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Analytic Solution of a Stochastic Richards Equation driven by Brownian motion

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Abstract. In this paper we present a new result on an analytical study of stochastic logistic equations of Richards-type for population growth. First, we introduce the Richards equation as a generalization of the classical Verhulst equation which allows a flexibility in the sigmoid shape of the solution curve. Next, the stochastic equation is formed by adding a multiplicative white noise in the corresponding deterministic Richards equation. Our goal is to solve the stochastic Richards equation and investigate some of the qualitative properties of the solution. As a main result, an exact expression for the solution of the stochastic Richards equation is obtained by using tools from the Itô calculus. Some qualitative aspects of the solution, such as long time behavior and noise-induced transition, will be also discussed within the framework of diffusion processes theory. We also give a simulation of the solution of the stochastic Richards equation to illustrate the role of the so-called allometric parameter.

1. Introduction

One of the most famous model in population dynamics is the logistic equation which was first proposed by P F Verhulst in 1838. It has been extensively studied due to its theoretical and practical significance. The logistic equation, also known as the Verhulst equation, is given by the ordinary differential equation

$dN(t) = rN(t)\left(1 - \dfrac{N(t)}{K}\right)dt$, where $N(t)$ is the population size (number of individuals in the

population) at time t, r is the intrinsic growth rate, and $K > 0$ is the carrying capacity/saturation level.

This equation is also sometimes called the Verhulst-Pearl equation following its rediscovery by R Pearl in 1920's. See references [1–3], among others, for further information on the subject of mathematical population dynamics. In 1959, F J Richard in [4] proposed the following modification of the logistic equation to model growth of biological populations:

$$dN(t) = rN(t)\left(1 - \left(\frac{N(t)}{K}\right)^{\alpha}\right)dt \tag{1}$$

with the initial condition $N(0) = N_0$. Here we assume that $0 < N_0 < K$. The allometric parameter $\alpha > 0$ provides a measure of flexibility in the curvature of the sigmoid shape exhibited by the resulting solution curve. In other words, α is the exponent of deviation from the standard logistic curve. The Richards equation is interesting for modeling purposes, as it allows the modeler to adjust the value of α to be as close as possible to its observations. Note that (1) is a Bernoulli equation of order $\alpha + 1$ and the solution is given by

$$N(t) = K \left(1 + \left(\left(\frac{K}{N_0} \right)^\alpha - 1 \right) \exp(-\alpha r t) \right)^{-\frac{1}{\alpha}} \tag{2}$$

It is obvious that for the case $\alpha = 1$ (2) reduces to the solution of the Verhulst equation. The equation (1) possesses two equilibrium solutions: $N(t) = 0$ which is unstable and $N(t) = K$ which is asymptotically stable. The long time behavior of the solution is given by $\lim_{t \to \infty} N(t) = K$. For a more comprehensive study on (deterministic) logistic-type models including Verhulst and Richards equations, we refer to [5] and references therein. The aim of the present paper is to solve a stochastic version of equation (1) and to study some of the qualitative properties of the solution of (1).

2. Stochastic Richards Equations

Population dynamics is better understood when we incorporate the influence of the uncertainty/random factors which cannot be determined in advance, such as natural disasters (flood, fire, volcano eruption, earthquake, and so on), diseases, predation, hunting, and many others. A mathematical idealization for these external random perturbations is the so-called white noise. In the applied science white noise is commonly taken as the time derivative of the Brownian motion. It is natural to assume that the strength of the noise is proportional to the population size. Hence, we construct the stochastic Richards model by inserting the multiplicative noise term in the deterministic model (2) to obtain a randomized equation

$$dN(t) = rN(t) \left(1 - \left(\frac{N(t)}{K} \right)^\alpha \right) dt + \sigma \, noise \, N(t) dt.$$

In the following we shall write N_t instead of $N(t)$ to emphasize that N_t is, in general, no longer a deterministic function but a random variable. Now, we view the noise as the Gaussian white noise which is $\frac{dB_t}{dt}$ and we get the stochastic differential equation in the Itô sense

$$dN_t = rN_t \left(1 - \left(\frac{N_t}{K} \right)^\alpha \right) dt + \sigma N_t dB_t, \tag{3}$$

where σ is the diffusion coefficient which measures the size of the fluctuation of the noise. A stochastic Richards equation, which substantially differs from our equation (3), has been introduced and studied in [6–8]. The equation considered in those works contains quadratic random part and, hence, cannot be solved explicitly within the Itô interpretation. Thus, the focus of study was on the qualitative behavior such as existence, uniqueness, and stability of the solution.

To solve stochastic differential equation (3) we need some results from Itô's stochastic calculus which will be summarized below. For more details and proof see for example [9,10]. First, recall that a stochastic process $(X_t)_{t \geq 0}$ is called *adapted* if there exists a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ such that for each $t \geq 0$ the random variable X_t is defined on $(\Omega, \mathcal{F}, \mathbb{P})$ as well as $\mathcal{F}_t$-measurable. In the rest of paper by $\mathbb{E}$ we denote the expectation with respect to the probability measure $\mathbb{P}$. An adapted stochastic process $(X_t)_{t \in [0,T]}$ is called an Itô process if it can be written in the form

$$X_t = X_0 + \int_0^t b(s) ds + \int_0^t \sigma(s) dB_s, \quad X_0 = x_0, \, t \in [0,T]$$

where b and σ are adapted stochastic processes satisfying $\int_0^T |b(s)| ds < \infty$ and $\int_0^T \sigma(s)^2 ds < \infty$. The covariation of two Itô processes

$$X_t = x_0 + \int_0^t b(s)\,ds + \int_0^t \sigma(s)\,dB_s \ \text{ and } \ Y_t = y_0 + \int_0^t d(s)\,ds + \int_0^t \rho(s)\,dB_s$$

is the stochastic process $\left(\langle X, Y \rangle_t\right)_{t \in [0,T]}$ where $\langle X, Y \rangle_t := \int_0^t \sigma(s)\rho(s)\,ds$. The process $\langle X \rangle_t := \langle X, X \rangle_t$ is

called the quadratic variation of X.

Theorem 1 [9,10]. *If* $\left(X_t\right)_{t \in [0,T]}$ *is an Itô process and* $F \in C^{1,2}([0,T] \times \mathbb{R})$, *then*

$$F(T, X_T) - F(0, X_0) = \int_0^T \frac{\partial F}{\partial x}(t, X_t)\,dX_t + \int_0^T \frac{\partial F}{\partial t}(t, X_t)\,dX_t + \frac{1}{2}\int_0^T \frac{\partial^2 F}{\partial x^2}(t, X_t)\,d\langle X \rangle_t.$$

Let $I = [0,T]$ or $I = [0,\infty)$. Consider a stochastic differential equation

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dB_t, \ X_0 = x_0, \ t \in I \tag{4}$$

or in the integral form

$$X_t = x_0 + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dB_s, \ t \in I. \tag{5}$$

The first integral is the usual Lebesgue integral while the second is the Itô integral. A continuous stochastic process $\left(X_t\right)_{t \in I}$ is called a solution of the equation (4) in the interval I if, for all $t \in I$, it satisfies (5) with probability one. A linear stochastic differential equation is an equation of the form

$$dX_t = \left(a_1(t)X_t + a_2(t)\right)dt + \left(b_1(t)X_t + b_2(t)\right)dB_t, \ X_0 = x_0, \tag{6}$$

where a_i and $b_i\,(i=1,2)$ are deterministic function, bounded on every finite interval $[0,T]$. If a_i and $b_i\,(i=1,2)$ are constants, then (6) is called autonomous; if $a_i = b_i = 0$ then it is called homogeneous.

Theorem 2 [9]. *The stochastic process*

$$X_t = \Phi_t\left(x_0 + \int_0^t (a_2(s) - b_1(s)b_2(s))\Phi_s^{-1}\,ds + \int_0^t b_2(s)\Phi_s^{-1}\,dB_s\right), \ t \geq 0,$$

where $\Phi_t := exp\left(\int_0^t \left(a_1(s) - \frac{1}{2}b_1^2(s)\right)ds + \int_0^t b_1(s)\,dB_s\right)$ *is the solution of the linear stochastic differential equation (6).*

3. Results and Discussion

This section contains the main results of this paper. The first result is the derivation of the exact solution of the stochastic Richards equation.

Theorem 3. *The solution of the stochastic Richards equation (3) is given by*

$$N_t = N_0 exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right)\left(1 + \left(\frac{N_0}{K}\right)^\alpha r\alpha \int_0^t exp\left(\alpha\left(\left(r - \frac{1}{2}\sigma^2\right)s + \sigma B_s\right)\right)ds\right)^{-\frac{1}{\alpha}}.$$

Proof: Let $X_t := \frac{N_t}{K}$. Then, we obtain $dX_t = rX_t\left(1 - X_t^\alpha\right)dt + \sigma X_t\,dB_t$ with the initial condition

$X_0 = \frac{N_0}{K}$. Let $F(t,x) := x^{-\alpha}$ and, hence $\frac{\partial F}{\partial x} = -\frac{\alpha}{x^{\alpha+1}}, \frac{\partial^2 F}{\partial x^2} = \frac{\alpha(\alpha+1)}{x^{\alpha+2}}$, and $\frac{\partial F}{\partial t} = 0$. The quadratic

variation of X_t is given by $\langle X \rangle_t = \int_0^t \sigma^2 X_s^2 ds$ which means $d\langle X \rangle_t = \sigma_t^2 X_t^2 dt$. By using the notation

$Y_t := X_t^{-\alpha}$ and applying Itô formula (Theorem 1) we obtain

$$dY_t = -\frac{\alpha}{X_t^{\alpha+1}} dX_t + \frac{1}{2} \frac{\alpha(\alpha+1)}{X_t^{\alpha+2}} dX_t = \left(\left(\frac{1}{2}\alpha(\alpha+1)\sigma^2 - r\alpha \right) Y_t + r\alpha \right) dt - \alpha\sigma Y_t dB_t.$$

This is a linear stochastic differential equation with integrating factor

$\Phi_t = \exp\left(\left(\frac{1}{2}\alpha\sigma^2 - r\alpha \right) t - \alpha\sigma B_t \right)$. Therefore, by Theorem 2, the solution of the linear stochastic

differential equation in Y_t is

$$Y_t = \exp\left(\alpha \left(\left(\frac{1}{2}\sigma^2 - r \right) t - \sigma B_t \right) \right) \left(Y_0 + r\alpha \int_0^t \exp\left(\alpha \left(\left(r - \frac{1}{2}\sigma^2 \right) s + \sigma B_s \right) \right) ds \right).$$

Rewriting the last expression in term of X_t yields

$$X_t = \exp\left(\left(r - \frac{1}{2}\sigma^2 \right) t + \sigma B_t \right) \left(\left(\frac{1}{X_0} \right)^\alpha + r\alpha \int_0^t \exp\left(\alpha \left(\left(r - \frac{1}{2}\sigma^2 \right) s + \sigma B_s \right) \right) ds \right)^{-\frac{1}{\alpha}}$$

Finally, the solution of (3) is given by

$$N_t = N_0 \exp\left(\left(r - \frac{1}{2}\sigma^2 \right) t + \sigma B_t \right) \left(1 + \left(\frac{N_0}{K} \right)^\alpha r\alpha \int_0^t \exp\left(\alpha \left(\left(r - \frac{1}{2}\sigma^2 \right) s + \sigma B_s \right) \right) ds \right)^{-\frac{1}{\alpha}} \quad (7)$$

The proof is complete. ∎

Note that if we let $\sigma = 0$ in (7), then it is easy to see that (7) reduces (2). This fact means that when the random influence in the stochastic Richards model is very small, it is negligible and it suffices to work with the deterministic model.

Now we want to compute the approximate mean and approximate variance of the solution process

$(N_t)_{t \geq 0}$. Note that N_t in (7) can be expressed as $N_t = N_0 \left(\Phi_t^{-1} \right)^{\frac{1}{\alpha}} \left(1 + \left(\frac{N_0}{K} \right)^\alpha r\alpha \int_0^t \Phi_s^{-1} ds \right)^{-\frac{1}{\alpha}}$. Since

the randomness of the N_t is encoded in the term Φ^{-1}, it is quite natural to approximate the mean of N_t

by the one of Φ^{-1}. Since the random variable B_t is normally distributed with mean zero and variance

t, we have

$$\mathbb{E}\left(\Phi_t^{-1} \right) = \mathbb{E}\left(\exp\left(\alpha \left(\left(r - \frac{1}{2}\sigma^2 \right) t + \sigma B_t \right) \right) \right) = \left(\exp\left(\left(r - \frac{1}{2}\sigma^2 \right) t \right) \exp\left(\frac{1}{2}\sigma^2 \mathbb{E}\left(B_t^2 \right) \right) \right)^\alpha = \exp(\alpha r t).$$

Hence,

$$\mathbb{E}(N_t) \approx N_0 \left(\frac{\mathbb{E}\left(\Phi_t^{-1} \right)}{1 + \left(\frac{N_0}{K} \right)^\alpha r\alpha \int_0^t \mathbb{E}(\Phi_s^{-1}) ds} \right)^{\frac{1}{\alpha}} = N_0 \left(\frac{\exp(\alpha r t)}{1 + \left(\frac{N_0}{K} \right)^\alpha r\alpha \int_0^t \exp(\alpha r t) ds} \right)^{\frac{1}{\alpha}}$$

$$= N_0\left(\left(\frac{N_0}{K}\right)^{\alpha}+\left(1-\left(\frac{N_0}{K}\right)^{\alpha}\right)\exp(-\alpha rt)\right)^{-\frac{1}{\alpha}}=K\left(1+\exp(-\alpha rt)\left(\left(\frac{K}{N_0}\right)^2-1\right)\right)^{-\frac{1}{\alpha}}.$$

Then, the approximate long time behavior of the solution is described by $\lim_{t\to\infty}\mathbb{E}(N_t)\approx K$. This means that, in average, the population size converges near to the carrying capacity. By a similar argument, we compute the variance of N_t. Note that

$$\mathbb{E}\left(\left(\Phi_t^{-1}\right)^2\right)=\mathbb{E}\left(\exp\left(\alpha\left(\left(2r-\sigma^2\right)t+2\sigma B_t\right)\right)\right)=\left(\exp\left(\left(2r-\sigma^2\right)t\right)\exp\left(\frac{1}{2}\mathbb{E}\left(\left(2\sigma B_t\right)^2\right)\right)\right)^{\alpha}$$

$$=\left(\exp\left(\left(2r-\sigma^2\right)t\right)\exp\left(2\sigma^2\mathbb{E}\left(B_t^2\right)\right)\right)^{\alpha}=\exp\left(\alpha\left(2r+\sigma^2\right)t\right).$$

As a consequence

$$\mathrm{Var}\left(\Phi_t^{-1}\right)=\exp\left(\alpha\left(2r+\sigma^2\right)t\right)-\exp\left(2\alpha rt\right)=\exp(2\alpha rt)\left(\exp\left(\alpha c^2 t\right)-1\right).$$

Therefore,

$$\mathrm{Var}(N_t)\approx N_0\left(\frac{\mathrm{Var}\left(\Phi_t^{-1}\right)}{1+\left(\frac{N_0}{K}\right)^{\alpha}r\alpha\int_0^t\mathrm{Var}(\Phi_s^{-1})ds}\right)^{\frac{1}{\alpha}}=N_0\left(\frac{\exp(2\alpha rt)(\exp\left(\alpha c^2 t\right)-1)}{1+\left(\frac{N_0}{K}\right)^{\alpha}r\alpha\int_0^t\exp(2\alpha rt)(\exp\left(\alpha c^2 t\right)-1)ds}\right)^{\frac{1}{\alpha}}$$

$$=K\left(\frac{\exp\left(\alpha\sigma^2 t\right)-1}{\exp(-2\alpha rt)\left(\left(\frac{K}{N_0}\right)^{\alpha}-\frac{r}{2r+\sigma^2}+\frac{1}{2}\right)+\left(\frac{r}{2r+\sigma^2}\exp\left(\alpha\sigma^2 t\right)-\frac{1}{2}\right)}\right)^{\frac{1}{\alpha}}.$$

In the following we simulate the solution of the stochastic Richards equation by using the Euler-Maruyama scheme in MAPLE. The goal is to compare the behavior of the stochastic Richards equations with different allometric parameters. We set $N_0=20$, $r=0.5$, $K=400$, and $\sigma=0.01$. The horizontal axis represents time (t) and the vertical axis represents the size of population (N_t).
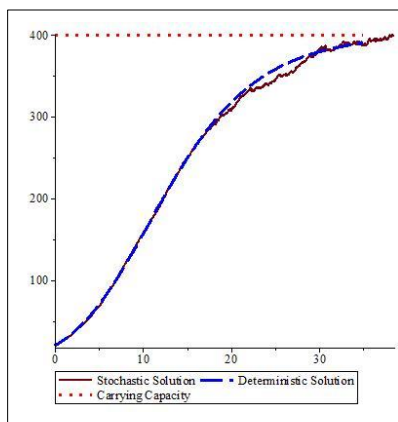
Figure 1. Simulation of the solution of the stochastic Richards equation with $\alpha=0.3$
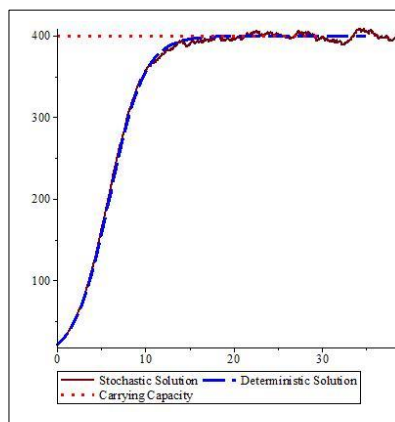
Figure 2. Simulation of the solution of the stochastic Richards equation with $\alpha=1$
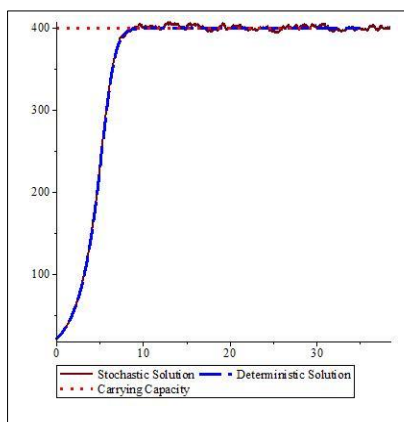
Figure 3. Simulation of the solution of the stochastic Richards equation with $\alpha=3$

In the simulations above we see that the stochastic solutions fluctuate around the deterministic solutions. It is also possible for stochastic solutions to have values above the carrying capacity due to the incessant and irregular movement of Brownian motion. Of course, the value of σ should not too big, otherwise the stochastic solutions will be very wild and deviate largely from the deterministic solutions. As we can see above Richards curve for $\alpha = 3$ is steeper than Verhulst curve (Richards curve for $\alpha = 1$) which is in turn steeper than Richards curve for $\alpha = 0.3$. Hence, the values of α determine the shape and steepness of the solutions curve (both in the case of deterministic and stochastic) as well as the speed of convergence to the equilibrium solution $N_t = K$.

Next, we shall discuss some qualitative aspects of the solution of the stochastic Richards equation. We recall some basic notions from the diffusion processes theory and refer to [11,12] for details. A solution X of the stochastic differential equation (4) is called a diffusion process. The coefficients b and σ are called the drift and diffusion coefficient of process X, respectively. If the b and σ do not depend on t, X is called a (time)-homogeneous diffusion process. In order to emphasize the dependence of X on the initial point x, we denote it X^x, and by a diffusion process X we mean the whole family of solutions $\{X^x, x \in \mathbb{R}\}$. Any diffusion process X is a Markov process and its transition density function will be denoted by $p(t, x, y)$. The generator of a homogeneous diffusion process $X = \{X^x\}$ is the operator A defined by $Af(x) = \lim_{t \to 0} \dfrac{\mathbb{E}f(X_t^x) - f(x)}{t}$, $x \in \mathbb{R}$ on a set of real-valued function f.

A stationary density of a diffusion process X with transition density $p(t, x, y)$, $t > 0, x, y \in \mathbb{R}$ is a density function $p_0(y), y \in \mathbb{R}$, satisfying the equation $p_0(y) = \int_{\mathbb{R}} p(t, x, y) p_0(x) dx, t > 0, x \in \mathbb{R}$. In fact, under rather general conditions, the stationary density of a diffusion process has a certain attraction property: the distribution of a diffusion process X_t^x starting from any initial point x eventually stabilized in the sense that the density of X_t^x becomes close to the stationary one. Precisely speaking, as $t \to \infty$ we have $p(t, x, y) \to p_0(y), x, y \in \mathbb{R}$. The sample paths of a diffusion process X may have a certain closedness with respect to some interval $(a, b) \subset \mathbb{R}$: starting at any point $x \in (a, b)$, it stays in (a, b) forever. Then, a and b are called *unattainable boundaries* of X. For the solution (7) of the Richards equation (3) such interval is $(0, \infty)$. In such a situation, it makes sense to consider the stationary density in the interval (a, b).

A diffusion process X with transition density $p(t, x, y)$ is said to have a stationary density p_0 in the interval (a, b) if $\int_a^b p(t, x, y) dy = 1$, $t > 0, x \in (a, b)$ and $p_0(y) = \int_a^b p(t, x, y) p_0(x) dx, t > 0, y \in (a, b)$.

There are two types of unattainable boundaries of diffusion process. In the first case $X_t^x \to a$, or $X_t^x \to b$ as $t \to \infty$; then a and b is called an *attracting boundary* of X. In this case, X has no stationary density in the interval (a, b) since the limit distribution of the process is concentrated at point a or b. In the other case, the process X^x, when started at any point $x \in (a, b)$, infinitely often visits all points of the interval (a, b); then a and b are called the *natural boundaries* of X.

Proposition 4 [11,12]**.** Let X be a diffusion process with generator A and transition density $p = p(t, x, y)$ which has a stationary density $p_0 = p_0(y)$ in the interval (a, b). Suppose that p and p_0

are continuous functions having continuous partial derivatives $\dfrac{\partial p}{\partial t}, \dfrac{\partial p}{\partial y}, \dfrac{\partial^2 p}{\partial y^2}, \dfrac{\partial p_0}{\partial y}, and \dfrac{\partial^2 p_0}{\partial y^2}$. Suppose

also $\sigma(x) > 0$, $x \in (a,b)$. Then, the stationary density p_0 of X is of the form

$$p_0(y) = \frac{N}{\sigma^2(y)} \exp\left(2\int_k^y \frac{b(u)}{\sigma^2(u)} du \right), \ y \in (a,b)$$

where k is an arbitrary point from (a,b) and N is the normalizing constant such that $\int_a^b p_0(y) dy = 1$.

To analyze how the stationary density varies under a proportional increase in noise, we assume that the diffusion coefficient is proportional to some function, i.e. $\sigma(x) = \sigma g(x)$ (this includes the case of stochastic Richards equation). From (7) we see that if $N_0 > 0$, then also $N_t > 0$, for all $t > 0$. If $r < \dfrac{1}{2}\sigma^2$

, then $\left(r - \dfrac{1}{2}\sigma^2 \right) t + \sigma B_t = t\left(\left(r - \dfrac{1}{2}\sigma^2 \right) + \sigma \dfrac{B_t}{t} \right) \to -\infty$ as $t \to \infty$, since by the Brownian law of large

number, we have $\lim_{t \to \infty} \dfrac{B_t}{t} = 0$. Hence, $\lim_{t \to \infty} N_t = 0$. In other words, whenever the random influence is too

strong compared to the growth rate, then the population will vanish eventually. Thus, for $r < \dfrac{1}{2}\sigma^2$ the

point 0 is an attracting boundary and ∞ is a natural boundary. As a consequence, we can expect the

existence of stationary density p_0 in the interval $(0,\infty)$ only for $r \geq \dfrac{1}{2}\sigma^2$. By Proposition 4 it must be

of the form (choose $k = 1$)

$$p_0(x) = \frac{N}{\sigma^2 x^2} \exp\left(2\int_1^x \frac{ru\left(1 - \left(\frac{u}{K}\right)^\alpha\right)}{\sigma^2 u^2} \right) = \frac{N}{\sigma^2 x^2} \exp\left(2\int_1^x \left(\frac{r}{\sigma^2 u} - \frac{r}{\sigma^2 K} u^{\alpha-1} \right) du \right)$$

$$= \frac{N}{\sigma^2 x^2} \exp\left(\frac{2r}{\sigma^2}\left(\ln x + \frac{1}{\alpha K}(1 - x^\alpha) \right) \right) = \frac{N}{\sigma^2} \exp\left(\frac{2r}{\alpha K \sigma^2} \right) x^{\frac{2r}{\sigma^2} - 2} \exp\left(-\frac{2r}{\alpha K \sigma^2} x^\alpha \right).$$

The function p_0 is integrable on $(0,\infty)$ if and only if $\dfrac{2r}{\sigma^2} - 2 > -1$, that is, for $r > \dfrac{1}{2}\sigma^2$. Thus, a

noise-induced transition occurs whenever $r = \dfrac{1}{2}\sigma^2$. This means that the diffusion process described by

the Richards equation has no stationary density for $r \leq \dfrac{1}{2}\sigma^2$ and has a stationary density for $r > \dfrac{1}{2}\sigma^2$. In

terms of the generalized function theory we say that in the case $r \leq \dfrac{1}{2}\sigma^2$, the stationary density $p_0(x)$

is given by the Dirac delta function $\delta(x)$. Moreover, we have the following result on the limiting behavior of $p_0(x)$ as x approaches zero:

$$\lim_{x \to 0} p_0(x) = \begin{cases} \infty & for \ \dfrac{1}{2}\sigma^2 < r < \sigma^2 \\[2ex] \dfrac{N}{\sigma^2}\exp\left(\dfrac{2}{\alpha K}\right) & for \ \ r = \sigma^2 \\[2ex] 0 & for \ \ r > \sigma^2 \end{cases} \ .$$

We see that the behavior of the stationary density are different for $r < \sigma^2 < 2r$ and $\sigma^2 < r$. Thus, we can say that at $r = \sigma^2$ there is also a noise-induced transition.

4. Conclusion

We have solved a stochastic Richards equation analytically by using the Itô theory for stochastic differential equation. The simulation gives an insight on how the solution behaves according to the allometric parameter. Moreover, we analyze the qualitative aspects of the solution by using tools from diffusion processes theory. The provided analytic solution could be very useful for researchers for testing the existing and new numerical methods for the solution of stochastic differential equations. As a future work we will consider the study of the stochastic Richards equation driven by another stochastic process such as fractional Brownian motion or Levy process.

5. Acknowledgments

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