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# Finite Difference Methods for Simulation of Water Waves Generated by Moving Topography

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# Finite Difference Methods for Simulation of Water Waves Generated by Moving Topography

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**Abstract**. In this paper, we consider water waves excited by moving topography. This problem usually occurs in the tsunami generation and propagation. We use the shallow water wave model governing this wave generation and propagation problem. Two finite difference methods are tested to solve the problem. The first is the forward in time centred in space finite difference method. The second is the Adams-Bashforth--Adams-Moulton method. We obtain that the second gives more flexibility in terms of the choice of time step value.

## 1. Introduction

Water waves in the ocean can be generated by a number of sources, such as wind, lunar influence, ship movement, bottom motion (moving topography), etc. This paper considers water waves that are generated by moving topography. The moving topography can initiate water to move, which is called the wave generation, and it is then followed by the wave propagation.

Water surface waves generated by moving topography have been an interest of a number of authors, as these can be used to simulate tsunami generation and propagation. Tinti, et al. [1] studied tsunami generation by submarine slides using the shallow-water theory. Didenkulova, et al. [2] derived some analytical solutions for a basin of variable depth relating to tsunami waves generated by submarine landslides. Jamin et al. [3] conducted experiments on the generation of surface waves by a moving topography. Case studies on this topic have also been conducted, such as by Grilli, et al. [4] regarding the coastal tsunami hazard model from submarine mass failures. These previous studies indicate that a reliable numerical method is needed in the simulation of surface wave generation and propagation.

In order to fulfil the mentioned need, this paper investigates the performance of popular finite difference methods used to solve the generation and propagation problems of surface waves due to moving topography. Two numerical methods are presented and their performances are discussed. We take the forward in time centred in space (FTCS) and the Adams-Bashforth--Adams-Moulton (ABAM) methods.

The remainder of this paper is organised simply as follows. Section 2 provides the mathematical model governing water waves. Section 3 contains the numerical methods under consideration for solving the mathematical model. Numerical results are given in Section 4. Finally, we draw some concluding remarks in Section 5.

## 2. Mathematical model

The mathematical model for water waves generated by moving topography is [1]:

$$\eta_t + [uH]_x = -h_t \,, \tag{1}$$

$$u_t + g\eta_x = 0 \,. \tag{2}$$

Here $t$ is the time variable, $x$ is the space variable, $\eta = \eta(x,t)$ is the level of water surface, $u = u(x,t)$ is water velocity, $h = h(x,t)$ is the bed topography, and $g$ denotes the acceleration due to gravity. In addition, $H(x,t) = \eta(x,t) + h(x,t)$. The bed topography is assumed to be impermeable, but it depends on variables $x$ and $t$. An illustration of water waves and their variables is given in Figure 1.
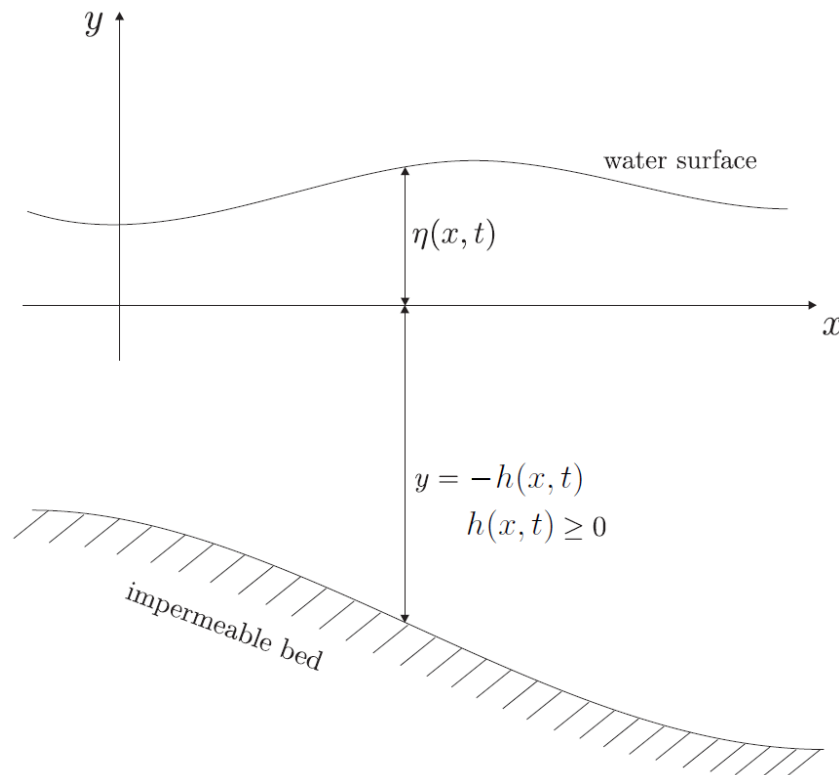


**Figure 1.** Illustration of water waves and their variables.

For smooth solutions, the model (1) and (2) is actually a rewriting of using the potential function relation to the velocity $u = \partial\phi/\partial x$, where $\phi$ is the potential function. This simplifies equation (2), so equations (1) and (2) can also be written as

$$\eta_t + [H\phi_x]_x = -h_t \,, \tag{3}$$

$$\phi_t + g\eta = 0 \,. \tag{4}$$

In this paper, we solve equations (3) and (4) using finite difference methods. The methods are presented in the next section.

### 3. Finite difference methods

We present two numerical finite difference methods. The first is the forward in time centred in space (FTCS) finite difference method. The second is the Adams-Bashforth--Adams-Moulton (ABAM) finite difference method.

Let us assume that the space domain $a \leq x \leq b$ is discretised using a uniform spatial step size $\Delta x$. Then we have spatial points $x_i = a + i\Delta x$, where $i = 0, 1, 2, \ldots$. In addition, we assume that the time domain $t \geq 0$ is discretised using a uniform temporal step size $\Delta t$. Then we have temporal points $t^n = n\Delta t$, where $n = 0, 1, 2, \ldots$. Using the finite difference framework, we use $\eta_i^n \approx \eta(x_i, t^n)$, $H_i^n \approx H(x_i, t^n)$, $h_i^n \approx h(x_i, t^n)$, $\phi_i^n \approx \phi(x_i, t^n)$, etc.

The FTCS scheme being considered is a two-step method. The first step is

$$\eta_i^* = \eta_i^n - \frac{\Delta t}{4\Delta x^2}[H_{i+1}^n(\phi_{i+2}^n - \phi_i^n) - H_{i-1}^n(\phi_i^n - \phi_{i-2}^n)] - \frac{1}{2}(h_i^{n+1} - h_i^{n-1}), \tag{5}$$

$$\phi_i^* = \phi_i^n - \Delta t \, g \, \eta_i^n. \tag{6}$$

The second step is

$$\eta_i^{**} = \eta_i^* - \frac{\Delta t}{4\Delta x^2}[H_{i+1}^*(\phi_{i+2}^* - \phi_i^*) - H_{i-1}^*(\phi_i^* - \phi_{i-2}^*)] - \frac{1}{2}(h_i^{n+1} - h_i^{n-1}), \tag{7}$$

$$\phi_i^{**} = \phi_i^* - \Delta t \, g \, \eta_i^*. \tag{8}$$

After those two steps, we take the average for the values at time $t^{n+1}$, as follows

$$\eta_i^{n+1} = \frac{1}{2}(\eta_i^{**} + \eta_i^n), \tag{9}$$

$$\phi_i^{n+1} = \frac{1}{2}(\phi_i^{**} + \phi_i^n). \tag{10}$$

The ABAM method consists of two steps too [5-6]. The Adams-Bashforth step is

$$\eta_i^* = \eta_i^n + \frac{\Delta t}{12}(23F_i^n - 16F_i^{n-1} + 5F_i^{n-2}), \tag{11}$$

$$\phi_i^* = \phi_i^n + \frac{\Delta t}{12}(23G_i^n - 16G_i^{n-1} + 5G_i^{n-2}). \tag{12}$$

Here

$$F_i^n = -\frac{1}{4\Delta x^2}[H_{i+1}^n(\phi_{i+2}^n - \phi_i^n) - H_{i-1}^n(\phi_i^n - \phi_{i-2}^n)] - \frac{1}{2\Delta t}(h_i^{n+1} - h_i^{n-1}), \tag{13}$$

$$G_i^n = -g\eta_i^n. \tag{14}$$

Using the predicted values $\eta_i^*, \phi_i^*$, we evaluate $F_i^{n+1}$ and $G_i^{n+1}$. Then, we use the Adams-Moulton step, as follows

$$\eta_i^{**} = \eta_i^* + \frac{\Delta t}{12}(5F_i^{n+1} + 8F_i^n - F_i^{n-1}), \tag{15}$$

$$\phi_i^{**} = \phi_i^* + \frac{\Delta t}{12}(5G_i^{n+1} + 8G_i^n - G_i^{n-1}). \tag{16}$$

After those two steps, we take the average for the values at time $t^{n+1}$, as follows

$$\eta_i^{n+1} = \frac{1}{2}(\eta_i^{**} + \eta_i^n), \tag{17}$$

$$\phi_i^{n+1} = \frac{1}{2}(\phi_i^{**} + \phi_i^n). \tag{18}$$

We use these FTCS and ABAM methods to simulate water waves generated by the motion of water bed topography. We present our simulations in the next section.

## 4. Numerical results
We present our numerical results and discussion. When quantities are not written with their units, we assume that they have SI units with the MKS system.

In our simulations, we assume that initially water surface and its bed topography are horizontal. At time $t = 0$, water is still and its depth is $H_0 = 4000$ everywhere. We consider the space domain $|x| \le$ 80,000. The acceleration due to gravity is $g = 9.81$ ms$^{-2}$. Suppose that during the first $T = 10$ seconds, the bottom topography is perturbed, so the water depth is defined as

$$H(x,t) = H_0 - f(x)S(t) \tag{19}$$

where

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| \le 10,000 \\ \\ 0 & \text{if} \quad |x| > 10,000 \end{cases} \tag{20}$$

and

$$S(t) = \begin{cases} 0.5(1 - \cos(\pi t/T)) & \text{if} \quad t \le T \\ \\ 1 & \text{if} \quad t > T. \end{cases} \tag{21}$$

The space domain is discretised using the spatial step size $\Delta x = 25$. The time domain is discretised using the temporal step sizes $\Delta t = \Delta x/1000, \Delta x/500$, and $\Delta x/250$. We will investigate which numerical method is more flexible with respect to the temporal step size.

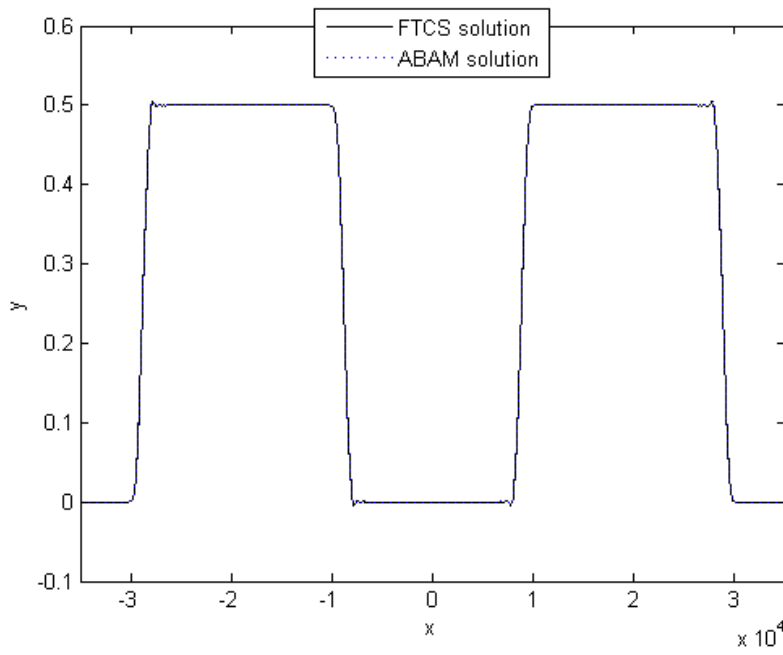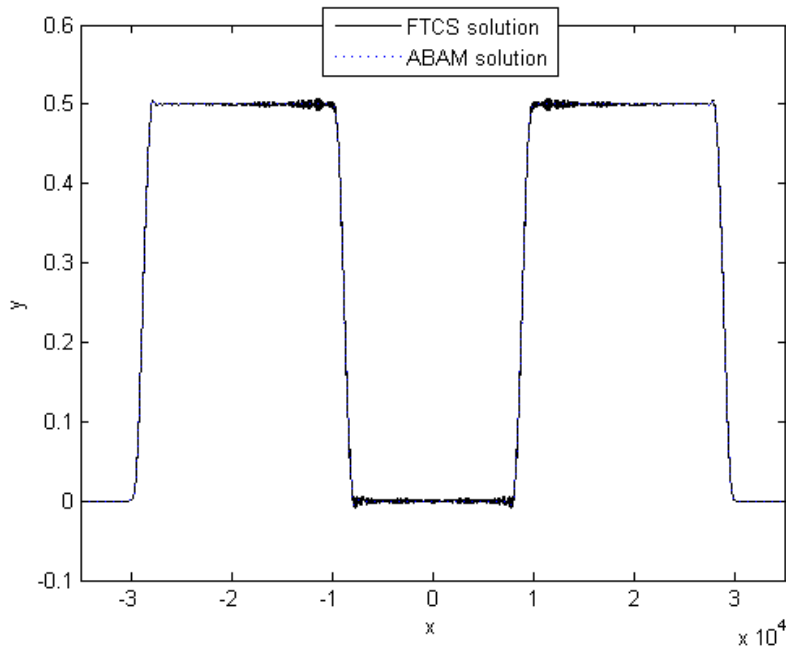**Figure 2.** Numerical solution at time $t = 100$ using $\Delta t = \Delta x / 1000$.



**Figure 3.** Numerical solution at time $t = 100$ using $\Delta t = \Delta x / 500$.
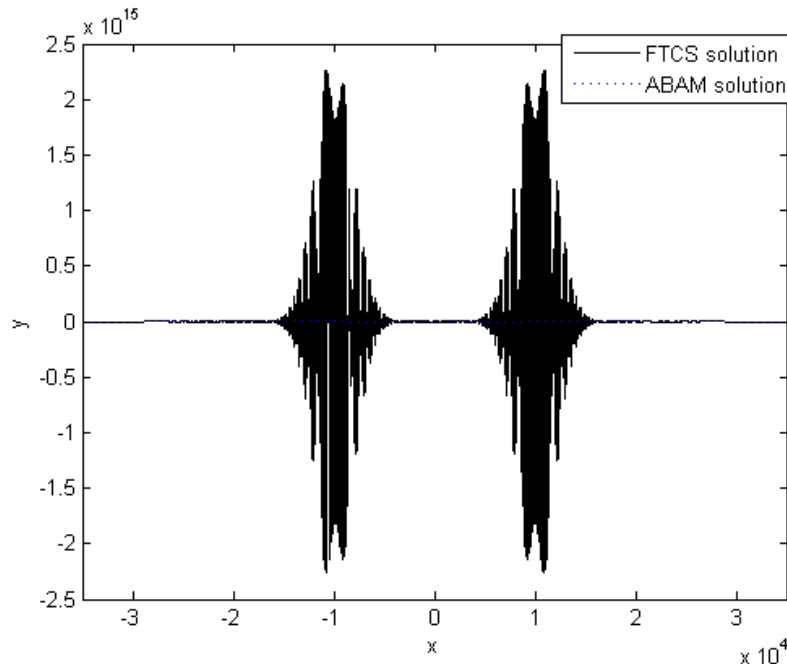
**Figure 4.** Numerical solution at time $t = 100$ using $\Delta t = \Delta x / 250$.

Numerical simulations are stopped for time $t = 100$. Figures 2-4 show the numerical solution for the water surface at time $t = 100$ using the temporal step sizes $\Delta t = \Delta x / 1000, \Delta x / 500$, and $\Delta x / 250$ respectively. In Figure 2, both the FTCS and ABAM methods produce the same results (with negligible discrepancies). Figure 3 shows that while the ABAM method gives a very much similar to the results in Figure 2, the FTCS method starts producing artificial oscillations. Completely incorrect results are generated by the FTCS method when the time step (the temporal step size) is taken smaller, as shown in Figure 4. In Figure 4, we observe that when the time step is $\Delta t = \Delta x / 250$, the FCTS is unstable, whereas the ABAM method is still able to solve the problem.

## 5. Conclusion

We have investigated the performance of the forward in time centred in space (FTCS) and Adams-Bashforth--Adams-Moulton (ABAM) methods in solving problems of water waves generated by moving topography. Both methods are in the finite difference family. We obtain that the ABAM method gives more flexibility in terms of the time step choice. The FTCS method is still able to solve the problems, but the time step should be taken smaller than that for the ABAM method.

## Acknowledgements

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