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Variational estimate method for solving autonomous ordinary differential equations

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Abstract. In this paper, we propose a method for solving first-order autonomous ordinary differential equation problems using a variational estimate formulation. The variational estimate is constructed with a Lagrange multiplier which is chosen optimally, so that the formulation leads to an accurate solution to the problem. The variational estimate is an integral form, which can be computed using a computer software. As the variational estimate is an explicit formula, the solution is easy to compute. This is a great advantage of the variational estimate formulation.

1. Introduction

A number of methods are available in the literature for solving ordinary differential equations, either autonomous or non-autonomous [1-5]. Some analytical methods are available in the work of Haberman [1]. Variational iteration methods involving Lagrange multiplier were discussed by He and Wu [2], Inokuti et al. [3], Mungkasi et al. [4-5], and still many others. In this paper, we shall solve first-order autonomous ordinary differential equations of the form.

$$x'(t) + f(x) = 0 \quad (1)$$

with the initial condition

$$x(0) = \alpha. \quad (2)$$

We limit our discussion to one-dimensional problems with variable x is dependent on variable t. The discussion could be extended to higher dimensions, but it is out of the scope of this paper. Some real-world problems are governed by equations (1) with the initial condition (2), such as investment problems and the dynamics of population of a species [1]. Therefore, finding a solving method for this problem that is reliable is our interest. In this paper, we propose a variational estimate method for solving the problem (1)-(2). The method can be considered as an alternative solver to the usual numerical methods, such as those discussed by Mungkasi and Christian [6]. We note that the term of variational estimate itself has also been used in physics [7-12], engineering [13-15], and statistics [16]. The rest of this paper is organised as follows. We provide a general formulation for the variational estimate method in Section 2. Computational results and discussion are given in Section 3. We conclude the paper in Section 4 with some remarks.

2. Variational estimate method

In this section, we follow the work of Inokuti et al. [3] to derive a general formulation of the variational estimate method. Let us consider problem (1)-(2). Suppose that we want to compute $x(T)$, that is the solution to the problem at point $t = T$. We assume that t is the free variable and x is the dependent variable with respect to t. The variational estimate is given by

$$x(T)_{\mathrm{est}} = x(T) - \int_0^T \lambda(t)[x'(t) + f(x)]\, dt\,. \tag{3}$$

With this variational estimate, the differential equation (1) is treated as a condition on $x(t)$ that must be satisfied for all $t \in [0, T]$. To get the solution estimate $x(T)$, we can set $x(t)$ to be the initial condition. We note that choosing $x(t) = x(0)$ to be set into the variational estimate formula may not always lead to the most accurate results. However, it shall still produce a quite accurate solution to the problem.

3. Computational results and discussion

In this section, we write our computational results and some discussions.
We consider $f(x) = x^2$ and $\alpha = 1$, then we have the following ordinary differential equation

$$x'(t) + x(t)^2 = 0 \tag{4}$$

with the initial condition

$$x(0) = 1\,. \tag{5}$$

This problem is the same as the one considered by He and Wu [2] as well as Inokuti et al. [3]. He and Wu [2] solved the problem using the variational iteration method, but their standard variational iteration method has a demerit as they come to unnecessary calculation for the solution. Inokuti et al. [3] provided the solution only at point $t = 1$. We solve problem (4)-(5) using the variational estimate method for the domain of $t \in [0,1]$. We follow the step by step of Inokuti et al. [3]. We estimate the solution at $t = T$ using the variational estimate

$$x(T)_{\mathrm{est}} = x(T) - \int_0^T \lambda(t)[x'(t) + x(t)^2]\, dt\,. \tag{6}$$

We take a trial function $x(t) = x_0(t) + \delta x(t)$, where $x(0) = 1$ such that $\delta x(0) = 0$. With these data, we obtain

$$x(T)_{\mathrm{est}} = x_0(T) + \delta x(T) - \int_0^T \lambda(t)[x_0'(t) + \delta x'(t)]\, dt$$
$$- \int_0^T \lambda(t)\left[x_0(t)^2 + 2x_0(t)\,\delta x(t) + \delta x(t)^2\right] dt. \tag{7}$$

Calculating the optimal Lagrange multiplier, we obtain

$$\lambda(t)=\exp\left(2\int_1^x x(\tau)\ d\tau\right). \tag{8}$$

Therefore, the variational estimate formula for the problem is

$$x(T)_{est} = x(T) - \int_0^T \exp\left(2\int_1^x x(\tau)\ d\tau\right)[x'(t) + x(t)^2]\,dt\,. \tag{9}$$

This simplifies to

$$x(T)_{est} = \frac{1}{2}(1 + e^{-2T}) \tag{10}$$

which is the estimate solution x at point $t = T$.

Table 1. Variational estimate solution and the exact solution.

t	Estimate solution	Exact solution	Absolute error
0.0	1.0000	1.0000	0.0000
0.1	0.9094	0.9091	0.0003
0.2	0.8352	0.8333	0.0018
0.3	0.7744	0.7692	0.0052
0.4	0.7247	0.7143	0.0104
0.5	0.6839	0.6667	0.0173
0.6	0.6506	0.6250	0.0256
0.7	0.6233	0.5882	0.0351
0.8	0.6009	0.5556	0.0454
0.9	0.5826	0.5263	0.0563
1.0	0.5677	0.5000	0.0677

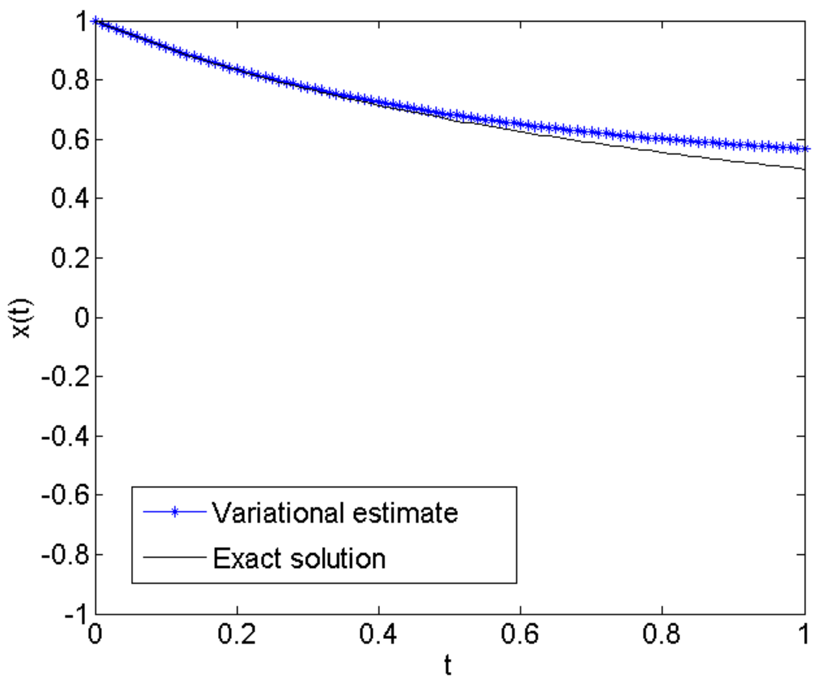

Figure 1. Computational results of variational estimate solution and the exact solution.

In order to investigate the accuracy of our results, we recall that the exact solution to problem (4)-(5) is given by

$$x(t) = \frac{1}{1+t}. \tag{11}$$

Table 1 summarises some computational results with respect to the exact solution. Figure 1 shows the graphic comparison between the variational estimate solution and the exact solution. Even though the absolute error gets larger as the free variable increases, the error is still quite small. We obtain that the variational estimate solution is accurate and is reliable. Therefore, we believe that it will be successful to be used to solve other first-order autonomous ordinary differential equation problems.

4. Conclusion

We have proposed a variational estimate method for solving first-order autonomous ordinary differential equation problems. The variational estimate solution is quite accurate. It is a promising method for ordinary differential equations with the initial condition is known. Our research focuses on one-dimensional autonomous problems. Future research direction could seek for a variational estimate method for solving non-autonomous and/or higher dimensional problems.

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