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Numerical solutions to a parabolic model of two-layer fluids

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Abstract. In this paper, we consider the motion of two layers of fluids having different viscosity values. The motion is driven by a moving surface, but the bottom is fixed. An exact analytical solution for unsteady state cases is not available. Therefore, a numerical method should be used for the solution to unsteady state cases of the problem. In this work, we propose a finite volume numerical method to find the numerical velocity of the problem. We use the Lax-Friedrichs formulation for the flux calculation. Our numerical results show that fluids move following the motion of the surface. In addition, the fluid at the top layer moves faster than the bottom layer fluid. These behaviour is correct with respect to the physical problem under consideration.

1. Introduction

Numerical simulation has been widely used by mathematical modellers to investigate physical phenomena. The phenomena is first modelled into a system of equations. The system is then solved numerically, as the exact analytical solution is generally difficult to find.

This paper considers the problem of two layers of fluids moving in one direction. We assume that oil layer is on the top of water layer. The two layers are in between a moving surface and a fixed bottom. The moving surface has a constant velocity. This problem was introduced by Caldwell and Ng [1], who used a finite difference method to obtain its numerical solution.

In this work, we provide an alternative numerical solver by proposing a finite volume method to find the numerical solution to the problem. The finite volume method works well as long as the mathematical model can be written in a conservative form. We implement the Lax-Friedrichs formulation [2-3] to compute numerical fluxes of the conservative form of the model. Our numerical scheme is explicit. As time tends to infinity, the motion of fluids is steady. This is confirmed by the exact as well as the numerical solution. The numerical solution agrees quite well with the exact steady state solution.

The rest of this paper is organised as follows. We present the problem formulation in Section 2. Numerical method is proposed in Section 3. We provide numerical results in Section 4. The paper is concluded in Section 5.

2. Problem formulation

This section introduces the equations for the motion of the two-layer fluids.

Consider two horizontal plates with a distance of 10 cm and between those two plates there are oil and water layers, as shown in Figure 1. The top plate moves to the right with a constant velocity. The bottom plate is fixed. We assume that starting at time $t = 0$, the top plate moves with constant velocity 7 cm/s.



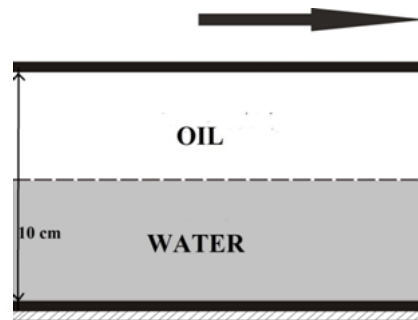


Figure 1. Two plates spaced 10 cm apart.

This problem has been modelled mathematically by Caldwell and Ng [1] as:

$$\frac{\partial v_{\text{water}}}{\partial t} = \mu_{\text{water}} \frac{\partial^2 v_{\text{water}}}{\partial x^2}, \quad (1)$$

and

$$\frac{\partial v_{\text{oil}}}{\partial t} = \mu_{\text{oil}} \frac{\partial^2 v_{\text{oil}}}{\partial x^2}. \quad (2)$$

Here $v_{\text{water}}(x, t)$ is the velocity of water, $v_{\text{oil}}(x, t)$ is the velocity of oil, μ_{water} is the viscosity of water, and μ_{oil} is the viscosity of oil. The free variables are time t and space x . The space x is the fluid height measured from the bottom. At the oil-water interface, we have the following relations:

$$v_{\text{oil}} = v_{\text{water}}, \quad (3)$$

and

$$\mu_{\text{oil}} \frac{\partial v_{\text{oil}}}{\partial x} = \mu_{\text{water}} \frac{\partial v_{\text{water}}}{\partial x}. \quad (4)$$

The simulation is conducted to determine the velocity of fluids at any time $t > 0$, especially for a large time t , when the steady state is achieved.

3. Numerical method

In this section we present a complete mathematical model and the finite volume method that we propose to solve the problem.

The mathematical model is given in the following two sets of initial-boundary value problems:

$$\left\{ \begin{array}{l} \frac{\partial v_{\text{water}}}{\partial t} = \mu_{\text{water}} \frac{\partial^2 v_{\text{water}}}{\partial x^2}, 0 \leq x \leq 6 \\ v_{\text{water}}(x, 0) = 0 \\ v_{\text{water}}(0, t) = 0 \\ v_{\text{water}}(6, t) = v_{\text{oil}}(6, t) \\ \mu_{\text{oil}} \frac{\partial v_{\text{oil}}}{\partial x} \Big|_{x=6} = \mu_{\text{water}} \frac{\partial v_{\text{water}}}{\partial x} \Big|_{x=6} \end{array} \right. \quad (5)$$

$$\left\{ \begin{array}{l} \frac{\partial v_{\text{oil}}}{\partial t} = \mu_{\text{oil}} \frac{\partial^2 v_{\text{oil}}}{\partial x^2}, 6 \leq x \leq 10 \\ v_{\text{oil}}(x, 0) = 0 \\ v_{\text{oil}}(10, t) = 7 \\ v_{\text{oil}}(6, t) = v_{\text{water}}(6, t) \\ \mu_{\text{oil}} \frac{\partial v_{\text{oil}}}{\partial x} \Big|_{x=6} = \mu_{\text{water}} \frac{\partial v_{\text{water}}}{\partial x} \Big|_{x=6} \end{array} \right. \quad (6)$$

The problem (5) and (6) can be solved separately, because there are some conditions that relate to each other.

In order to use the finite volume method, we rewrite equations (5) and (6) as:

$$\frac{\partial v_{\text{water}}}{\partial t} - \mu_{\text{water}} \frac{\partial^2 v_{\text{water}}}{\partial x^2} = 0, \quad (7)$$

$$\frac{\partial v_{\text{oil}}}{\partial t} - \mu_{\text{oil}} \frac{\partial^2 v_{\text{oil}}}{\partial x^2} = 0. \quad (8)$$

Therefore, we have

$$\frac{\partial v_{\text{water}}}{\partial t} - \frac{\partial}{\partial x} \left(\mu_{\text{water}} \frac{\partial v_{\text{water}}}{\partial x} \right) = 0, \quad (9)$$

$$\frac{\partial v_{\text{oil}}}{\partial t} - \frac{\partial}{\partial x} \left(\mu_{\text{oil}} \frac{\partial v_{\text{oil}}}{\partial x} \right) = 0. \quad (10)$$

In compact forms, we write equations (9) and (10) as

$$v_{\text{water}_t} + (-\mu_{\text{water}} v_{\text{water}_x})_x = 0, \quad (11)$$

$$v_{\text{oil}_t} + (-\mu_{\text{oil}} v_{\text{oil}_x})_x = 0. \quad (12)$$

Equations (11) and (12) are in the form of conservation laws, that is, in their conservative forms:

$$v_{\text{water}_t} + f(v_{\text{water}})_x = 0, \quad (13)$$

$$v_{\text{oil}_t} + f(v_{\text{oil}})_x = 0. \quad (14)$$

where $f(v_{\text{water}}) = -\mu_{\text{water}} v_{\text{water}_x}$ and $f(v_{\text{oil}}) = -\mu_{\text{oil}} v_{\text{oil}_x}$ respectively.

Equation (13) is solved using the finite volume method with an explicit numerical scheme:

$$V_{\text{water}_i}^{n+1} = V_{\text{water}_i}^n - \frac{\Delta t}{\Delta x} \left[F_{\text{water}_{i+\frac{1}{2}}}^n - F_{\text{water}_{i-\frac{1}{2}}}^n \right]. \quad (15)$$

Here,

$$V_{\text{water}_i}^n \approx v_{\text{water}}(x_i, t^n) \quad \text{and} \quad F_{\text{water}_{i+1/2}}^n \approx f(v_{\text{water}}(x_{i+1/2}, t^n))$$

are the conserved quantity and the flux function, respectively. Here Δt is time step and Δx is space step, i is notation for the space index and n is notation for the time index. To compute fluxes in equation (15), we use the Lax-Friedrichs formulation:

$$F_{\text{water}_{1-\frac{1}{2}}}^n = \frac{F_{\text{water}_i}^n - F_{\text{water}_{i-1}}^n}{2} - \frac{\Delta x}{2\Delta t} (V_{\text{water}_i}^n - V_{\text{water}_{i-1}}^n), \quad (16)$$

and

$$F_{\text{water}_{1+\frac{1}{2}}}^n = \frac{F_{\text{water}_{i+1}}^n - F_{\text{water}_i}^n}{2} - \frac{\Delta x}{2\Delta t} (V_{\text{water}_{i+1}}^n - V_{\text{water}_i}^n). \quad (17)$$

Analogously, equation (14) is solved using the finite volume method with an explicit scheme:

$$V_{\text{oil}_i}^{n+1} = V_{\text{oil}_i}^n - \frac{\Delta t}{\Delta x} \left[F_{\text{oil}_{i+\frac{1}{2}}}^n - F_{\text{oil}_{i-\frac{1}{2}}}^n \right]. \quad (18)$$

To compute fluxes in equation (18), we use the Lax-Friedrichs formulation:

$$F_{\text{oil}_{1-\frac{1}{2}}}^n = \frac{F_{\text{oil}_i}^n - F_{\text{oil}_{i-1}}^n}{2} - \frac{\Delta x}{2\Delta t} (V_{\text{oil}_i}^n - V_{\text{oil}_{i-1}}^n), \quad (19)$$

and

$$F_{\text{oil}_{1+\frac{1}{2}}}^n = \frac{F_{\text{oil}_{i+1}}^n - F_{\text{oil}_i}^n}{2} - \frac{\Delta x}{2\Delta t} (V_{\text{oil}_{i+1}}^n - V_{\text{oil}_i}^n). \quad (20)$$

The finite volume schemes (15)-(17) and (18)-(20) are iterated with consideration of the initial and boundary conditions given in equations (5)-(6). The iterations result in numerical solutions for the fluid velocities. Note that another approach of finite volume method is available, for example, the relaxation system [4-6], but it doubles the number of equations to be solved. Our finite volume method in this paper is simpler than those relaxation approach.

4. Numerical results

In this section we present some representatives of our numerical results. The numerical results are compared with a known analytical exact solution.

As the benchmark, we recall the analytical exact solution obtained by Caldwell and Ng [1] as follows. First considering

$$\frac{d^2 v_{\text{water}}}{dx^2} = 0, \quad 0 \leq x \leq 6, \quad (21)$$

with

$$v_{\text{water}}(0) = 0, \quad (22)$$

we get

$$v_{\text{water}}(x) = Ax + B. \quad (23)$$

Because $v_{\text{water}}(0) = 0$, then $B = 0$. Equation (23) becomes:

$$v_{\text{water}}(x) = Ax, \quad 0 \leq x \leq 6. \quad (24)$$

The steady state case for oil requires

$$\frac{d^2 v_{\text{oil}}}{dx^2} = 0, \quad 6 \leq x \leq 10, \quad (25)$$

with

$$v_{\text{oil}}(10) = 7, \quad (26)$$

$$v_{\text{oil}}(6) = v_{\text{water}}(6), \quad (27)$$

$$\mu_{\text{oil}} \left. \frac{\partial v_{\text{oil}}}{\partial x} \right|_{x=6} = \mu_{\text{water}} \left. \frac{\partial v_{\text{water}}}{\partial x} \right|_{x=6}. \quad (28)$$

Then we get:

$$v_{\text{oil}}(x) = Cx + D, \quad 6 \leq x \leq 10. \quad (29)$$

The conditions (26) and (27) lead to

$$10C + D = 7, \quad (30)$$

$$6C + D = 6A. \quad (31)$$

Because $\frac{\partial v_{\text{oil}}}{\partial x} = C$ and $\frac{\partial v_{\text{water}}}{\partial x} = A$, we write

$$\mu_{\text{oil}}(C) = \mu_{\text{water}}(A). \quad (32)$$

Equation (30)-(32) can be solved to obtain:

$$A = \frac{7\mu_{\text{oil}}}{6\mu_{\text{oil}} + 4\mu_{\text{water}}} \quad (33)$$

$$C = \frac{7\mu_{\text{water}}}{6\mu_{\text{oil}} + 4\mu_{\text{water}}} \quad (34)$$

$$D = \frac{21(\mu_{\text{oil}} - \mu_{\text{water}})}{3\mu_{\text{oil}} + 2\mu_{\text{water}}} \quad (35)$$

Finally, the exact steady state solutions are:

$$v_{\text{water}}(x) = \frac{7\mu_{\text{oil}}}{6\mu_{\text{oil}} + 4\mu_{\text{water}}} x, \quad 0 \leq x \leq 6, \quad (36)$$

$$v_{\text{oil}}(x) = \frac{7\mu_{\text{water}}}{6\mu_{\text{oil}} + 4\mu_{\text{water}}} x + \frac{21(\mu_{\text{oil}} - \mu_{\text{water}})}{3\mu_{\text{oil}} + 2\mu_{\text{water}}}, \quad 6 \leq x \leq 10. \quad (37)$$

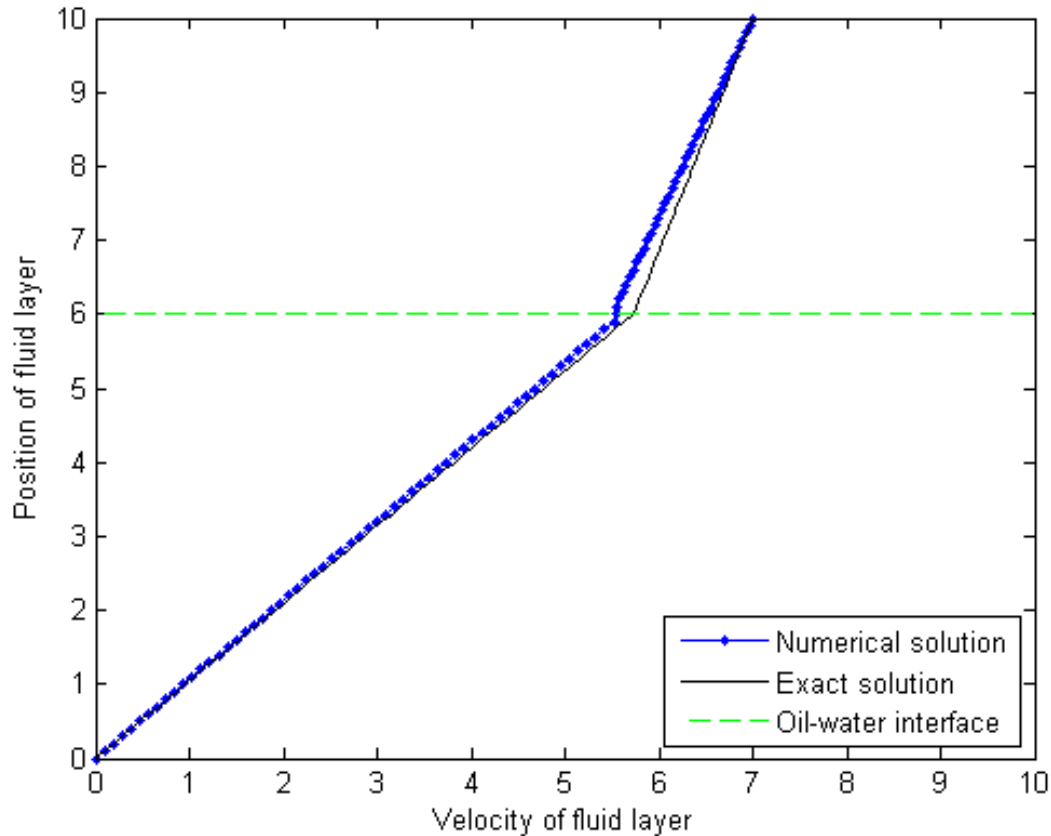


Figure 2. Large time behaviour of velocity of two fluid layers, at time $t = 100$ s.

Numerical setting in this simulation is as follows. We assume that $\mu_{\text{water}} = 1$ cp and $\mu_{\text{oil}} = 3$ cp. The cell width or also known as the space step is $\Delta x = 0.1$. The time step is $\Delta t = 0.001 \cdot \Delta x$. The simulation is stop at the final time $t = 100$ s. Here we want to see that change of velocity of two-fluid layers from $x = 0$ to $x = 10$ when the steady state is achieved. Note that the x -axis is vertical, instead of horizontal.

Our numerical results show that the numerical method solves the problem successfully. The main source of numerical error is at the non-smooth transition between the oil and water layers, that is, the oil-water interface. At this interface, the fluid viscosity is discontinuous. However, we can have smaller error if we take smaller cell width and smaller time step. A representative of our numerical results is shown in Figure 2. The numerical results is correct physically. The velocity values get larger from the bottom to the top surface. This phenomena is identified in both the analytical and numerical solutions.

5. Conclusion

We have proposed a finite volume numerical method to solve the problem of two layers of fluids driven by a horizontally moving surface with fixed bottom. The numerical method can be used to solve both steady and unsteady state problems. Accurate solution can be obtained using a fine numerical mesh, that is, fine cell width and fine time step. Our research results are limited to one-dimensional problems. Future research direction could be an extension to higher dimensional problems.

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