Journal of Physics: Conference Series

**PAPER** • **OPEN ACCESS**

# Max-plus algebraic modeling of three crossroad traffic queue systems with one underpass

To cite this article: R U Hurit and A Rudhito 2019 *J. Phys.: Conf. Ser.* **1307** 012013

IOP Conf. Series: Journal of Physics: Conf. Series **1307** (2019) 012013 doi:10.1088/1742-6596/1307/1/012013

# Max-plus algebraic modeling of three crossroad traffic queue systems with one underpass

**R U Hurit[1*], A Rudhito[2]**

[1]Masters Student of Mathematics Education Study Program, Sanata Dharma University, Yogyakarta, Indonesia
[2] Lecturer at Mathematics Education Study Program, Sanata Dharma University, Yogyakarta, Indonesia

*uronhurit@gmail.com

**Abstract**. This research aims to model traffic flow queuing system of three crossroads, namely the Monjali, Kentungan, and Gejayan crossroads in Yogyakarta with one underpass at the Kentungan crossroad using max-plus algebra. This research is based on data of traffic flow density, especially cars and traffic light duration data on the three crossroads. Then, a crossing condition scheme is created and represents the direction of car movement especially in the direction from Monjali to Gejayan. Then the model is compiled which includes the waiting time of cars when the red traffic light is on, the time cars leave the crossroad during the green traffic light is on, and the travel time of cars between crossroad with max-plus algebra, so that the total vehicle time can be determined from Monjali to the Gejayan crossroad . The results show that the queueing system can be modelled using max-plus algebra. Furthermore, it can be determined the waiting time, the time to leave crossroad, and the total time needed to cross from the Monjali crossroad to exit the Gejayan crossroad. The resulting model can reduce congestion on the lines of the intersection of Monjali, Kentungan and Gejayan.

## 1. Introduction

In this revolutionary era, the demands for transportation are increasing, there need to be new or alternative routes added. The significant increase in the volume of transportation equipment that is not comparable to the expansion of roads will cause transportation problems such as traffic jams, delays, or even road accidents. The alternative system is becoming increasingly important as a means of public transportation, which is expected to be able to overcome the occurring problems [1]. Transportation movement issues, especially congestion problems in urban areas, often occur at crossroads. An intersection is a meeting place for road sections and places where traffic conflicts occur, functioning as a place for vehicles to change traffic direction movements. Intersections are a very important part of road network, this is due to their influence on the movement and safety of vehicle traffic flows. One of the methods used to overcome this congestion is by building underpasses. Kentungan-Gejayan underpass is one of the alternative traffic systems to facilitate the density that occurs in the area.

Maxplus algebra (the set of all real numbers R equipped with max and plus operations) can be well-used to model and analyze algebraically network problems, such as problems: scheduling (project) and queuing systems, more details can be seen in [2]. The advantage of max-plus algebra is that the calculation only uses two operations, namely maximum operation and addition operation.

## 2. Max-plus algebra

Max-plus algebra is an algebraic structure on the set of real numbers comprising the sum of max-plus algebra and multiplication of max-plus algebra [2]. Max-Plus algebra is a set $\boldsymbol{R} \cup \{-\infty\}$ and **R** is a set of all real numbers equipped with maximum operation, denoted by $\oplus$ and addition operation denoted by $\otimes$. The general form of max-plus algebra with max-plus linear equation system $A \otimes x = b$ can be a reference in solving algebraic applications with regard to properties $A \otimes x = b$ in matrix and its vectors

[2]. In addition, the theory of graphs is also a tool in describing traffic light conditions into a directed graph. Graphs in max-plus algebra are discussed [3].

The following is discussed the basic of max-plus algebra concept and its relation to graph theory which will be used in the research discussion. More complete material can be seen in[4]. Given $\boldsymbol{R}_\varepsilon: = \boldsymbol{R}_\varepsilon \cup \{\varepsilon\}$ with R is the set of all real numbers and $\varepsilon := -\infty$. in $\boldsymbol{R}_\varepsilon$ the following operations are defined: $\forall a, b \in \boldsymbol{R}_\varepsilon a \oplus b := max(a, b)$ and $a \otimes b := a + b$.

$(R_\varepsilon, \oplus, \otimes)$ is an idempotent commutative semiring with a neutral element $\varepsilon = -\infty$ and a unit element $e = 0$, namely that $\forall a, b \in \boldsymbol{R}_\varepsilon$ applies:

i) $a \oplus b = b \oplus a, (a \oplus b) \oplus c = a \oplus (b \oplus c), a \oplus \varepsilon = a.$
ii) $(a \otimes b) \otimes c = a \otimes (b \otimes c), a \otimes e = a + 0 = a = 0 + a = e \otimes a.$
iii) $a \otimes \varepsilon = \varepsilon \otimes a$ .
iv) $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c), a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c).$
(v) $a \otimes b = b \otimes a \ dan \ a \oplus a = a.$

Furthermore $(\boldsymbol{R}_\varepsilon, \oplus, \otimes)$ is a semifield, namely that $(\boldsymbol{R}_\varepsilon, \oplus, \otimes)$ is a commutative semiring where for every $a \in \boldsymbol{R}$ there is $-a$ so that it applies $a \otimes (-a) = 0$. $\boldsymbol{R}_{max} := (\boldsymbol{R}_\varepsilon, \oplus, \otimes)$ is called max-plus algebra, which is then simply written as $\boldsymbol{R}_{max}$. In the case of an operation sequence (if the sign is not not written down), the operation $\otimes$ has a higher priority than the operation $\oplus$. Rank $k$ from element $x \in \boldsymbol{R}$ notated by $x^{\otimes k}$ is defined as follows: $x^{\otimes 0}$ : = 0 dan $x^{\otimes k} := x \otimes x^{\otimes k-1}$, and definition as $\varepsilon^{\otimes 0}$ : = 0 and $\varepsilon^{\otimes k} := \varepsilon$, to $k = 1, 2, ....$ Operations $\oplus$ and $\otimes$ on the $\boldsymbol{R}_{max}$ can be extended to inner matrix operations $\boldsymbol{R}_{max}^{mxn}$. Especially for matricesin $\boldsymbol{R}_{max}^{mxn}$ we define

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}$$

matrix $E \in \boldsymbol{R}_{max}^{nxn}, (E)_{i,j} := \begin{cases} 0, jika\ i = j \\ \varepsilon, jika\ i \neq j \end{cases}$ and matrix $\varepsilon\ \epsilon \boldsymbol{R}_{max}^{mxn}\ (\varepsilon)_{i,j} := \varepsilon \forall i, j. (\boldsymbol{R}_{max}^{mxn} \otimes, \oplus)$ is an idempotent semiring with a neutral matrix element $\mathcal{E}$ and matrix unit elements $E$. Rank $k$ from matrix $A \in \boldsymbol{R}_{max}^{nxn}$ in max-plus algebra is defined by: $A^{\otimes 0} = E_n$ dan $A^{\otimes k} = A \otimes A^{\otimes k-1}$ $k = 1, 2, \ldots$ . Due to the idempotent property of the operation $\oplus, \forall \in \boldsymbol{R}_{max}^{nxn}$ is valid, for

$$(\boldsymbol{E} \oplus \boldsymbol{A})^{\boldsymbol{q}} = \boldsymbol{E} \oplus \boldsymbol{A} \oplus \ldots \oplus \boldsymbol{A}^{\boldsymbol{q}}.$$

### *2.1. Max-plusalgebraic model of traffic lighttime*
Traffic lights configuration can be done by modeling traffic light time using max-plus algebra. In this article, it is discussed the timing of traffic lights at the intersection of Monjali, Kentungan and Gejayan. This roadway consists of 4 two-way lanes, each consisting of two lanes. The visualization of the intersection can be seen in figure 1.

### *2.2. Queue Network*
Observe a closed tandem network with n single servants [4]. The basic assumptions in this network are as follows:

1. Queue buffer capacity is infinite
2. The queue works with the First-in First-Out (FIFO) principle.
3. Transfer of customers from a queue to the next queue does not require time.
4. Customers must pass through the queue from start to finish in a row to receive the service of each waiter.


**Figure 1.** Kentungan intersection with underpass.

## 3. Research methods

In this process the researcher directly observes the place of research and searches for information needed in the research process until details are obtained so that the research process runs smoothly and gets the expected results [5]. The research method is based on literature studies and field studies, which include theoretical studies, field observations for real problems, mathematical modeling and analysis by conducting simulations [6]. The steps taken are conducting field observations on the object of the research, namely at the Kentungan intersection, monjali and gejayan, which uses existing traffic light. The chosen place is eastward towards the west of Gejayan. The choice of the place is to find the harmony of the data used in the study and to ensure that the data can be examined using the Max-Plus Algebraic tool [7].

Types of data collected include, among others, old time traffic lights from the east and west, number of cars passing, car density data,the distance between cars, car length and car track length. The method used in collecting research data is direct observation and recording to retrieve primary data from the research site, namely the duration of red, yellow, and green lights on each intersection [8]. From the observation process, researchers can carry out the analysis using the help of the max-plus algebra tool to obtain a queuing system model to get a good solution on the use of the Kentungan underpass.

## 4. Results and discussions

This research is located at the Kentungan, Monjali and Gejayan intersections in Yogyakarta. The Kentungan intersection is the path that connects between Monjali street and Gejayan street. Map of intersection can be seen in figure 2.

This study consists of several steps, namely the first step to retrieve data on the duration of green and red lights, as well as data on the length of the vehicle queue that are interrelated. Based on research in the field, data is obtained as in tables 1–4.

The steps of data analysis are as follows:

a. Research starts by drawing the traffic system under study in a form of a graph that explains the condition of the system. The graph is then determined by the system direction rules that apply from the beginning of the system to the end of the system. From the directed graph then the rules of synchronization are arranged in the traffic system to determine the characteristics and values of each current at the intersection which will be a reference in compiling the equation.

b. Search for equations based on the rules of max-plus algebra $\boldsymbol{x}\ (\boldsymbol{k}\ +\ \mathbf{1})\ =\ \boldsymbol{A}\otimes\boldsymbol{x}\ (\boldsymbol{k})$. The rules of applying max-plus algebra especially in the application of the queue system and scheduling have the form $\boldsymbol{x}\ (\boldsymbol{k}\ +\ \mathbf{1})\ =\ \boldsymbol{A}\otimes\ \boldsymbol{x}\ (\boldsymbol{k})$. The system is a reference for adjusting the intersection model and forming a matrix according to the equations that have been obtained. Adjustment of the equation with the system $\boldsymbol{x}\ (\boldsymbol{k}\ +\ \mathbf{1})\ =\ \boldsymbol{A}\otimes\boldsymbol{x}\ (\boldsymbol{k})$ aims to obtain a matrix that can describe the condition of the traffic system in accordance with the conditions in the field[4].
c. Calculate eigenvalues and eigenvectors in the max-plus algebra matrix model. The next process is to find eigenvalues and eigenvectors on the matrix that describe the condition of the traffic system using a special algorithm on max-plus algebra which was the initial discussion in this study, namely by using the power algorithm [6].
d. Determine the travelling time from the Monjali crossroad to Gejayan crossroad.

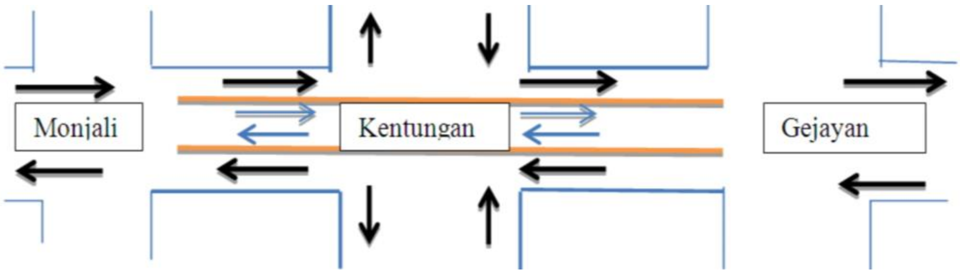


**Figure 2**. Map of the intersection that uses the underpass.

**Table 1.**Time duration data lights red, and green.

| Monjali | Red(second) | Green(second) | Yellow(second) |
|---|---|---|---|
| South | 110 | 33 | 3 |
| East | 112 | 41 | 3 |
| North | 120 | 26 | 2 |
| West | 145 | 38 | 3 |

**Table 2.** Time duration data lights red, and green.

| Kentungan | Red(second) | Green(second) | Yellow(second) |
|---|---|---|---|
| South | 175 | 20 | 3 |
| East | 137 | 60 | 3 |
| North | 180 | 31 | 4 |
| West | 140 | 63 | 3 |

**Table 3.** Time duration data lights red, and green

| Gejayan | Red(second) | Green(second) | Yellow(second) |
|---|---|---|---|
| South | 133 | 24 | 4 |
| East | 133 | 45 | 3 |
| North | 131 | 53 | 4 |
| West | 133 | 53 | 3 |

**Table 4.** Data on track length, length of car and number of vehicles.

| Red light to | Car track length (m) | Many vehicles | The length of the car |
|---|---|---|---|
| 1 | 50 | 12 | 4.5 |
| 2 | 45 | 10 | 4.5 |
| 3 | 45 | 10 | 4.5 |
| 4 | 47 | 10 | 4.5 |
| 5 | 60 | 13 | 4.5 |
| 6 | 58 | 13 | 4.5 |
| 7 | 47 | 10 | 4.5 |
| 8 | 58 | 13 | 4.5 |
| 9 | 53 | 12 | 4.5 |
| 10 | 51 | 10 | 4.5 |
| 11 | 54 | 12 | 4.5 |
| 12 | 58 | 13 | 4.5 |
| 13 | 58 | 13 | 4.5 |

Based on the data analysis steps above, the next step is to develop a mathematical model of the case above based on Max-Plus algebraic rules. Before compiling a mathematical model, the variables used in forming a mathematical model are explained first.

Figure 3 shows a network of closed series queue systems, where the network queue will be simulated for the queuing system at the intersection of Monjali towards the Gejayan intersection with one underpass.

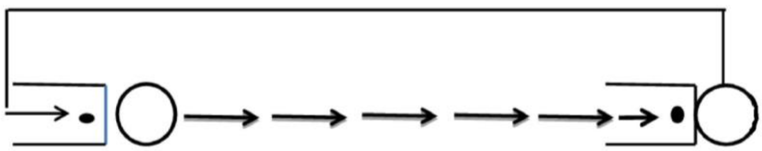

**Figure 3.** Queue network.

Variables defined as follows:

$a_i(k)$ : Upon arrival of the first car queue unit at the i intersection
$d_i(k)$ : The departure time for the first car queue unit from the i intersection
$t_i$ : Duration of red lights at the first intersection when the car queue unit is at the front
$th_i$ : Duration of green light at the first intersection when the car queue unit is at the front
$tj_i$ : Duration of travel on the 1st street

Based on the data obtained in the field, we apply the following assumptions:

1. The initial conditions for all red lights are the same for each intersection.
2. Fixed vehicle volume with the view that there are other vehicles going in the direction of the road that will be modeled when there is an exit from the first straight area in Monjali, Kentungan, or Gejayan.

3. Simulation model for 1 queue unit.
4. The queue buffer capacity is infinite.
5. Queues work according to First-in First-Out (FIFO) principle.
6. The transfer of customers from a queue to the next queue does not require time.
7. Customers must pass through the queue from beginning to end in sequence to receive the service of each waiter.

The purpose of these assumptions to facilitate the making of the max plus model and obtain accuracy in calculations.

Based on the above example, a mathematical model will be made as follows. Mathematical model without underpasses

$$d_1(k) = \max(t_1 + a_1(k), t_1 + d_1(k-1),$$
$$d_2(k) = \max(th_1 + tj_1 + t_2 + a_2(k), th_1 + tj_1 + t_2 + d_2(k-1)$$
$$d_3(k) = \max(th_2 + tj_2 + t_3 + a_3(k), th_2 + tj_2 + t_3 + d_3(k-1)$$

From the mathematical model without the underpass, a mathematical model with an underpass will be created

$$d_1(k) = \max(t_1 + a_1(k), t_1 + d_1(k-1),$$
$$d_2{}'(k) = \max(th_1 + tj_1{}' + t_3 + a_2(k), th_1 + tj_1{}' + t_3 + d_2{}'(k-1)$$

The above model can be modelled in the Max-plus algebra as follows

$$d_1(k) = \max(t_1 \otimes a_1(k) \oplus t_1 \otimes d_1(k-1)),$$
$$d_2{}'(k) = \max(th_1 \otimes tj_1{}' \otimes t_3 \otimes a_2(k) \oplus th_1 \otimes tj_1{}' \otimes t_3 \otimes d_2{}'(k-1)$$

Based on the data obtained, the queue dynamics at the service ready for the queue are as follows. Models without underpasses:

$$d_1(k) = \max(145 + a_1(k), 145 + d_1(k-1)),$$
$$d_2(k) = \max(35 + 180 + 140 + a_2(k), 35 + 180 + 180 + d_2(k-1)$$
$$d_3(k) = \max(70 + 180 + 180 + a_3(k), 70 + 180 + 180 + d_3(k-1)$$

where

$$a_1(k) = d_1(k-1)$$
$$a_2(k) = d_2(k-1)$$
$$a_3(k) = d_3(k-1)$$

Model without underpasses becomes

$$d_1(k) = \max(145 + a_1(k), 145 + d_1(k-1)),$$
$$d_2{}'(k) = \max(35 + 360 + 180 + a_2(k), 35 + 360 + 180 + d_2{}'(k-1)$$

where

$$a_1(k) = d_2{}'(k-1) \quad (1)$$
$$a_2(k) = d_1(k-1) \quad (2)$$

We rewritte in max-plus algebra as

$$d_1(k) = (145 \otimes a_1(k) \oplus 145 \otimes d_1(k-1), \quad (3)$$
$$d_2{}'(k) = (572 \otimes a_2(k) \oplus 572 \otimes d_2(k-1). \quad (4)$$

By substituting Eqs.(1)–(4), we obtain the following equations:

$$d_1(k) = (145 \otimes d_2{}'(k-1) \oplus 145 \otimes d_1(k-1),$$
$$d_2{}'(k) = (572 \otimes d_1(k-1) \oplus 572 \otimes d_2{}'(k-1)$$

The mathematical model is then converted into a matrix to facilitate the calculation of the output of mathematical models. Matrix form of the mathematical models above is

$$\begin{bmatrix} d_1(k) \\ d_2{}'(k) \end{bmatrix} \begin{bmatrix} 145 & 145 \\ 572 & 572 \end{bmatrix} \otimes \begin{bmatrix} d_1(k) \\ d_2{}'(k) \end{bmatrix}$$

The calculations using Matlab give the results represented in table 5. Based on the data generated, the time needed for 1 unit queue to pass the intersection from Monjali to Gejayan is 4721 second. Every vehicle that arrives at the intersection will wait for the red light and will exit the intersection simultaneously at the first red light. Based on the calculations in the table above it can be concluded that the waiting time of 1 queue unit to the next queue is 5 minutes while waiting for the length of the red light at the intersection. The model produced with one queue unit at the intersection using an underpass is quite efficient.

**Table 5.** Calculation value of the travel time of a car.

| | Time (in second) | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\boldsymbol{d_1(k)}$ | 145 | 290 | 862 | 1434 | 2006 | 2578 | 3150 | 3772 | 4294 |
| $\boldsymbol{d_2{}'(k)}$ | 145 | 717 | 1289 | 1861 | 2433 | 3005 | 3577 | 4149 | 4721 |

## 5. Conclusion and suggestions

The duration of the traffic lights on the intersection can be modelled using max-plus algebra. From the Max-plus algebra model and the data obtained from direct observation, it can be determined that the eigenvectors and eigenvalues are used to determine the travel time of the vehicle every 1 round. In this study, data on the duration of green, red and yellow lights were used at Monjali, Gejayan, and Kentungan intersections. The vehicle will require a travelling time of 4721 second to pass the intersection from Monjali to Gejayan.The steps used to get the results of the calculation are compiling the model, changing the mathematical model in the form of a matrix, doing calculations and analyzing the results of calculations. With the creation of this model it is expected to reduce the queue length at the intersection of Monjali, Kentungan and Gejayan.

## References

[1] Farhi N, Goursat M and Quadrat J-P 2005 *Proc. IEEE Conf. Decision Control*
[2] Rudhito M A 2016 *Max-Plus Algebra and Its Application* (Yogyakarta: Sanata University Press)
[3] Schutter B D and Boom T 2000 *Model Predictive Control for max-linear Discrete-event systems: Extended report & addendum* (Deflt University of Technology)
[4] Krivulin N K 1995 *Math. Comput. Model.* **22** 25
[5] Lall B K and Kristy C J 2003 *Fundamentals of Transportation Engineering* (Jakarta: Penerbit Erlangga)
[6] Farhi N, Goursat M and Quadrat J-P 2013 *The Traffic Phases of Road Networks* Online: https://www.academia.edu/26033960/The_Traffic_Phases_of_Road_Networks
[7] Paskalia P, Sari M and Anggita 2016 *Proc. of the National Algebra Seminar, its Application and Learning* p 16
[8] Fahim K, Subchan and Subiono 2013 *Jurnal Teknik Pomits* **1**1