

# STUDENTS' MATHEMATICAL REASONING IN EXPLORING FUNCTIONS AND ITS DERIVATIVE

Yosep Dwi Kristanto<sup>1)</sup> and Dewa Putu Wiadnyana Putra

Department of Mathematics Education, Universitas Sanata Dharma

<sup>1)</sup>yosepdwikristanto@usd.ac.id

## Abstract

Mathematical reasoning has been identified as an important vehicle in investigating students' understanding about mathematical concepts and procedures. The students' mathematical reasoning can be examined by using mathematical tasks in which the students justify and generalize mathematical ideas. Therefore, the aims of this study are to analyze (1) the features of mathematical tasks that have been worked by the students, and (2) students' reasoning in solving the tasks. The mathematical tasks used in this study are in the topic of functions and its derivative. The findings suggest that the provided tasks have features that demand the students' reasoning. The tasks are also consistent to its initial set up during its implementation. However, we found that the students still lack graphical understanding. The recommendation about how calculus learning should be in promoting graphical understanding will be explained as well.

**Keywords:** Mathematical Reasoning, Mathematical Task, Conceptual Knowledge, Derivative, Function.

## Introduction

Mathematical reasoning is important for students in learning and doing mathematics. Students use reasoning in many forms of learning. They use reasoning to justify their arguments to other students or teacher. They also employ their reasoning to generalize mathematical ideas.

Students' mathematical reasoning can be promoted using mathematical tasks. Stein and colleagues (1996) defined mathematical task as problem(s) that focuses students' attention on a particular mathematical idea. It can be a set of problems or a single complex problem. Francisco and Maher (2005) claim that giving students opportunity to work on the complex task can provoke their mathematical reasoning. When students deal with complex task, they need to decompose complex system into simpler subsystems to which they understand. When students understand the task deeply, they can justify and generalize mathematical ideas.

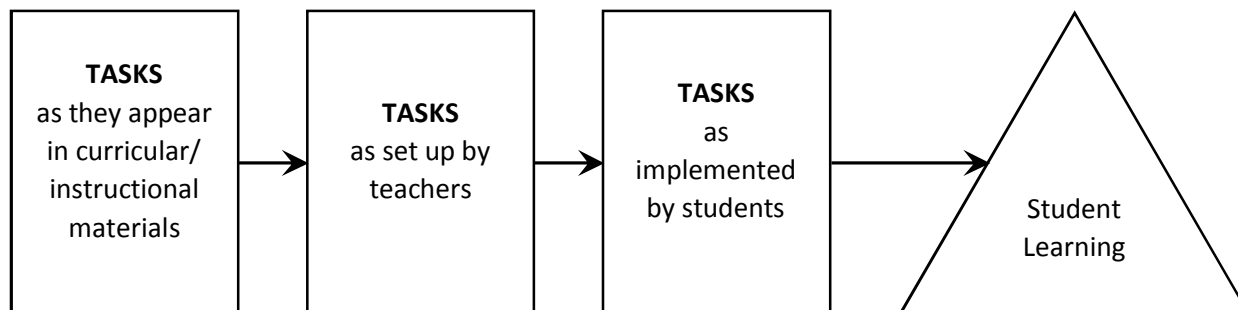
Since not all mathematical tasks can stimulate students' mathematical reasoning, the present study has following questions to be answered:

- Q1: What features do mathematical tasks that have been worked by students have?*
- Q2: How does students' mathematical reasoning?*

We argue that identifying task's features is necessary in order to give readers understanding the context of the present study.

### A. The Framework of Mathematical Task

In present study, we use mathematical task framework proposed by Stein and Smith (2011). The framework divide tasks to be three phases through which students pass: first, the emergence of the tasks in curricular or instructional material; second, configuration or announcement the tasks by teacher; and at last, its implementation to students in classroom. The framework of mathematical tasks is shown in Figure 1.



**Figure 1** The Mathematics Tasks Framework

There is possibility that tasks in curricular/instructional materials different with the tasks that developed and announced by teacher. The experiences of teacher in solving and using the tasks can predict how the tasks will be accessed by students (Liljedahl, Chernoff, & Zazkis, 2007). As such, those teacher experiences and students' context can affect the design of the tasks. Teacher sorts and selects parts of curricular/instructional material's tasks that ensemble to students learning. The change of the tasks from curricula material to teacher set up is corroborated by the finding of Stein, Grover and Henningsen (1996). They contend that students tend to engage in working from ground-breaking materials and/or from teacher-developed materials than from a textbook series. Furthermore, students' engagement also come from their activity in doing mathematical tasks that demand them to use multiple-solution strategies, multiple representations, and explanation or justification of their answers.

Change can also happen from teacher set up tasks to implementation phase. Stain and her colleagues (1996) found that high-level cognitive demanding tasks are less likely consistent between teacher set up and implementation by students. Besides, those kinds of tasks are essential to promote the students' thinking and reasoning. In or order to keep the tasks in consistent manner during the implementation, it is important for teacher to have pedagogical affordances within the classroom context (Liljedahl, Chernoff & Zazkis, 2007).

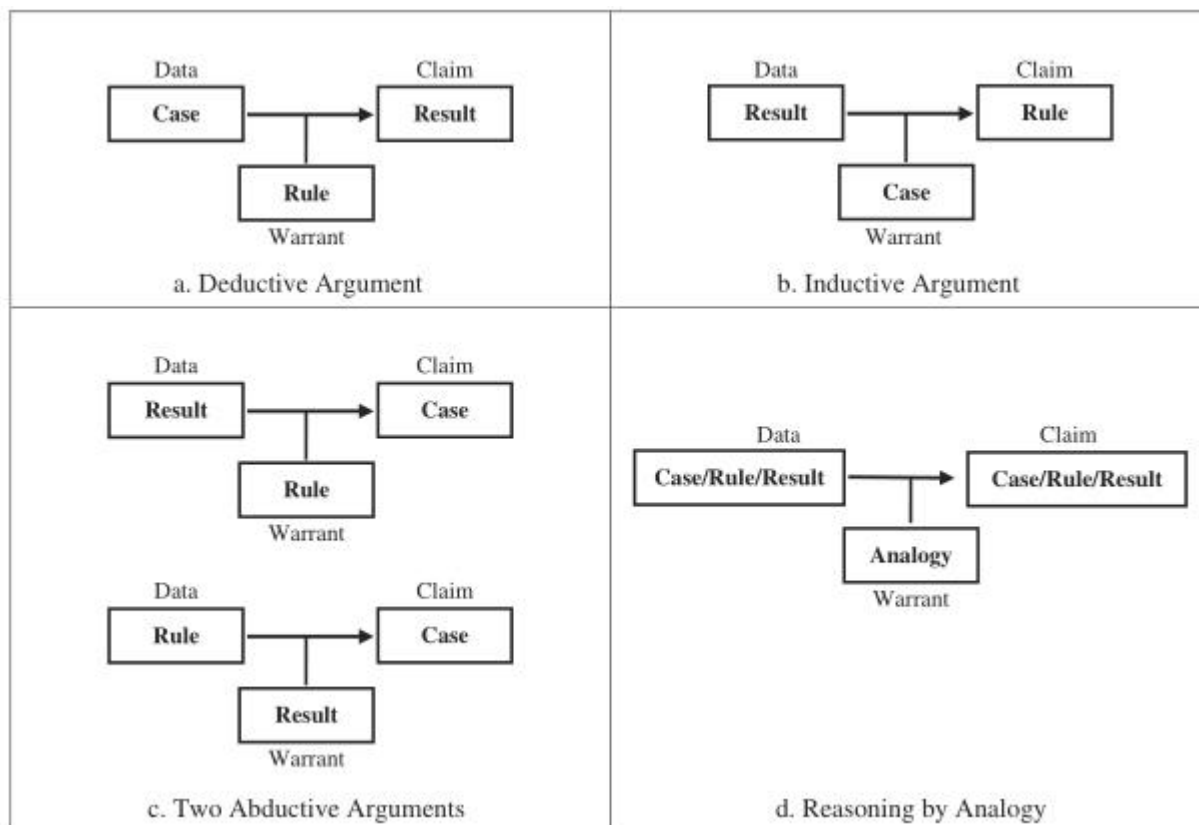
### B. Mathematical Reasoning

Literature give difference perspective on mathematical reasoning. Thompson (1996) described reasoning as "purposeful inference, deduction, induction, and association in the areas of quantity and structure." National Council of Teachers of Mathematics (2009) defined it as "the process of drawing conclusions on the basis of evidence or stated assumptions." When students work with

data or premises, they will use their reasoning with certain assumption to make a conclusion. Students also use reasoning in explaining their conjecture in order to reassure interlocutors.

Moshman (2004) defined reasoning as “epistemically self-constrained thinking.” He explained that making inferences is a natural process in which everyone engages, but when people are aware of and “constrain their inferences with the intent of conforming to what they deem to be appropriate inferential norms,” they engage in reasoning. Conner and colleagues (2014) agreed with this definition and described mathematical reasoning as purposeful inference about mathematical entities or relationships.

Mathematical reasoning can emerge in social setting, for example students’ group discussion. Therefore, we introduce collective argumentation. Collective argumentation is characterized as “multiple people working together to establish a claim” (Conner et. al., 2014). A helpful diagrammatic method to examine the kinds of reasoning encountered in collective argumentation can be understood by using Toulmin’s (1958/2003) model, see Figure 2.



**Figure 2** Toulmin-style diagrams of arguments reflecting different kinds of reasoning.

When engaging in deductive reasoning, one constructs conclusions as the logical consequence of aforementioned assumptions or conditions. As illustration, a student may claim that One exterior and one interior angle make a half circle. Since a half circle has 180 degrees, he concludes that the interior angle is 180 degrees minus the exterior angle.

When students engage in inductive reasoning, they draw abstractions or generalizations from individual observations. Abductive reasoning as making “an inference which allows the construction of a claim starting from an observed fact.” Similarly, Reid (2010) characterized the structure of abductive reasoning as the reverse of deductive reasoning. Reasoning by analogy requires developing a claim based on noticing similarities between corresponding cases (Reid, 2010).

## Method

The present study is a case study that use qualitative data. Subjects of the study were first year undergraduate students in a private university that come from difference places in Indonesia. They were the students of first author in Differential Calculus class that held twice a week where each meeting employed two hours class.

First author developed tasks in the topic of Derivative. This process began with searching problem in calculus textbooks. The problems to be considered were the problems that afforded students opportunities to explore mathematical ideas and concepts in meaningful ways. After the problems searching process, the chosen problem was one from calculus textbook by Briggs (2013: 159). The problem was as follows.

Suppose the line tangent to the graph of  $f$  at  $x = 2$  is  $y = 4x + 1$  and suppose  $y = 3x - 2$  is the line tangent to the graph of  $g$  at  $x = 2$ . Find an equation of the line tangent to the following curves at  $x = 2$ .

(a)  $y = f(x)g(x)$

(b)  $y = f(x)/g(x)$

After the problems were chosen, the next step was modifying that problems so that it would suit for the students. This process yielded the following problems.

Suppose the tangent line to the graph of  $f$  at  $x = 2$  is  $y = 3x - 2$  and suppose  $y = x + 1$  is the tangent line to the graph of  $g$  at  $x = 2$ . Sketch possible graphs of  $f$  and  $g$  on the coordinate plane. Why do you sketch the graphs of  $f$  and  $g$  like that? Give your reason.

The problem we present above was the same for all students. That problem was followed by another problem that not same among students. However, the latter also had the similarity: it demanded students to use Rule of Differentiation, e. g. Power Rule, Constant Multiple Rule, Sum Rule, Product and Quotient Rule.

The problems were announced in the classroom and students in the class were divided into thirteen small groups. Each group consisted 3 – 4 students. The group formation was determined by first author that considered the students’ achievement on the previous quizzes, so that each group consisted high- and low-achievers. It was intended that students can discuss one another in their groups actively.

Finally, two groups were chosen to be interviewed by first author right after the class finished and each group was represented by two students to be reviewed at once. We selected two students in each interview so that students were not nervous when interviewed by their lecturer. Besides, it was important to study how their collective argument. The interview technique to be

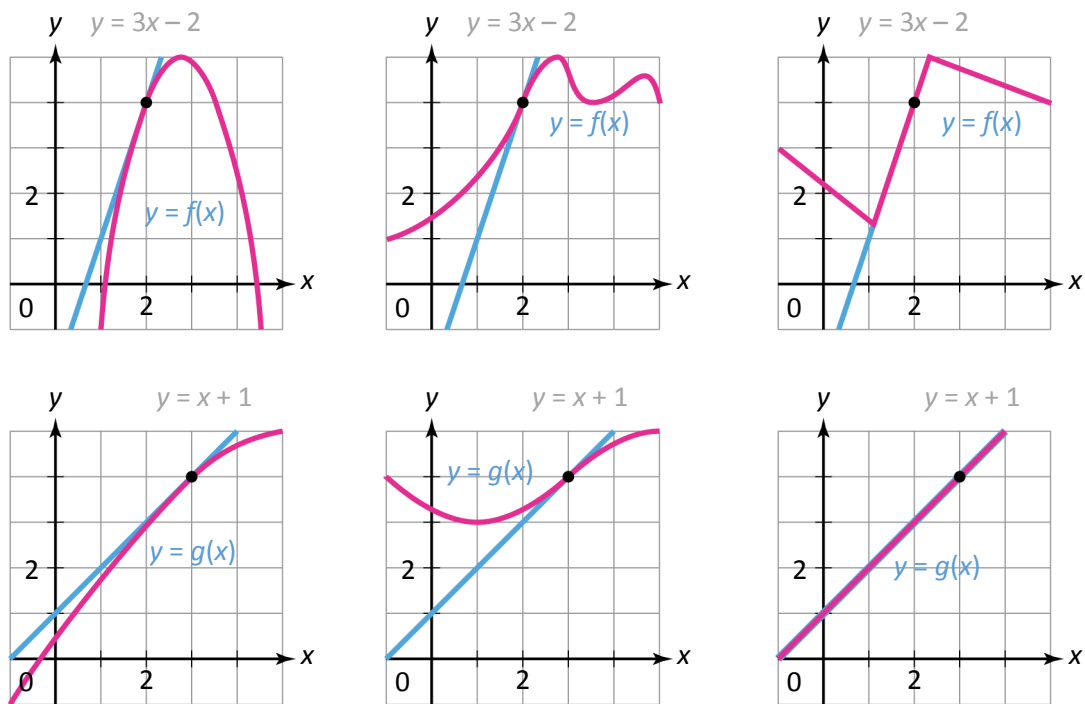
used in this present study was semi-structured interview. In general, the interview was used to know the students' mathematical reasoning for the given tasks.

## Results and Discussion

The present study aims are to analyze the features of mathematical tasks that have been worked by the students, and students' reasoning in solving the tasks. In answering first question, we describe the mathematical tasks and its implementation by students. Then, we use the data from students' worksheet to analyze students' mathematical reasoning.

### C. Description of Mathematical Tasks

The tasks consist of two problems, e.g. sketching possible function graphs and using The Derivative Rule. The number of possible solution strategies for the first problems is more than one. The first problem is a problem which does not define clearly what the question asks for, therefore allowing many possible solutions, so it is open-ended problem (Kwon, Park & Park, 2006). In fact, the first problem has infinite number of solutions provided they satisfies to conditions, e.g. they must be the graph of function, if they meet The Vertical Line Test, and they must have tangent line that mentioned in the problem. Examples of alternative solutions of the problem is shown in Figure 3.



**Figure 3** Alternative solutions of first problem

Many alternative solutions of the first problem show that it can be represented in different ways in the students' thinking. Since the given tangent line have gradient not equal to zero, the tangent line can intersect graph of  $f$  or  $g$  in two ways. It intersects at an inflection point or at a point that not inflection point. When tangent line intersect graph not at an inflection point, it can coincide with the graph. Biza & Zachariades (2010) found that tangent line that intersect the graph at an

inflection point and tangent line that coincides with the graph are among the most challenging cases for students.

The first problem demands students' reasoning since it asks students to give argument why they sketch the graph like they did. This question can stimulate discourse among students in a group. In this situation, dialogic and dialectic conversations can emerge. Therefore, the problem requires communication between student to one another. There can be communication between students and instructor as well.

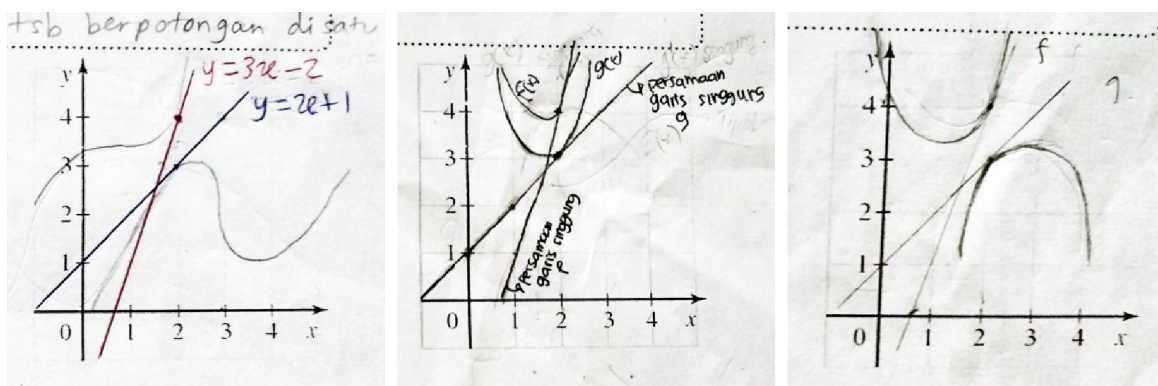
Second problem, the one that requires student to apply Derivative Rules, has exactly one solution but can be carried out by different strategies. In the first strategy, students can apply the Derivative Rules and then substitute the given value of variable to obtain an answer. Second, students begin by substituting the value of variable and then followed by applying the Derivative Rules.

This problem emphasizes the visualization of function. It requires students to interpret what the tangent line is and determine the derivative of functions based the information from the tangent line. Habre & Abboud (2006) observed that the algebraic representation of a function still dominated students' thinking for most students. Therefore, this problem can be challenge for students to think in unpopular way, so it stimulates communication among students, or between students and instructor.

#### D. Task Implementation

Tasks implementation by students is discussed in this section. The discussion is based on the students' work.

*Solution strategies and Representation.* After the tasks implementation, there are four correct answers that can be categorized into three types. The types of students' answers are shown in Figure 4. The first one, students sketch cubic-function-like graph for  $f$  and  $g$ . The other two are the parabola, but having different characteristic. Therefore, there is consistency between tasks set up and implementation.



**Gambar 4** Types of students' answers for first problem

All of the graphs that have been sketched by students, however, just represent on case. All of them intersect with its tangent line at a point other than inflection point. There is no graph by

students that coincide with the tangent line as well. It shows that those two cases are challenge for students in present study. This is consistent with Habre & Abboud's (2006) findings.

For the second problem, there are eight correct answers in which different strategies are encountered. It is conforming to the tasks set up. The different strategies that appear in the students' answer for the second problem are shown in Figure 5. Upper image shows that students find the derivative of the combined function respect to variable first, and then they determine the derivative at certain point. On the other hand, the students also can solve the problem by applying the Derivative Rules and substituting a value into variable at once, as shown in lower image. Furthermore, Figure 5 show different symbolic representation of derivative, e.g. Leibniz (upper image) and Lagrange's notation (lower image) of differentiation.

Misalkan  $h(x) = f(x)g(x) - x^2 + 5$ . Tentukan  $h'(2)$ .

$$h(x) = f(x)g(x) - x^2 + 5$$

$$h'(x) = (f(x)g'(x) + f'(x)g(x)) - 2x$$

$$h'(2) = (f(2)g'(2) + f'(2)g(2)) - 2(2)$$

$$h'(2) = (4 \cdot 1 + 3 \cdot 3) - 4$$

$$h'(2) = (4 + 9) - 4$$

$$h'(2) = 13 - 4 = 9$$
  

Misalkan  $h(x) = f(x)g(x)$ . Tentukan  $h'(2)$ .

$$h'(2) = f(2) \frac{d}{dx} g(2) + g(2) \frac{d}{dx} f(2)$$

$$= 4 \cdot 1 + 3 \cdot 3$$

$$= 13$$

**Figure 5** Different strategies and symbolic representation of second problem

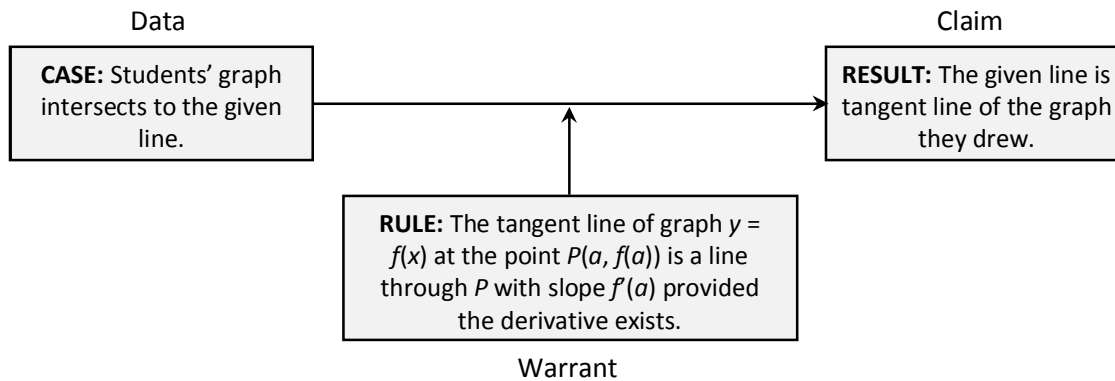
*Communication.* Both problems in the task have demanded communication among students in the group. The problems also encourage students to communicate to the instructor. For example, students ask to instructor about how to apply the Derivative Rule if the graph of functions and their tangent line are given.

### E. Students' Mathematical Reasoning

Students' mathematical reasoning in the present study is identified by students' worksheets and interview. Their reasoning that emergence in first problem are similar. They argue that the graph they drew have tangent line that same with the given line since the graph and the line are intersect each other. The represented students' reasoning that written on their worksheet is as follows.

The reason we draw the curve as shown in the figure is because it is one of possibilities that the curve intersects the given tangent line.

The argument that students use in the reasoning above is deductive. They use the definition of tangent line to show that a given line on figure is a tangent line of the graph they drew. In this case, the data is students' graph, the warranty is the definition of tangent line, and the claim is their conclusion. The deductive argument of students for first problem is shown in Figure 6 below.



**Figure 6** Example of deductive argument by students

However, not all students aware of tangent line definition in supporting their argument. They prefer to use informal terms, i.e. “touch” and “stick,” in describing the graph and its tangent line relation. As such, they thought that tangent line always intersects the corresponding graph exactly at one point. Yet it is not the case. They ignored the case that tangent line can cut the graph at more than one point. For example,  $x = 9$  is tangent line of  $f(x) = x^3 - 6x^2 + 9$ , but this tangent line intersects the graph at  $(0, 9)$  and  $(6, 9)$ . This misunderstanding about tangent line can be shown in the following interview.

- S1 : We draw the graph of  $f$  like that (pointed the graph) so that the graph can touch the given line at  $x = 2$ .  
 I : What do you mean about touching?  
 S2 : I mean it's intersect at one point.

For second problem, almost all of students cannot directly solve the problem. They asked the instructor for further explanation. However, instructor not give them explanation. Rather, he give students scaffolding on how solving the problem. After the discussion process, the students' reasoning can be shown in the following interview.

- I : At first, what is your procedure to solve the problem?  
 S2 : Applying the Derivative Rule for multiplication, Pak.  
 I : What do the rule say?  
 S2 : The derivative of  $fg$  is equal to the sum of  $f$  multiplied by the derivative of  $g$  and the multiplication of the derivative of  $f$  and  $g$ .  
 I : Okay, and then explain your answer.  
 S2 : We look at the graph. If  $x$  that equal to 2 is substituted into the equation (pointed the tangent line equation of  $f$ ), we get  $y = 4$  and the derivative of  $g$  is equal to the gradient of its tangent line, and then add the result to that  $g(x)$ , when  $x = 2$  we get  $y = 3$  then multiply by the derivative of  $f(x)$  that is equal to the gradient of its tangent line, i.e. 3.

Based on the students' reasoning for second problem, we conclude that students use the deductive reasoning. The data of their argument are described as follows:

Data	Warrant	Claim
• The derivative of	Derivative Rule:	$h'(2) = 13$



<p>function at certain point equal to the gradient of its tangent line that passing through the point.</p> <ul style="list-style-type: none"> <li>• The value of function at certain point equal to the value of its tangent line at the point.</li> </ul>	$\frac{d}{dx}[f(x)g(x)]$ $= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$	
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### Conclusion

The present study found that the mathematical tasks' set up is consistent to its implementation. The characteristics of the tasks are open ended and multi representation both in set up and implementation. However, students' answers to first problem are not representative for all cases. Students not presented the case that tangent line intersect the graph of function at an inflection point. Also, they not exposed the case when tangent line coincides the corresponding graph. It is consistent to Biza & Zachariades (2010) finding that those two cases are most challenging problems in the topic of tangent line.

We found that students can use the deductive arguments in their reasoning. They use the data that they found from the given problem together with the rule in making conclusion. However, there are students that simplify their argument too much. This simplification caused their argument not correct. The students have difficulties in finding the information from the graph as well. In other words, they still lack of graphical understanding.

Based on our findings, we suggest that instructors should conduct calculus learning that focus on theoretical analysis. It is consistent with Asiala, Cottrill, Dubinsky & Schwingendorf (1997) findings. They found that the students whose course was based on the theoretical analysis of learning may have had more success in developing a graphical understanding of a function and its derivative, than students from traditional courses.

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