mathcal{V}}_{i-1}^n)}{2} - \frac{\Delta z}{2\Delta t}(\bar{\mathcal{V}}_i^n - \bar{\mathcal{V}}_{i-1}^n). \tag{2.4.21}$$

From equations (2.4.19)-(2.4.21), we get the numerical scheme for (A, u) system (2.3.3) as follows

$$A_i^{n+1} = A_i^n - \frac{\Delta t}{\Delta z}\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right) \tag{2.4.22}$$

with the formulation of the Lax-Friedrichs flux

$$F_{i+\frac{1}{2}}^n = \frac{1}{2}(A_{i+1}^n u_{i+1}^n + A_i^n u_i^n) - \frac{\Delta z}{2\Delta t}(A_{i+1}^n - A_i^n), \tag{2.4.23}$$

$$F_{i-\frac{1}{2}}^n = \frac{1}{2}(A_i^n u_i^n + A_{i-1}^n u_{i-1}^n) - \frac{\Delta z}{2\Delta t}(A_i^n - A_{i-1}^n), \tag{2.4.24}$$

and

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta z}\left(\mathcal{F}_{i+\frac{1}{2}}^n - \mathcal{F}_{i-\frac{1}{2}}^n\right) \tag{2.4.25}$$

with the formulation of the Lax-Friedrichs flux

$$\mathcal{F}_{i+\frac{1}{2}}^n = \frac{1}{2}\left[\frac{(u_{i+1}^n)^2}{2} + \frac{p_{i+1}^n}{\rho} + \frac{(u_i^n)^2}{2} + \frac{p_i^n}{\rho}\right] - \frac{\Delta z}{2\Delta t}(u_{i+1}^n - u_i^n), \tag{2.4.26}$$

$$\mathcal{F}_{i-\frac{1}{2}}^n = \frac{1}{2}\left[\frac{(u_i^n)^2}{2} + \frac{p_i^n}{\rho} + \frac{(u_{i-1}^n)^2}{2} + \frac{p_{i-1}^n}{\rho}\right] - \frac{\Delta z}{2\Delta t}(u_i^n - u_{i-1}^n). \tag{2.4.27}$$

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3. Results and discussion

In this section, we simulate the numerical scheme of each system using the MATLAB programming language. For all simulations, we use the same coefficient values, initial and boundary conditions. Then, we investigate the numerical results, and determine which system is better numerically.

3.1. Numerical settings

For the first numerical simulation, we take the length of human artery $l_1 = 15$ cm and $t \in [0,\ 0.035]$ seconds. We assume the Young's modulus E is constant, so it implies that β is also constant. In other words, the values of β and E do not change at every $z \in [0,\ 15]$ and the value of $\frac{d\beta}{dz}$ equals to zero. Moreover, we take $\Delta z = 0.005$ and $\Delta t = 0.002\Delta z$. Methods are stable as long as Δz and Δt satisfy the CFL (Courant-Friedrichs-Lewy) condition [4]. All simulations have the length unit in cm and the time unit in second.

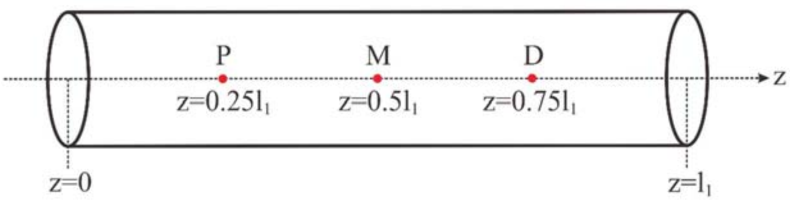

Figure 3. Illustration of human artery at $t = 0$.

Then, the layout of numerical simulation is explained as follows. Figure 3 shows the illustration of human artery at $t = 0$ and $z \in [0,\ 15]$. In this simulation, we set the P, M, and D points as the monitoring points to investigate the pressure variety. These P, M, and D points are proximal, medium, and distal points, respectively. Consider that P point is the nearest point from the heart, and D point is the farthest point from the heart. Table 1 shows the coefficient values used for this numerical simulation (see [2]).

Table 1. The coefficient values for numerical simulation [2].

Coefficient	Value
Blood density, ρ	1 gram/cm^3
Young's modulus, E_0	3x10^6 dyne/cm^2
Artery wall thickness, h	0.05 cm
Initial artery cross section area, A_0	$\pi 0.5^2$ cm^2

After that, we set the initial and boundary conditions to each of A, Q, u, and p. We are given initial values $A(z,0) = A_0$, $Q(z,0) = 0$, $u(z,0) = 0$, and $p(z,0) = 0$ for every $z \in (0,\ 15)$. The boundary values for this simulation are taken as follows. At the left boundary, we give an impulse input in the form of a single sine wave with a small time period,

$$p(0,t) = 10^3 \sin\left(\frac{\pi t}{0.0025}\right). \tag{3.1.1}$$

At the left boundary values of A and Q, let us consider the characteristic variables of (A, Q) system, W_1 and W_2, that have been explained in [2]

$$W_2 = \frac{Q}{A} - 2\sqrt{\frac{2\beta}{\rho}}A^{\frac{1}{4}}, \tag{3.1.2}$$

$$W_1 = W_2 + 4\sqrt{\frac{2}{\rho}}\left(\sqrt{p + \beta\sqrt{A_0}}\right), \tag{3.1.3}$$

$$A = \left(\frac{\rho}{\beta}\right)^2 \frac{(W_1 - W_2)^4}{4^5}, \tag{3.1.4}$$

$$Q = A\frac{W_1 + W_2}{2}. \tag{3.1.5}$$

The left boundary values of A and u are given as follows

$$A = \left(\frac{p + \beta\sqrt{A_0}}{\beta}\right)^2, \tag{3.1.6}$$

$$u = \sqrt{\frac{8\beta}{\rho}}\left(A^{\frac{1}{4}} - A_0^{\frac{1}{4}}\right). \tag{3.1.7}$$

The left boundary value of u in equation (3.1.7) is an equation of the wave characteristic speed (see [6]). Finally, each of right boundary values of A, Q, u, and p equals to the corresponding value of the nearest neighbour in the domain.

3.2. *Numerical results*

In this section, we show our numerical results. We take $T = 0.035$ seconds.

Numerical results of the (A, Q) system are shown in Figures 4-7. Based on the results that we obtain, with a constant Young's modulus, the artery cross section area is directly proportional to the blood pressure (see Figure 4 and Figure 5). Furthermore, the amplitude of blood pressure decreases with respect to space and time (see Figure 6 and Figure 7). We conclude that it is caused by the dissipation of the numerical method. As long as the solution is continuous, we also conclude that smaller cell width leads to smaller amplitude's decrement.

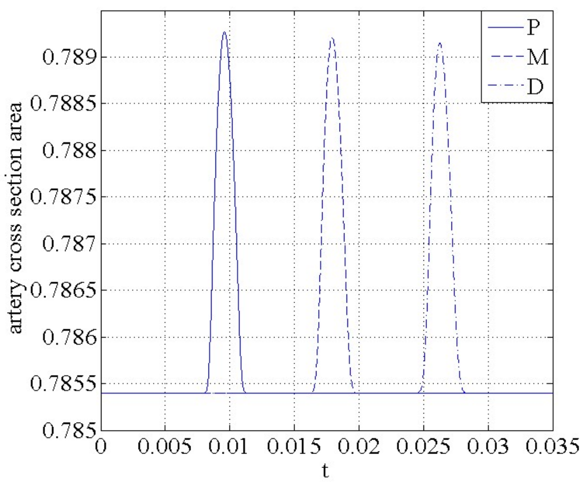

Figure 4. Graphic of artery cross section area of the (A, Q) system with respect to time.

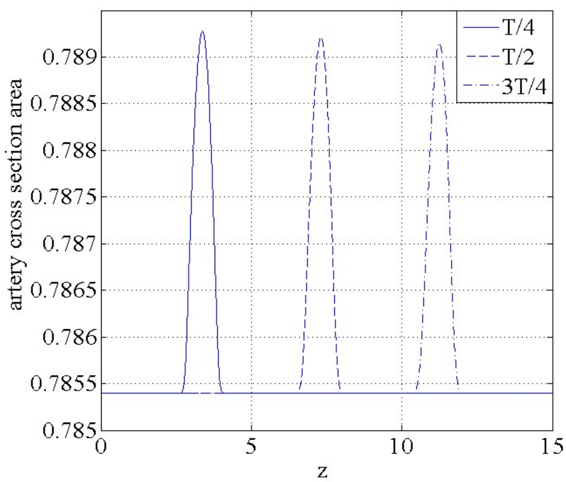

Figure 5. Graphic of artery cross section area of the (A, Q) system with respect to space.

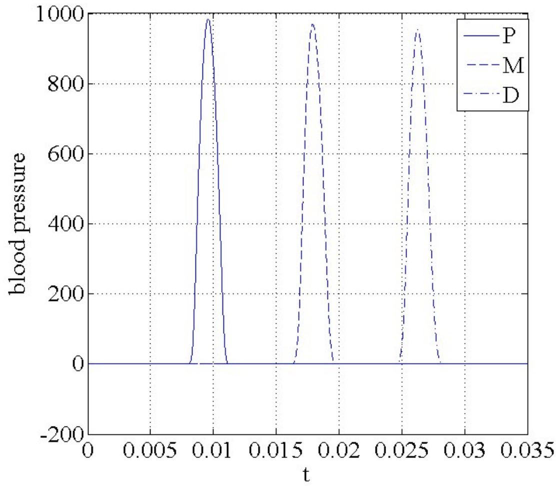

Figure 6. Graphic of blood pressure of the (A, Q) system with respect to time.

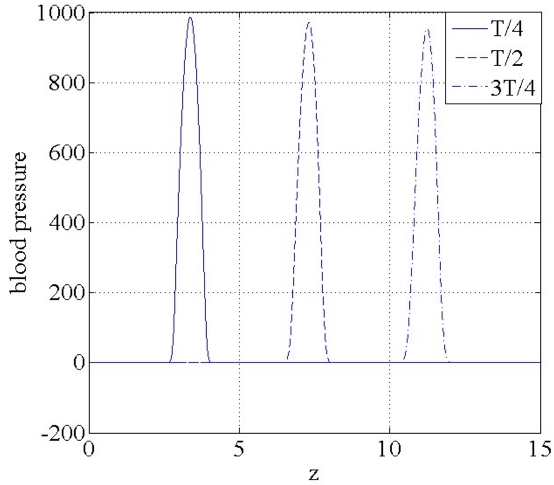

Figure 7. Graphic of blood pressure of the (A, Q) system with respect to space.

Numerical results of the (A, u) system are illustrated in Figures 8-11. Like the previous discussion, the artery cross section area is directly proportional to the blood pressure (see Figure 8 and Figure 9). Furthermore, the amplitude of the blood pressure decreases with respect to space and time (see Figure 10 and Figure 11). Again we conclude that it is caused by the dissipation of the numerical method. As long as the solution is continuous, we also conclude that smaller cell width results in smaller amplitude's decrement.

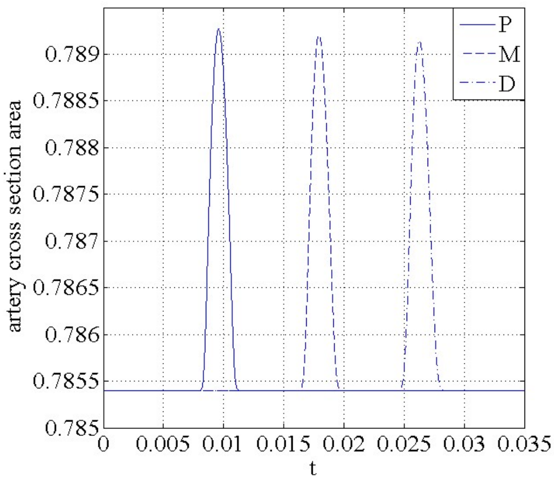

Figure 8. Graphic of artery cross section area of the (A, u) system with respect to time.

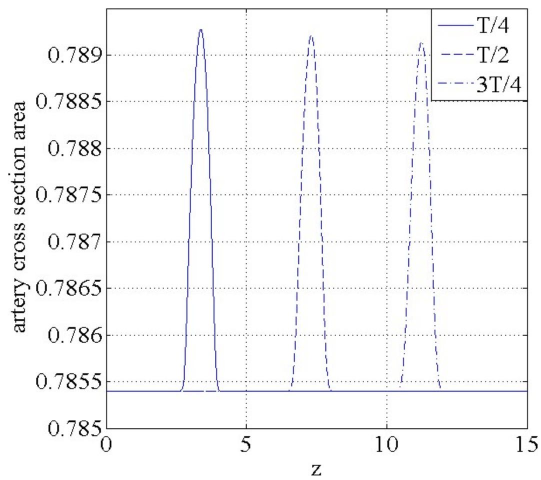

Figure 9. Graphic of artery cross section area of the (A, u) system with respect to space.

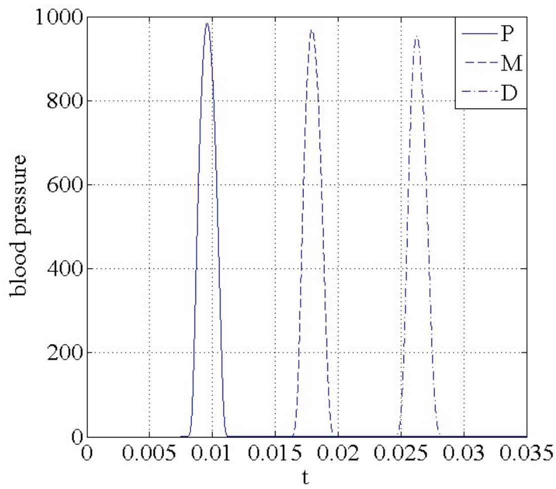

Figure 10. Graphic of blood pressure of the (A, u) system with respect to time.

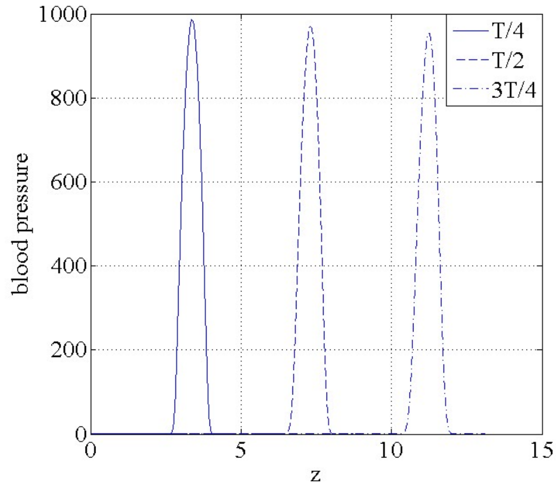

Figure 11. Graphic of blood pressure of the (A, u) system with respect to space.

The numerical results of the (A, Q) system and those of the (A, u) system look similar. However, there is a difference between those numerical results. The blood pressure of the (A, Q) system has negative values (see Figure 12), whereas the values of the blood pressure of the (A, u) system are non-negative (see Figure 13).

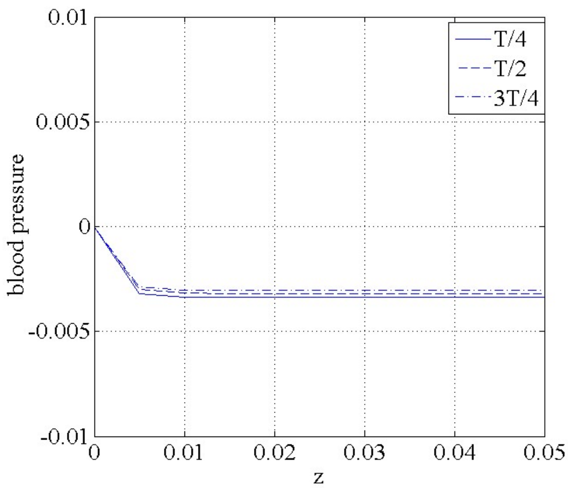

Figure 12. Magnification of blood pressure's graphic of the (A, Q) system with the space.

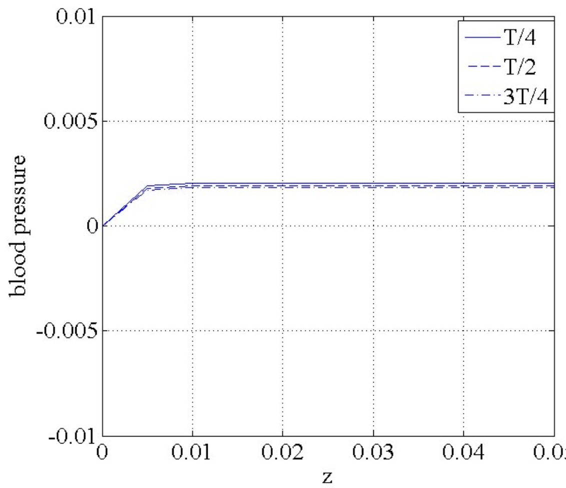

Figure 13. Magnification of blood pressure's graphic of the (A, u) system with the space.

3.3. Residuals

In this subsection, we compute the residual of A for each system to determine which system is better numerically. A system is said to be better numerically, if its residual is closer to zero.

We compute the residual of A with this formula [5]

$$E_{i+1/2}^{n-1/2} = \frac{\Delta z}{2}\left[q_i^n - q_i^{n-1} + q_{i+1}^n - q_{i+1}^{n-1}\right] + \frac{\Delta t}{2}\left[f\left(q_{i+1}^{n-1}\right) - f\left(q_i^{n-1}\right) + f\left(q_{i+1}^n\right) - f\left(q_i^n\right)\right], \tag{3.3.1}$$

where in this formula (3.3.1) we take $q_i^n = A_i^n$ to each of system, $f(q_i^n) = Q_i^n$ to the (A, Q) system, and $f(q_i^n) = A_i^n u_i^n$ to the (A, u) system.

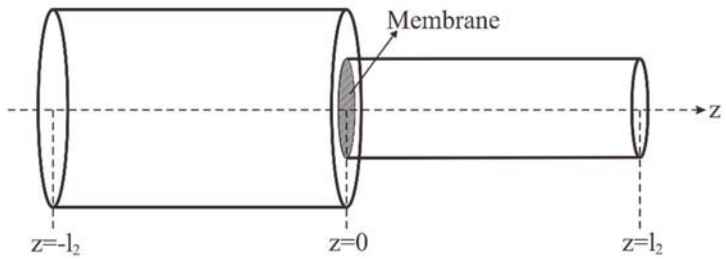

Figure 14. Illustration of a human artery with a membrane at $t = 0$.

The numerical layout to compute the residual of both systems is as follows. We assume to have human artery in the interval $[-l_2, l_2]$ with $l_2 = 2$. The initial condition is shown in Figure 14. There is a membrane at the artery cross section $S(0,0)$ at time $t = 0$. Then, at time $t > 0$, the membrane is removed completely, so blood can flow through.

Let us consider initial and boundary conditions for this simulation as follows. We are given a discontinuous initial function A

$$A(z,0) = \begin{cases} 10, \text{ if } z \le 0, \\ 5, \text{ if } z > 0, \end{cases} \tag{3.3.2}$$

for all $z \in (-2, 2)$. The initial values of Q, u, and p are zero, for all $z \in (-2, 2)$. Then, we are given the left and right boundary values of A

$$A(-2, t) = 10, \tag{3.3.3}$$
$$A(2, t) = 5, \tag{3.3.4}$$

for all $t > 0$. The left and right boundary values of Q, u, and p are zero for all $z \in (-2, 2)$. The time domain and coefficient values are taken the same as the previous simulation.

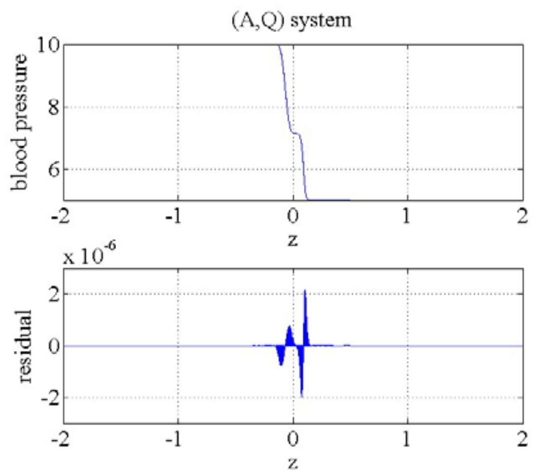

Figure 15. Blood pressure and residual of A of the (A, Q) system.

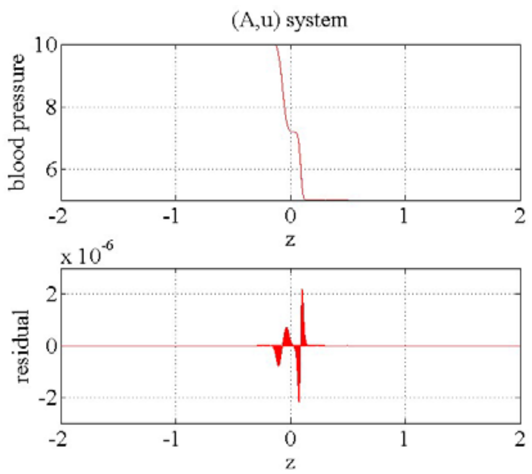

Figure 16. Blood pressure and residual of A of the (A, u) system.

Residuals of A are plotted in Figure 15 and Figure 16, for (A, Q) and (A, u) systems, respectively. The blood pressure's graphic looks alike, because the difference of the residual is small. Figure 15 shows that the lower bound of the residual of A using the (A, Q) system is larger than or equal to -2×10^{-6}. Figure 16 shows that the lower bound of the residual of A using the (A, u) system is smaller than

-2×10^{-6}. The upper bound of the residual of A using both systems look similar, that is larger than 2×10^{-6}.

4. Conclusion

Using our finite volume numerical method, the blood pressure pulse propagates from high pressure to lower one, as expected. The form of the propagating pulse does not change, but the amplitude decreases due to the dissipation of the numerical method. We conclude that the (A, Q) system is better numerically than the (A, u) system, because the absolute of the residual of A using the (A, Q) system is smaller than the absolute of the residual of A using the (A, u) system. Our results are limited to one-dimensional problems, but could be extended for multi-dimensional problems in future research.

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