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# Performance of the Lax-Wendroff finite volume method for solving the gravity wave-model equations

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**Abstract.** The gravity wave-model equations are simplifications of the Saint-Venant equations by neglecting the convective term. This neglect is realistic as long as the gravity effect is much more significant than the convective effect in the system. In this paper, we present the performance (behaviour) of the standard Lax-Wendroff finite volume method used to solve the gravity wave-model equations. This is the first work in discussing the aforementioned method's performance in solving the gravity wave-model equations. We obtain that the standard Lax-Wendroff method is suitable for solving problems without discontinuity in the solution. When there is a discontinuity, the standard Lax-Wendroff method produces artificial oscillation in the solution.

## 1. Introduction

One of mathematical models for free-surface flows is the gravity wave-model equations [1]. Gravity wave-model equations are simplifications of the Saint-Venant equations by neglecting the convective terms [1-5]. The Saint-Venant (shallow water) equations consist of two partial differential equations, namely the mass equation and momentum equation [6-8]. The Saint-Venant equations involving horizontal topography are:

$$\frac{\partial}{\partial t} h + \frac{\partial}{\partial x} q = 0 \quad (1)$$

and

$$\frac{\partial}{\partial t} q + \frac{\partial}{\partial x} \left( \frac{q^2}{h} \right) + \frac{g}{2} \frac{\partial}{\partial x} h^2 = 0 \quad (2)$$

where  $h$  is fluid depth;  $q$  is unit-discharge;  $g$  is the gravitational acceleration;  $t$  is the time variable and  $x$  is the space variable. By neglecting the convective term of the Saint-Venant equations, we obtain the gravity wave-model equations, which can be written as

$$\frac{\partial}{\partial t} h + \frac{\partial}{\partial x} q = 0 \quad (3)$$

and

$$\frac{\partial}{\partial t} q + \frac{g}{2} \frac{\partial}{\partial x} h^2 = 0. \quad (4)$$



All quantities are assumed to be in SI units.

Real-world problems that have been modeled mathematically needs to be solved. The difficulty in determining the analytical solutions to the mathematical model can be overcome by doing numerical calculations to get the approximate solution of the model. In this paper, we discuss about the numerical solution to the gravity wave-model equations using the Lax-Wendroff finite volume method. On the one hand, variants of the Lax-Wendroff method have been broadly applied in solving numerous problems of hyperbolic conservation laws [9-15]. On the other hand, the gravity wave-model equations were attempted to be solved numerically by several authors, such as Apriani and Mungkasi [16] as well as Martins et al. [17]. However, an accurate and efficient solver for the gravity wave-model equations still needs to be sought. The aim of this paper is to find such solver. Therefore, we investigate the performance of the Lax-Wendroff finite volume method used to solve the gravity wave-model equations.

The rest of this paper is organised as follows. We present the Lax-Wendroff finite volume method for solving the gravity wave-model equations in Section 2. Numerical results are provided in Section 3 together with the discussion about them. We close the paper with concluding remarks in Section 4.

## 2. Lax-Wendroff finite volume method

The gravity wave-model equations (3) and (4) are conservation laws having the form

$$\bar{U}_t + \bar{f}(\bar{U})_x = \bar{0} \quad (5)$$

where

$$\bar{U}_t \equiv \frac{\partial \bar{U}(x,t)}{\partial t} \text{ and } \bar{f}(\bar{U})_x \equiv \frac{\partial \bar{f}(\bar{U})}{\partial x}. \quad (6)$$

Here

$$\bar{U} = \begin{pmatrix} h \\ q \end{pmatrix} \quad (7)$$

And

$$\bar{f}(\bar{U}) = \begin{pmatrix} q \\ \frac{g}{2} h^2 \end{pmatrix}. \quad (8)$$

We can write the gravity wave-model equations as

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} q \\ \frac{g}{2} h^2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (9)$$

In another form, gravity wave-model equations are

$$h_t + f'_1 q_x = 0 \quad (10)$$

And

$$q_t + f'_2 h_x = 0 \quad (11)$$

where  $f'_1 = 1$  and  $f'_2 = gh$ .

We can create a discrete form from (5) as follows [7, 8]:

$$\frac{Q_i^{n+1} - Q_i^n}{\Delta t} + \frac{F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n}{\Delta x} = 0 \quad (12)$$

And

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} \left( F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n \right) \quad (13)$$

where  $Q_i^n \approx \bar{U}(x_i, t^n)$ ;  $F_{i+\frac{1}{2}}^n \approx \bar{f}\left(\bar{U}\left(x_{i+\frac{1}{2}}, t^n\right)\right)$ ;  $\Delta t$  is the time step;  $\Delta x$  is the space step.

The general standard Lax-Wendroff fluxes are

$$F_{i+\frac{1}{2}}^n = \frac{1}{2}(f(Q_{i+1}^n) + f(Q_i^n)) - \frac{a_{i+\frac{1}{2}}^n \Delta t}{2\Delta x}(f(Q_{i+1}^n) - f(Q_i^n)) \quad (14)$$

And

$$F_{i-\frac{1}{2}}^n = \frac{1}{2}(f(Q_i^n) + f(Q_{i-1}^n)) - \frac{a_{i-\frac{1}{2}}^n \Delta t}{2\Delta x}(f(Q_i^n) - f(Q_{i-1}^n)) \quad (15)$$

where  $a_{i+\frac{1}{2}}^n = f'\left(\frac{Q_i^n + Q_{i+1}^n}{2}\right)$  and  $a_{i-\frac{1}{2}}^n = f'\left(\frac{Q_{i+1}^n + Q_i^n}{2}\right)$ .

The discrete form for the law of mass conservation is

$$h_i^{n+1} = h_i^n - \frac{\Delta t}{\Delta x} \left( {}^h F_{i+\frac{1}{2}}^n - {}^h F_{i-\frac{1}{2}}^n \right). \quad (16)$$

The Lax-Wendroff fluxes for the law of mass conservation are

$${}^h F_{i+\frac{1}{2}}^n = \frac{1}{2}(q_{i+1}^n + q_i^n) - \frac{{}^h a_{i+\frac{1}{2}}^n \Delta t}{2\Delta x}(q_{i+1}^n - q_i^n) \quad (17)$$

And

$${}^h F_{i-\frac{1}{2}}^n = \frac{1}{2}(q_i^n + q_{i-1}^n) - \frac{{}^h a_{i-\frac{1}{2}}^n \Delta t}{2\Delta x}(q_i^n - q_{i-1}^n). \quad (18)$$

Here,  ${}^h F_1^n$  is the flux for the mass conservation. In addition,

$${}^h a_{i+\frac{1}{2}}^n = f_1' = 1 \quad \text{and} \quad {}^h a_{i-\frac{1}{2}}^n \Delta t = f_1' = 1.$$

The discrete form for the law of momentum conservation is

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x} \left( {}^q F_{i+\frac{1}{2}}^n - {}^q F_{i-\frac{1}{2}}^n \right). \quad (19)$$

The Lax-Wendroff fluxes for the momentum conservation are

$${}^q F_{i+\frac{1}{2}}^n = \frac{1}{2} \left( \left( \frac{g}{2} h^2 \right)_{i+1}^n + \left( \frac{g}{2} h^2 \right)_i^n \right) - \frac{{}^q a_{i+\frac{1}{2}}^n \Delta t}{2\Delta x} \left( \left( \frac{g}{2} h^2 \right)_{i+1}^n - \left( \frac{g}{2} h^2 \right)_i^n \right) \quad (20)$$

And

$${}^q F_{i-\frac{1}{2}}^n = \frac{1}{2} \left( \left( \frac{g}{2} h^2 \right)_i^n + \left( \frac{g}{2} h^2 \right)_{i-1}^n \right) - \frac{{}^q a_{i-\frac{1}{2}}^n \Delta t}{2\Delta x} \left( \left( \frac{g}{2} h^2 \right)_i^n - \left( \frac{g}{2} h^2 \right)_{i+1}^n \right). \quad (21)$$

Here,  ${}^q F_i^n$  is the flux for the momentum conservation. In addition,

$${}^q a_{i+\frac{1}{2}}^n = (f_2')_{i+\frac{1}{2}}^n = (gh)_{i+\frac{1}{2}}^n = \frac{(gh)_{i+1}^n + (gh)_i^n}{2}$$

and

$${}^q a_{i-\frac{1}{2}}^n = (f_2')_{i-\frac{1}{2}}^n = (gh)_{i-\frac{1}{2}}^n = \frac{(gh)_i^n + (gh)_{i-1}^n}{2}.$$

By using (14) and (15), for the thegravity wave-model where  $Q_i^n \approx \begin{bmatrix} h_i^n \\ q_i^n \end{bmatrix}$ ,  $F_i^n \approx \begin{bmatrix} q_i^n \\ \frac{g}{2} (h_i^n)^2 \end{bmatrix}$ ,  $\Delta t$  is the time step,  $\Delta x$  is the space step,  $a_{i+\frac{1}{2}}^n = f' \left( \frac{Q_i^n + Q_{i+1}^n}{2} \right)$  and  $a_{i-\frac{1}{2}}^n = f' \left( \frac{Q_{i+1}^n + Q_i^n}{2} \right)$  we obtain

$$F_{i+\frac{1}{2}}^n = \frac{1}{2} \left[ \frac{g}{2} \left( (h_{i+1}^n)^2 + (h_i^n)^2 \right) \right] - \frac{a_{i+\frac{1}{2}}^n \Delta t}{2\Delta x} \left[ \frac{g}{2} \left( (h_{i+1}^n)^2 - (h_i^n)^2 \right) \right] \quad (22)$$

And

$$F_{i-\frac{1}{2}}^n = \frac{1}{2} \left[ \frac{g}{2} \left( (h_i^n)^2 + (h_{i-1}^n)^2 \right) \right] - \frac{a_{i-\frac{1}{2}}^n \Delta t}{2\Delta x} \left[ \frac{g}{2} \left( (h_i^n)^2 - (h_{i-1}^n)^2 \right) \right] \quad (23)$$

as the Lax-Wendroff numerical fluxes in the vector form including the mass and momentum fluxes.

### 3. Numerical results and discussion

In this section we provide our numerical results and some discussions.

#### 3.1. Surface wave simulation

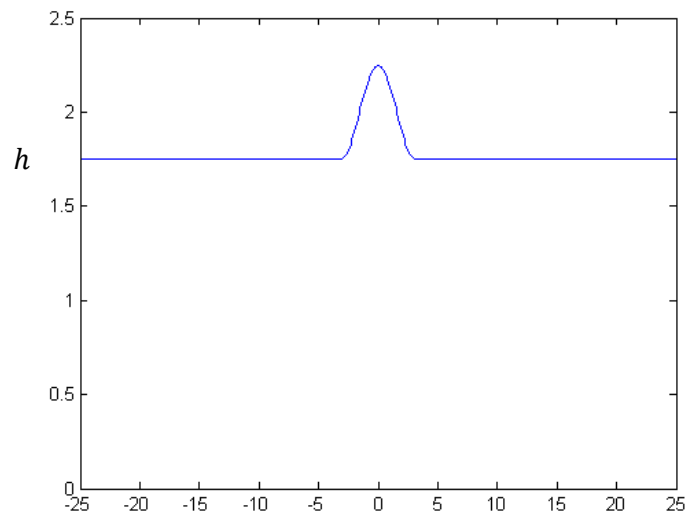
Surface wave is simulated by using the initial water depth function  $h = 2 + 0.25 \cos(x)$  if  $-\pi \leq x \leq \pi$  and 1.75 otherwise. We denote  $u = u(x, t)$  for the fluid velocity. Note that water discharge is  $q = uh$ . In this simulations we use the number of cells  $N = 500$ , space step  $\Delta x = 0.1$ , time step  $\Delta t = 0.001 \Delta x$ , and gravitational constant  $g = 9.81$ . The simulation is stopped at  $t = 5$ . The initial condition for fluid velocity is  $u(x, 0) = 0$  for all  $x$ . We set  $L = 25$ . Boundary conditions at  $x = -L$  are

$$h(-L, t) = 1.75, \quad u(-L, t) = 0$$

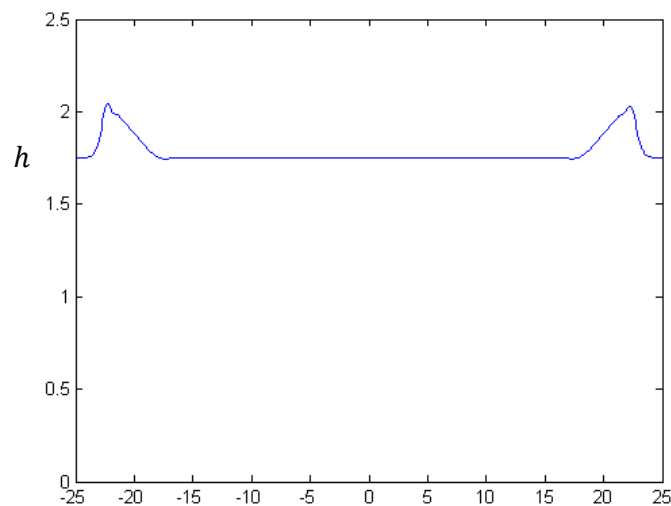
and those at  $x = L$  are

$$h(L, t) = 1.75, \quad u(L, t) = 0.$$

The initial water surface is shown in Figure 1. After we solve the problem using the Lax-Wendroff method, the water surface at  $t = 5$  is shown in Figure 2. These results are realistic as initially we have one bump of water in the middle of domain, as there is gravity effect, we then have two waves. One goes to the left and another one propagates to the right direction.



**Figure 1.** Illustration of the initial water surface. The horizontal axis is for  $x$  variable.



**Figure 2.** Simulated water surface at  $t = 5$ . The horizontal axis is for  $x$  variable.

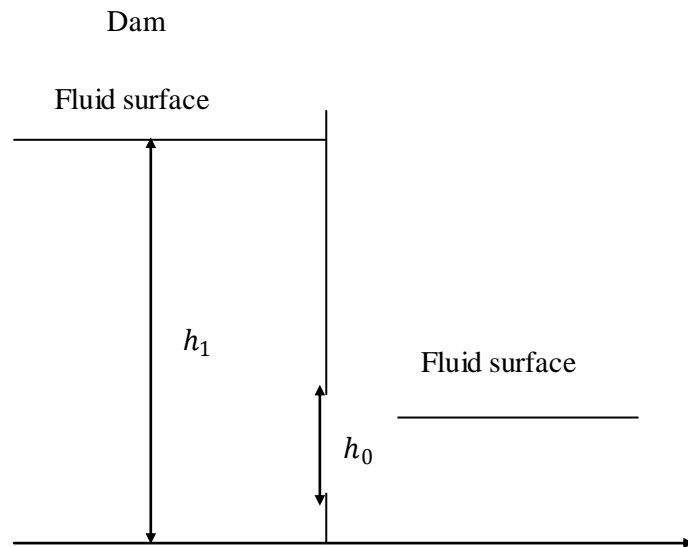
### 3.2. Dam-break simulation

In this simulation we use the number of cells  $N = 200$ , space step  $\Delta x = 0.1$ , time step  $\Delta t = 0.005\Delta x$ , gravitational constant  $g = 1$ . The simulation is stopped at  $t = 1$ . We assume the initial for fluid depth is  $h_1 = 10$  for all negative values of  $x$  and  $h_0 = 5$  for all positive values of  $x$ . The initial condition for fluid velocity is  $u(x, 0) = 0$ , for all  $x$ . We set  $L = 10$ . Boundary conditions at  $x = -L$  are

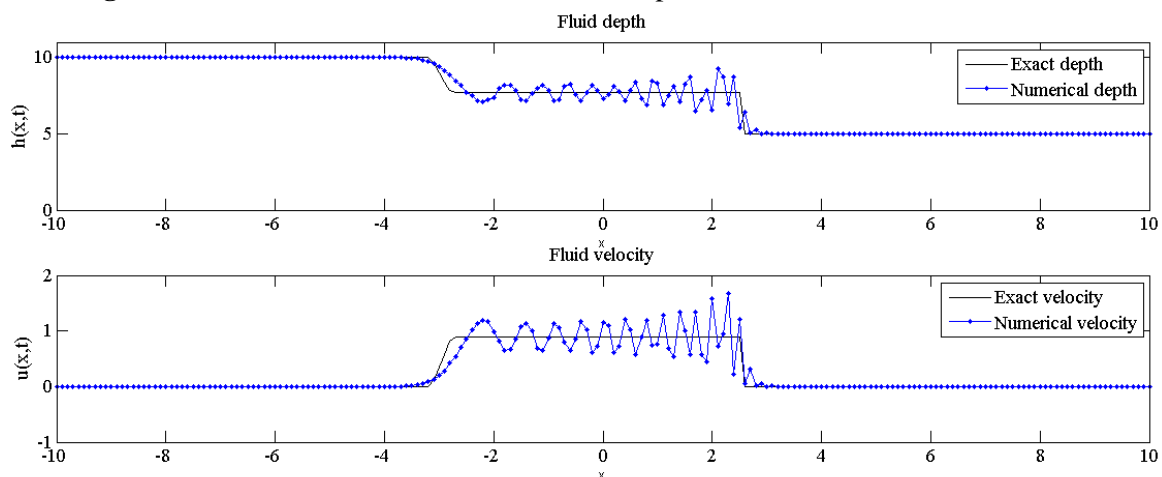
$$h(-L, t) = 10, \quad u(-L, t) = 0$$

and those at  $x = L$  are

$$h(L, t) = 5, \quad u(L, t) = 0.$$



**Figure 3.** Initial water surface for the dam break problem.



**Figure 4.** Analytical (exact) and numerical solutions for the dam break problem.

The initial condition of water surface for the dam break problem is given in Figure 3. There is discontinuity in the water depth. At time  $t = 0$ , we assume that the dam wall is completely removed. The simulation is conducted to observe how the Lax-Wendroff finite volume solution behaves for this discontinuous problem.

This test problem has been solved analytically by Martins et al [1]. We use the analytical (exact) solution to investigate the performance of the Lax-Wendroff finite volume method. We find that in comparison with the analytical solution, for this discontinuous problem, the Lax-Wendroff finite volume method produces artificial oscillation (see Figure 4), no matter how small we take the time step value. This is due to the order of accuracy of the method, which is larger than one. We do not implement any flux or slope limiter in the numerical method.

#### 4. Conclusion

The Lax-Wendroff finite volume method has been used to solve the gravity wave-model equations. The method is a standard method without any flux or slope limiter. We obtain that the method can be solved successfully for a smooth problem, but produces artificial oscillations in solving a

discontinuous problem. Future research direction is to seek a way to alleviate artificial oscillations in solving discontinuous problems.

### References

- [1] Martins R, Leandro J and Djordjević S 2016 Analytical solution of the classical dam-break problem for the gravity wave-model equations *Journal of Hydraulic Engineering***142**06016003
- [2] Aronica G, Tucciarelli T and Nasello C 1998 2D multilevel model for flood wave propagation in flood-affected areas *Journal of Water Resources Planning and Management***124**210
- [3] Bates P D, Horritt M S, Fewtrell T J 2010 A simple inertial formulation of the shallow water equations for efficient two-dimensional flood inundation modelling *Journal of Hydrology***387** 33
- [4] de Almeida GA M, Bates P, Freer J E and Souvignet M 2012 Improving the stability of a simple formulation of the shallow water equations for 2-D flood modeling *Water Resources Research***48** W05528
- [5] Seyoum S D, Vojinovic Z, Price R K and Weesakul S 2012 Coupled 1D and noninertia 2D flood inundation model for simulation of urban flooding *Journal of Hydraulic Engineering***138** 23
- [6] Chung W and Kang Y 2006 Classifying river waves by the Saint Venant equations decoupled in the Laplacian frequency domain *Journal of Hydraulic Engineering***132**666
- [7] LeVeque R J, 2004 *Finite Volume Methods for Hyperbolic Problems* (Cambridge: Cambridge University Press)
- [8] LeVeque R J, 1992 *Numerical Methods for Conservation Laws* (Berlin: Birkhäuser)
- [9] Bürger R, Kenettinkara S K and Zorío D 2017 Approximate Lax–Wendroff discontinuous Galerkin methods for hyperbolic conservation laws *Computers & Mathematics with Applications***74** 1288
- [10] Fridrich D, Liska R and Wendroff B 2017 Cell-centered Lagrangian Lax–Wendroff HLL hybrid method for elasto-plastic flows *Computers & Fluids***157** 164
- [11] Lee K, Leong Z Q and Nguyen H D 2017 Characterisation of waves via the Lax–Wendroff method *Journal of Ocean Engineering and Marine Energy***3** 247
- [12] Li T, Shu C-W and Zhang M 2017 Stability analysis of the inverse Lax–Wendroff boundary treatment for high order central difference schemes for diffusion equations *Journal of Scientific Computing***70** 576
- [13] Manshoor B, Khalid A, Sapit A, Zaman I and Nizam A 2017 Numerical solution of Burger's equation based on Lax-Friedrichs and Lax-Wendroff schemes *AIP Conference Proceedings***1831** 020025
- [14] Shu C-W 2017 Runge-Kutta and Lax-Wendroff discontinuous Galerkin methods for linear conservation laws *AIP Conference Proceedings***1863** 320010
- [15] Zorío D, Baeza A and Mulet P 2017 An approximate Lax–Wendroff-type procedure for high order accurate schemes for hyperbolic conservation laws *Journal of Scientific Computing***71** 246
- [16] Apriani C M and Mungkasi S 2017 A staggered grid finite difference method for solving the gravity wave-model equations *Journal of Physics: Conference Series***909** 012046
- [17] Martins R, Leandro J and Djordjević S 2015 A well balanced Roe scheme for the local inertial equations with an unstructured mesh *Advances in Water Resources***83** 351

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