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# On model of supply chain using max-plus algebra

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# On model of supply chain using max-plus algebra

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**Abstract.** Supply chain scheduling is used to manage the stability of the distribution of a commodity. In this paper, we will discuss the modelling of rice commodity supply chains using max-plus algebra. The rice distribution system data was obtained from the Yogyakarta Regional Logistics Bureau (Bulog) and its subsidiary company PT JPL. We get the supply chain model in the form of a matrix over max-plus algebra.

## 1. Introduction

Indonesia has been known as agricultural country. Rice is most important agricultural commodity. The demand of rice must be fulfilled by the supply of rice.Chain between supply-distribution-demand of rice has to stable. Supply chain is defined as a group of inter-connected participating companies that add value to a stream of transformed inputs from their source of origin to the end products or services that are demanded by designated end-customers [1]. There are four intrinsic flow of supply chain, i.e. material flow, infomation flow, finace flow, and comercial flow.

Indonesian Bureau of Logistics (Bulog) is a company which distibute rice in Indonesia. Bulog has 26 regional division who all of them has the same task to keep stability of rice commodity in Indonesia. Bulog was helped by it's subsidiary company, PT Jasa Prima Logistik (JPL) to distribute the commodity to people (customers). Bulog as a supplier and JPL as distributor cooperate to make the price of these commodity relatively stable. Bulog and JPL must make sure the distribution of the commodities are effective and efficient. Bulog made a schedule to distribute the commodity based on the demand of the customers. JPL help Bulog to prepare trucks to loading the commodity.

In [2][3][4] supply chain model of rice use causality diagram of supply chain, linear programming and LINDO, and goal programming model respectively. All the model use social variable approach and linear quantification.

Max-plus algebra is the system of algebra with two operation maximization and addition. The structure of this system is semiring [5]. Some work of tranportation such as scheduling at bus station, railway station, and airport using max-plus algebra [5][6]. Max-plus algebra also use to model supply chain of oil feedstock between two fuel terminal using tanker [6][7].

In this paper we will discuss of model supply chain of rice that distribute from Bulog and JPL using max-plus algebra.

## 2. Pleminaries

**Definition 2.1 Max-Plus Algebra [2]**

*Define a set* $\mathbb{R}_{max} = \mathbb{R} \cup \{\varepsilon\}$, *where* $\mathbb{R}$ *is the set of real number and* $\varepsilon := -\infty$. *For every* $a, b \in \mathbb{R}_{max}$ *define operations* $\oplus$ and $\otimes$ *by*

$$a \oplus b := \max\{a,b\} \text{ and } a \otimes b := a+b$$

*The set* $\mathbb{R}_{max}$ *with operations* $\oplus$ and $\otimes$ *is called Max-Plus Algebra and denoted by* $\mathfrak{R} = (\mathbb{R}_{max}, \oplus, \otimes)$. Clearly, $a \oplus \varepsilon = a$ and $a \oplus 0 = a$, for all $a \in \mathbb{R}_{max}$. The structure of $\mathfrak{R}$ is semiring with unity. Unit element in $\mathfrak{R}$ is $\varepsilon$ and neutral element is $e = 0$. Furthermore, $\mathfrak{R}$ is commutative semiring.

**Definition 2.2 Supremum on Max-Plus Algebra**
If set $\{a_i : i \in \mathbb{R}\}$ is countable set, with $a_i \in \mathbb{R}_{max}$ then

$$\bigoplus_{i\in\mathbb{R}} a_i = \bigoplus_{i=1}^{\infty} a_i = \sup_{i\in\mathbb{R}} a_i.$$

**Definition 2.3 Power on Max-Plus Algebra**
*Given* $a \in \mathbb{R}_{max}$ *and* $n \in \mathbb{R}$, *where* $\mathbb{R}$ *is the set of natural number, and define*

$$a^{\otimes n} := \underbrace{a + a + a + \cdots + a}_{n \text{ times}}$$

*If* $n = 0$, *we define* $a^{\otimes n} = e$.
We will generalize definition 2.2 for the power of $a \in \mathbb{R}_{max}$ to arbitrary $x \in \mathbb{R}$. We have,

$$a^{\otimes x} = xa$$

for all $x \in \mathbb{R}$.

Next, we will introduce matrix over max-plus algebra and basic operation on these matrix. We denote the set of all matrix $m \times n$ over max-plus algebra by $\mathbb{R}_{max}^{m\times n}$. The element of matrix $M \in \mathbb{R}_{max}^{m\times n}$ can be written as,

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \cdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where $a_{ij} \in \mathbb{R}_{max}$ for $0 < i \le m$ and $0 < j \le n$.
The sum of two matrices $A, B \in \mathbb{R}_{max}^{m\times n}$ denoted by $A \oplus B$, is defined by

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$

and for the $A \in \mathbb{R}_{max}^{m\times p}$ and $B \in \mathbb{R}_{max}^{p\times n}$ product of two matrices $A \otimes B$ is defined by

$$[A \otimes B]_{ij} = \bigoplus_{j=1}^{p} a_{ij} \otimes b_{jk} = \max_{j\in p}\{a_{ij} + b_{jk}\}$$

The zero matrix $\mathrm{E}$ is the matrix whose has entry $[\mathrm{E}]_{ij} = \varepsilon$ for all $i$ and $j$. The identity matrix $I \in \mathbb{R}_{max}^{n\times n}$ is the matrix whose has entry

$$[I]_{ij} = \begin{cases} 0, & i = j \\ \varepsilon, & i \ne j \end{cases}.$$

Clearly, the structure $(\mathbb{R}_{max}^{n\times n}, \oplus, \otimes)$ is semiring, with zero element $\mathrm{E}$ and unity $I$. The $n$th power of matrix $A \in \mathbb{R}_{max}^{m\times n}$ is defined by

$$A^{\otimes n} = \underbrace{A \otimes A \otimes \cdots \otimes A}_{n \text{ times}}$$

If $n = 0$ then $A^{\otimes n} = I^{n\times n}$.

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Base on definition 2.2 and definition of the power of matrix, we will define supremum of power of matrix by

$$A^{+} = \bigoplus_{i=1}^{\infty} A^{\otimes i}$$

The entry of $A^{+}$ is $\left[A^{+}\right]_{ij} = \sup\left\{\left[A^{k}\right]_{ij} : k \geq 0\right\}$.

Structure of max-plus algebra is semiring, which not always has an invers respect to addition. It means that we will find some problem to solve equation of linear equation $A \otimes x = b$. The solution of this equation does not always exists. But, inequality $A \otimes x \leq b$ always has solution. We will find out the greatest solution for vector $x$ such that satisfy the inequality $A \otimes x \leq b$. This solution is called principal solution and denoted by $x^*(A,b)$. The principal solution of this inequality is given the by the theorem bellow.

**Theorem 2.4**

*Suppose $A \in \mathbb{R}_{max}^{m\times n}$ is the matrix over max-plus algebra which every column contains at least one element does not equal to $\varepsilon$ and $b \in \mathbb{R}_{max}^{m}$, then*

$$\left[x^*(A,b)\right]_j = \min\left\{b_i - a_{ij} : i \in m \text{ and } a_{ij} > \varepsilon\right\}.$$

Theorem 2.4 has not given the solution of equation $A \otimes x = b$ but only the principal solution for inequality $A \otimes x \leq b$. The next lemma will give relationship between the solution existence of linear equation and the principal solution.

**Lemma 2.5**

*If equation $A \otimes x = b$ has solution then the principal solution of $A \otimes x \leq b$ is the solution of $A \otimes x = b$.*

For example, we have system of linear equation with two variables over max-plus algebra

$$\begin{cases} 3 \otimes x_1 \oplus 1 \otimes x_2 = 2 \\ 0 \otimes x_1 \oplus 4 \otimes x_2 = 5 \end{cases}$$

Matrix form of this system is $A \otimes x = b$, where $A = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $b = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

We have the principal solution for $A \otimes x \leq b$ is $x^*(A,b) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The principal solution also satisfy system $A \otimes x = b$. Hence, the solution of this system of linear equation is $x_1 = -1$ and $x_2 = 1$.

## 3. Method

The supply chain that will be modeled is supply chain of rice commodity in Yogyakarta, Indonesia. Data supply chain distribution was obtained by the Head of Supplier, Operations and Public Services Division of Yogyakarta Regional Division through interviews. Data related to consumer demand for rice is obtained from JPL's rice distribution realization data in June 2018. We conducted a literature study to develop mathematical models using max-plus algebra from the rice distribution supply chain in Yogyakarta.

## 4. Result and Discussion

Supply chain of rice in Bulog Divre Yogyakarta as follow the figure.

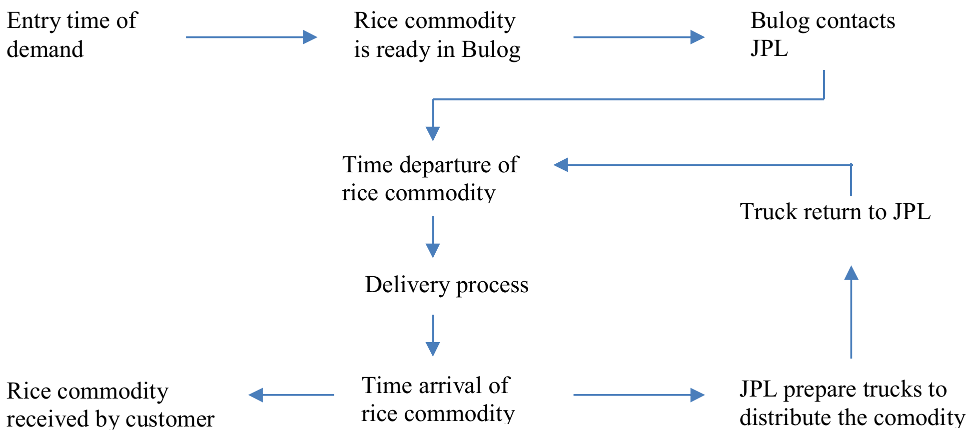


**Figure 1.** Supply Chain of Rice in Bulog Divre Yogyakarta

If $n$ is the number of truck then system of equation in max-plus algebra as follows.

$$\begin{cases} t_1(k) = p_r \otimes t_2(k-n) \otimes p_w \oplus u(k) \\ t_2(k) = p_d \otimes t_1(k) \\ y(k) = t_2(k) \end{cases} \quad (1)$$

with $t_1(k)$ and $t_2(k)$ are the time of rice departure from Bulog and the time of rice arrive to customer. Time spent by the truck to delivery is $p_d(k)$ and return to Bulog is $p_r(k)$. JPL need time to prepare the trucks $p_w(k)$. Finally, $u(k)$ and $y(k)$ are entry time of rice demand and time of rice supply arrive to customers. The model of equation system (1), in the matrix form as follows

$$\vec{t}(k) = A_0 \otimes \vec{t}(k) \oplus A_1 \otimes \vec{t}(k-n) \oplus B_0 \otimes u(k) \quad (2)$$

$$y(k) = C \otimes \vec{t}(k) \quad (3)$$

where,

$$\vec{t}(k) = \begin{bmatrix} t_1(k) \\ t_2(k) \end{bmatrix},\ A_0 = \begin{bmatrix} \varepsilon & \varepsilon \\ p_d & \varepsilon \end{bmatrix}, A_1 = \begin{bmatrix} \varepsilon & p_r \otimes p_w \\ \varepsilon & \varepsilon \end{bmatrix},\ B_0 = \begin{bmatrix} e \\ \varepsilon \end{bmatrix}, \text{ and } C = [\varepsilon \quad e].$$

We substitute $\vec{t}(k)$ in equation (2) with $\vec{t}(k)$ in left hand side, and we have

$$\vec{t}(k) = A_0 \otimes \left[ A_0 \otimes \vec{t}(k) \oplus A_1 \otimes \vec{t}(k-n) \otimes B_0 \otimes u(k) \right] \otimes A_1 \otimes \vec{t}(k-n) \otimes B_0 \otimes u(k)$$

$$= A_0^{\otimes 2} \otimes \vec{t}(k) \otimes \left( A_0 \otimes A_1 \otimes \vec{t}(k-n) \oplus A_1 \otimes \vec{t}(k-n) \right) \oplus \left( A_0 \otimes B_0 \otimes u(k) \oplus B_0 \otimes u(k) \right) \quad (4)$$

Next, we substitute again $\vec{t}(k)$ in equation (4) with $\vec{t}(k)$ in left hand side of equation (2) and we have

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$$\vec{t}(k) = A_0^{\otimes 3} \otimes \vec{t}(k) \oplus \bigoplus_{i=0}^{2} A_0 \otimes A_1 \otimes \vec{t}(k-n) \oplus \bigoplus_{i=0}^{2} A_0 \otimes B_0 \otimes u(k) \quad (5)$$

Analog, we substitute $\vec{t}(k)$ for *n* times for $n \to \infty$, and we have

$$\vec{t}(k) = A_0^{\otimes n} \otimes \vec{t}(k) \oplus \bigoplus_{i=0}^{n-1} A_0 \otimes A_1 \otimes \vec{t}(k-n) \oplus \bigoplus_{i=0}^{n-1} A_0 \otimes B_0 \otimes u(k) \quad (6)$$

We have $A_0^{\otimes i} = E$ for $i \geq 2$ and

$$\bigoplus_{i=0}^{n-1} A_0 \otimes A_1 \otimes \vec{t}(k-n) = I \oplus \left( \bigoplus_{i=1}^{\infty} A_0^{\otimes i} \otimes A_1 \right) \vec{t}(k-n) \text{ and}$$

$$\bigoplus_{i=0}^{n-1} A_0 \otimes B_0 \otimes u(k) = I \oplus \left( \bigoplus_{i=1}^{\infty} A_0 \otimes B_0 \right) u(k)$$

Hence, we can simplify equation (2) became

$$\vec{t}(k) = A \otimes \vec{t}(k-n) \oplus B \otimes u(k), \quad (7)$$

where $A = \left[ \bigoplus_{i=0}^{\infty} A_0^{\otimes i} \right] \otimes A_1$, $B = \left[ \bigoplus_{i=0}^{\infty} A_0^{\otimes i} \right] \otimes B_0$

We will determine matrix form of equation (3) by substitute $\vec{t}(k)$ in equation (7). Now, we have

$$y(k) = C \otimes \left( A \otimes \vec{t}(k-n) \oplus B \otimes u(k) \right) \quad (8)$$

Based on equation (7), we can conclude $\vec{t}(k-\alpha n) = A \otimes \vec{t}(k-(\alpha+1)n) \oplus B \otimes u(k-\alpha n)$ with $\alpha \in \mathbb{N} \cup \{0\}$. We repeat the substitute until *n* times for $n \to \infty$ and we have

$$y(k) = C \otimes A^{\otimes \alpha} \otimes \vec{t}(k-\alpha n) \oplus \left( \bigoplus_{i=0}^{\infty} C \otimes A^{\otimes i} \otimes B \otimes u(k-in) \right) \quad (9)$$

Then $y(k) = \bigoplus_{i=0}^{\infty} C \otimes A^{\otimes i} \otimes B \otimes u(k-in)$, since $k - \alpha n < n$.

$$Y(k) = H \otimes U(k) \quad (10)$$

where $H$ is lower triangular matrix with $[H]_{ij} = CA^{i-j}B$. Equation (7) and (10) are the models of supply chain of rice commodity using max-plus algebra.

**Simulation Model**

We used data of rice distribution in district of Kulon Progo, Yogyakarta on June 2018 to simulate the model of supply chain using Max-Plus Algebra.

The truck can carry maximum loads about 6,000 kg. For the simulation, we used the schedule of distribution on 21 June 2018, which is rice distributed to Pengasih and Kokap. Total demand was 95,970 kg. Truck spent 2.5 hour from Bulog to Pengasih and took 2 hour from Pengasih to Bulog. JPL need an hour to prepare the truck before used everyday. We estimated JPL need 8 truck to distribute the rice.

**Table 1. Demand and Schedule Distribution of rice in Kulon Progo**

| No | Sub-District of Kulon Progo | Total of Demand (Kg) | Schedule of Distribution |
|---|---|---|---|
| 1 | Pengasih | 52,480 | 21 June 2018 |
| 2 | Kokap | 43,490 | |
| 3 | Panjatan | 37,860 | 25 June 2018 |
| 4 | Sentolo | 57,140 | |
| 5 | Girimulyo | 28,100 | 26 June 2018 |
| 6 | Kalibawang | 39,820 | |
| 7 | Wates | 36,100 | |
| 8 | Samigaluh | 37,570 | 27 June 2018 |
| 9 | Nanggulan | 36,350 | |
| 10 | Temon | 23,920 | |
| 11 | Galur | 32,620 | 28 June 2018 |
| 12 | Lendah | 47,760 | |
| | **Total** | 473,210 | |

(Source Bulog Divre Yogyakarta June 2018)

**Table 2. Simulation Model on Schedule Distribution of Rice on 21 June 2018**

| Truck | Sub-District | Schedule Distribution | Vol (K Kg) | Time Delivery (hours) | Time Return (hours) | Prepare in JPL (hours) | Time Departure | Time Arrive | Decision |
|---|---|---|---|---|---|---|---|---|---|
| 1 | Pengasih | 21-Jun-18 | 6 | 2.5 | 2 | 1 | 7 | 10.5 | On-time |
| 2 | Pengasih | 21-Jun-18 | 6 | 2.5 | 2 | 1 | 7 | 10.5 | On-time |
| 3 | Pengasih | 21-Jun-18 | 6 | 2.5 | 2 | 1 | 7 | 10.5 | On-time |
| 4 | Pengasih | 21-Jun-18 | 6 | 2.5 | 2 | 1 | 7 | 10.5 | On-time |
| 5 | Pengasih | 21-Jun-18 | 6 | 2.5 | 2 | 1 | 7 | 10.5 | On-time |
| 6 | Pengasih | 21-Jun-18 | 6 | 2.5 | 2 | 1 | 7 | 10.5 | On-time |
| 7 | Pengasih | 21-Jun-18 | 6 | 2.5 | 2 | 1 | 7 | 10.5 | On-time |
| 8 | Pengasih | 21-Jun-18 | 6 | 2.5 | 2 | 1 | 7 | 10.5 | On-time |
| 1 | Pengasih & Kokap | 21-Jun-18 | 5.99 | 3 | 2.2 | 0 | 10.5 | 15.5 | On-time |
| 2 | Kokap | 21-Jun-18 | 6 | 2.5 | 2 | 0 | 10.5 | 15 | On-time |
| 3 | Kokap | 21-Jun-18 | 6 | 2.5 | 2 | 0 | 10.5 | 15 | On-time |
| 4 | Kokap | 21-Jun-18 | 6 | 2.5 | 2 | 0 | 10.5 | 15 | On-time |
| 5 | Kokap | 21-Jun-18 | 6 | 2.5 | 2 | 0 | 10.5 | 15 | On-time |
| 6 | Kokap | 21-Jun-18 | 6 | 2.5 | 2 | 0 | 10.5 | 15 | On-time |
| 7 | Kokap | 21-Jun-18 | 5.99 | 2.5 | 2 | 0 | 10.5 | 15 | On-time |
| 8 | Kokap | 21-Jun-18 | 5.99 | 2.5 | 2 | 0 | 10.5 | 15 | On-time |

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