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STRONGLY REGULAR MATRICES AND SIMPLE IMAGE SET IN INTERVAL MAX-PLUS ALGEBRA

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Abstract

Suppose that $\mathbb{R}$ is the set of real numbers and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$ with $\varepsilon = -\infty$. Max-plus algebra is the set $\mathbb{R}_\varepsilon$ that is equipped with two operations, maximum and addition. Matrices over max-plus algebra

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are matrices whose elements belong to $\mathbb{R}_\varepsilon$. The set $I(\mathbb{R})_\varepsilon = \{x = [\underline{x}, \overline{x}]\underline{x}, \overline{x} \in \mathbb{R}, \varepsilon < \underline{x} \leq \overline{x}\} \cup \{\varepsilon\}$ with $\varepsilon = [\varepsilon, \varepsilon]$ which is equipped with maximum and addition operations is known as interval max-plus algebra. Matrices over interval max-plus algebra are matrices whose elements belong to $I(\mathbb{R}_\varepsilon)$. The research about image set, strongly regular matrix and simple image set in max-plus algebra have been done. In this paper, we investigate image set, strongly regular matrix and simple image set in interval max-plus algebra.

1. Introduction

According to Baccelli et al. [1], max-plus algebra has been applied for modelling and analyzing problems in the area of planning, communication, production, queuing system with finite capacity, parallel computation and traffic. Max-plus algebra is the set of $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$ equipped with two operations, $\otimes$ (maximum) and $\otimes$ (addition), where $\mathbb{R}$ is the set of real numbers with $\varepsilon = -\infty$ (Baccelli et al. [1]). According to Tam [7], max-plus algebra is a linear algebra over semiring $\mathbb{R}_\varepsilon$ that is equipped with $\oplus$ (maximum) and $\otimes$ (addition) operations, while min-plus algebra is a linear algebra over semiring $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{\varepsilon'\}$, $\varepsilon' = \infty$ equipped with $\oplus'$ (minimum) and $\otimes'$ (addition) operations. Tam [7] also defined complete max-plus algebra and complete min-plus algebra. Complete max-plus algebra is a linear algebra over semiring $\mathbb{R}_{\varepsilon, \varepsilon'} = \mathbb{R}_\varepsilon \cup \{\varepsilon'\}$ equipped with $\oplus$ (maximum) and $\otimes$ (addition) operations, while complete min-plus algebra is a linear algebra over semiring $\mathbb{R}_{\varepsilon', \varepsilon} = \mathbb{R}_{\varepsilon'} \cup \{\varepsilon\}$ equipped with $\oplus'$ (minimum) and $\otimes'$ (plus) operations. Following the definition of max-plus algebra given by Baccelli et al. [1], we define min-plus algebra is the set of $\mathbb{R}_{\varepsilon'}$ equipped with $\oplus'$ (minimum) and $\otimes'$ (plus) operations. Besides that, we define complete max-plus algebra is the set $\mathbb{R}_{\varepsilon, \varepsilon'} = \mathbb{R}_\varepsilon \cup \{\varepsilon'\}$ equipped with $\oplus$ (maximum) and $\otimes$ (addition) operations, while complete min-plus algebra is the set $\mathbb{R}_{\varepsilon', \varepsilon} = \mathbb{R}_{\varepsilon'} \cup \{\varepsilon\}$ equipped with $\oplus'$ (minimum) and $\otimes'$ (plus) operations.

It shows that $\mathbb{R}_{\varepsilon,\varepsilon'} = \mathbb{R}_{\varepsilon} \bigcup \{\varepsilon'\} = \mathbb{R} \bigcup \{\varepsilon, \varepsilon'\} = \mathbb{R}_{\varepsilon'} \bigcup \{\varepsilon'\} = \mathbb{R}_{\varepsilon',\varepsilon}$. Furthermore, $\mathbb{R}_{\varepsilon,\varepsilon'}$ and $\mathbb{R}_{\varepsilon',\varepsilon}$ are written as $\overline{\mathbb{R}}$. It can be formed a set of matrices in the size $m \times n$ of the sets $\mathbb{R}_{\varepsilon}$, $\mathbb{R}_{\varepsilon'}$, and $\overline{\mathbb{R}}$. The set of matrices with components in $\mathbb{R}_{\varepsilon}$ is denoted by $\mathbb{R}_{\varepsilon}^{m\times n}$. The matrix $A \in \mathbb{R}_{\varepsilon}^{m\times n}$ is called *matrix over max-plus algebra*. The matrix over min-plus algebra, over complete max-plus algebra and over min-plus algebra can also be defined (Tam [7]).

In 2010, Tam [7] discussed the linear system in max-plus algebra, image set and strongly regular matrix. Tam [7] and Butkovic [2, 3] mentioned that the solution of the linear system in max-plus algebra is closely related to image set, simple image set and strongly regular matrix. Butkovic [3-5] have shown the characteristics of the strongly regular matrix, that is matrix has strong permanent.

Rudhito [6] has generalized max-plus algebra into interval max-plus algebra and fuzzy number max-plus algebra. Moreover, Siswanto et al. [8] have discussed linear equation in interval max-plus algebra. Based on the discussion on the max-plus algebra that is connected to the system of linear equation, in this paper we discuss image set, strongly regular matrix and simple image set in interval max-plus algebra. Furthermore, it will also discuss matrix has strong permanent. The matrix has strong permanent is the characteristic of strongly regular matrix.

Before discussing the result of the research, the necessary concepts are discussed. These are interval max-plus algebra, system of linear equation in max-plus algebra and system of linear equation and the system of linear inequation in interval max-plus algebra.

Related to the system of linear equation $A \otimes x = b$, Tam [7] defined the solution set of the system of linear equation and image set of matrix A.

Definition 1.1. Given $A \in \mathbb{R}_{\varepsilon}^{m\times n}$ and $b \in \mathbb{R}_{\varepsilon}^{m}$. The solution set of the system of linear equation $A \otimes x = b$ is $S(A, b) = \{x \in \mathbb{R}_{\varepsilon}^{n} \mid A \otimes x = b\}$ and the image set of matrix A is $Im(A) = \{A \otimes x \mid x \in \mathbb{R}_{\varepsilon}^{n}\}$.

Based on the above definition, the following theorem and corollary are given.

Theorem 1.2. *Given* $A \in \mathbb{R}_{\varepsilon}^{m\times n}$ *and* $b \in \mathbb{R}_{\varepsilon}^{m}$. *Vector* $b \in Im(A)$ *if and only if* $S(A, b) \neq \emptyset$.

Corollary 1.3. *Given* $A \in \mathbb{R}_{\varepsilon}^{m\times n}$ *and* $b \in \mathbb{R}_{\varepsilon}^{m}$. *Vector* $b \in Im(A)$ *if and only if* $A \otimes (A^{*} \otimes' b) = b$.

The following is the definitions of subspace and theorem which states that for $A \in R_{\varepsilon}^{m\times n}$, $Im(A)$ is a subspace.

Definition 1.4. Suppose that $S \subseteq \mathbb{R}_{\varepsilon}^{n}$. If for each $x, y \in S$ and for each $\alpha \in \mathbb{R}_{\varepsilon'}$, $\alpha \otimes x \in S$ and $x \oplus y \in S$, then S is known as subspace of max-plus algebra.

Theorem 1.5. *Suppose that* $A \in \mathbb{R}_{\varepsilon}^{m\times n}$. *If* $\alpha, \beta \in \mathbb{R}$ *and* $u, v \in Im(A)$ *then* $\alpha \otimes u \oplus \beta \otimes v \in Im(A)$.

In max-plus algebra there are the definitions of strongly linearly independent and strongly regular matrix.

Definition 1.6. The vectors $A_1, ..., A_n \in \mathbb{R}_{\varepsilon}^{m}$ are said to be *linearly dependent* if one of those vectors can be expressed as a linear combination of other vectors. Otherwise $A_1, ..., A_n$ are considered linearly independent.

Definition 1.7. The vectors $A_1, ..., A_n \in \mathbb{R}_{\varepsilon}^{m}$ are said to be *strongly linearly independent* if there is $b \in \mathbb{R}^{m}$ as such so that b can be uniquely expressed as a linear combination of $A_1, ..., A_n$.

Definition 1.8. For the vectors $A_1, ..., A_n \in \mathbb{R}_{\varepsilon}^{m}$ which are strongly linearly independent, if $m = n$, then the matrix $A = (A_1, ..., A_n)$ is known as strongly regular.

Below is the definition of simple image set.

Definition 1.9. Given $A \in \mathbb{R}_{\varepsilon}^{m \times n}$. The simple image set of matrix A is defined as $S_A = \{b \in \mathbb{R}^m \mid A \otimes x = b \text{ has unique solution}\}$.

Theorem 1.10. *Suppose $A \in \mathbb{R}_{\varepsilon}^{m \times n}$ is double $\mathbb{R}$-astic and $b \in \mathbb{R}^m$, then* $|S(A, b)| \in \{0, 1, \infty\}$.

Definition 1.11. *Suppose $A \in \mathbb{R}_{\varepsilon}^{m \times n}$, $T(A)$ is defined as* $\{|S(A, b)| \mid b \in \mathbb{R}^m\}$.

Theorem 1.12. *If $A \in \mathbb{R}_{\varepsilon}^{m \times n}$ is double $\mathbb{R}$-astic, then $T(A)$ is $\{0, \infty\}$ or* $\{0, 1, \infty\}$.

Based on Definition 1.7, the column vectors of A are strongly linearly independent if and only if $1 \in T(A)$. In max-plus algebra, a matrix that has strong permanent is also defined (Butkovic [4]).

Definition 1.13. Given $A \in \mathbb{R}_{\varepsilon}^{m \times n}$ and $\pi \in P_n$, where P_n is the set of all permutation of $N = \{1, 2, ..., n\}$. $\omega(A, \pi)$ is defined as $\prod_{j \in n}^{\otimes} A_{j, \pi(j)}$ and the permanent of matrix A is $\text{maper}(A) = \sum_{\pi \in P_n}^{\oplus} \omega(A, \pi)$.

Tam [7] defined that a matrix has strong permanent as given below.

Definition 1.14. The set of all permutation is written as $ap(A)$, and is defined as $ap(A) = \{\pi \in P_n \mid \text{maper}(A) = \omega(A, \pi)\}$. Matrix A is said to have *strong permanent* if $|ap(A)| = 1$.

Butkovic [4] and Tam [7] explained the connection between strong regular matrix and matrix has strong permanent as below.

Theorem 1.15. *Suppose that $A \in \mathbb{R}_{\varepsilon}^{m \times n}$ is double $\mathbb{R}$-astic. Matrix A is strongly regular matrix if and only if matrix A has strong permanent.*

In the following section, interval max-plus algebra, matrix over interval max-plus algebra and the system of linear equation in interval max-plus algebra is in (Rudhito [6], Siswanto et al. [8]) are presented.

Closed interval x in $\mathbb{R}_\varepsilon$ is a subset of $\mathbb{R}_\varepsilon$ in the form of $x = [\underline{x}, \overline{x}] = \{x \in \mathbb{R}_\varepsilon \mid \underline{x} \leq x \leq \overline{x}\}$. The interval x in $\mathbb{R}_\varepsilon$ is known as *interval max-plus*. A number $x \in \mathbb{R}_\varepsilon$ can be expressed as interval $[x, x]$.

Definition 1.16. $I(\mathbb{R})_\varepsilon = \{x = [\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \mathbb{R}, \varepsilon < \underline{x} \leq \overline{x}\} \cup \{\varepsilon\}$ is formed, with $\varepsilon = [\varepsilon, \varepsilon]$. On the set $I(\mathbb{R})_\varepsilon$, the operations $\overline{\oplus}$ and $\overline{\otimes}$ with $x \overline{\oplus} y = [\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}]$ and $x \overline{\otimes} y = [\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}]$ for every $x, y \in I(\mathbb{R})_\varepsilon$ are defined. Furthermore they are known as *interval max-plus algebra* and notated as $I(\Re)_{\max} = (I(\mathbb{R})_\varepsilon; \overline{\oplus}, \overline{\otimes})$.

Definition 1.17. The set of matrices in the size $m \times n$ with the elements in $I(\mathbb{R})_\varepsilon$ is notated as $I(\mathbb{R})_\varepsilon^{m\times n}$, namely:

$$I(\mathbb{R})_\varepsilon^{m\times n} = \{A = [A_{ij}] \mid A_{ij} \in I(\mathbb{R})_\varepsilon; \text{ for } i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n\}.$$

Matrices which belong to $I(\mathbb{R})_\varepsilon^{m\times n}$ are known as *matrices over interval max-plus algebra* or shortly *interval matrices*.

Definition 1.18. For $A \in I(\mathbb{R})_\varepsilon^{m\times n}$, matrix $\underline{A} = [\underline{A}_{ij}] \in \mathbb{R}_\varepsilon^{m\times n}$ and $\overline{A} = [\overline{A}_{ij}] \in \mathbb{R}_\varepsilon^{m\times n}$ are defined, as a lower bound matrix and an upper bound matrix of the interval matrix A, respectively.

Definition 1.19. Given that interval matrix $A \in I(\mathbb{R})_\varepsilon^{m\times n}$, and $\underline{A}$ and $\overline{A}$ are a lower bound matrix and an upper bound matrix of the interval matrix A, respectively. The matrix interval of A is defined, namely $[\underline{A}, \overline{A}] = \{A = \mathbb{R}_\varepsilon^{m\times n} \mid \underline{A} \leq A \leq \overline{A}\}$ and $I(\mathbb{R}_\varepsilon^{m\times n})_b = \{[\underline{A}, \overline{A}] \mid A \in (\mathbb{R})_\varepsilon^{m\times n}\}$.

Suppose that $A \in I(\mathbb{R})_\varepsilon^{m\times n}$ and $b = (b_1, ..., b_m) \in I(\mathbb{R})_\varepsilon^m$. The equation system $A \overline{\otimes} x = b$ is known as the *system of linear equation* in interval max-plus algebra. Furthermore, $A \approx [\underline{A}, \overline{A}]$, $x \approx [\underline{x}, \overline{x}]$ and $b \approx [\underline{b}, \overline{b}]$ with $[\underline{A}, \overline{A}] \in I(\mathbb{R}_\varepsilon^{n\times n})_b$, $[\underline{x}, \overline{x}], [\underline{b}, \overline{b}] \in I(\mathbb{R}_\varepsilon^{n})_b$. To find the solution of the system of linear equation in interval max-plus algebra, the solution of the system of linear equation in max-plus algebra $\underline{A} \otimes \underline{x} = \underline{b}$ dan $\overline{A} \otimes \overline{x} = \overline{b}$ should be determined.

Definition 1.20. Given the system of linear equation $A \overline{\otimes} x = b$ with $A \in I(\mathbb{R})_\varepsilon^{m\times n}$ and $b \in I(\mathbb{R})_\varepsilon^{m}$. $IS(A, b) = \{x \in I(\mathbb{R})_\varepsilon^{m} \mid A \overline{\otimes} x = b\}$ is defined.

Along with the discussion the system of linear equation in max-plus algebra, without loss of generality, the system of linear equation in interval max-plus algebra $A \overline{\otimes} x = b$ with $A \in I(\mathbb{R})_\varepsilon^{m\times n}$ double $I(\mathbb{R})$-astic and $b \in I(\mathbb{R})^m$ is discussed.

Theorem 1.21. *Suppose that* $A \in I(\mathbb{R})_\varepsilon^{m\times n}$ *double* $I(\mathbb{R})$*-astic and* $b \in I(\mathbb{R})^m$. *Vector* $y \in IS(A, b)$ *if and only if* $\underline{y} \in S(\underline{A}, \underline{b})$, $\overline{y} \in S(\overline{A}, \overline{b})$ *and* $\underline{y} \le \overline{y}$. *Furthermore y can be expressed as* $y \approx [\underline{y}, \overline{y}]$.

Each of the following two corollaries is a criterion of the system of linear equation which has solution and a criterion the system of linear equation which has unique solution. The notations of $M = \{1, 2, ..., m\}$ and $N = \{1, 2, ..., n\}$ are used to simplify.

Corollary 1.22. *Suppose* $A \in I(\mathbb{R})_\varepsilon^{m\times n}$ *is double* $I(\mathbb{R})$*-astic and* $b \in I(\mathbb{R})^m$, *then the following three statements are equivalent*:

a. $IS(A, b) \ne \emptyset$.

b. $\hat{y} \in IS(A, b)$.

c. $\bigcup_{j\in N} M_j(\underline{A}, \underline{b}) = M$ *and* $\bigcup_{j\in N} M_j(\overline{A}, \overline{b}) = M$.

Corollary 1.23. *Suppose* $A \in I(\mathbb{R})_{\varepsilon}^{m \times n}$ *is double* $I(\mathbb{R})$*-astic and* $b \in I(\mathbb{R})^{m}$, *then* $IS(A, b) = \{\hat{y}\}$ *if and only if*

a. $\bigcup_{j \in N} M_j(\underline{A}, \underline{b}) = M$ *and* $\bigcup_{j \in N} M_j(\overline{A}, \overline{b}) = M$.

b. $\bigcup_{j \in N'} M_j(\underline{A}, \underline{b}) \neq M$ *for every* $N' \subseteq N$, $N' \neq N$ *and* $\bigcup_{j \in N''} M_j(\overline{A}, \overline{b}) = M$ *for every* $N'' \subseteq N$, $N'' \neq N$.

2. Main Results

In this section, we present new results of the research. The results that are obtained, are related with the system of linear equation in interval max-plus algebra $A \overline{\otimes} x = b$, $A \in I(\mathbb{R})_{\varepsilon}^{m \times n}$ and $b \in I(\mathbb{R})^{m}$.

We will start the definition of the image set of matrix *A*.

Definition 2.1. Given $A \in I(\mathbb{R})_{\varepsilon}^{m \times n}$. The image set of matrix *A* is $IIm(A) = \{A \overline{\otimes} x \mid x \in I(\mathbb{R})_{\varepsilon}^{n}\}$.

Based on the definition of the image set of a matrix, a theorem which demonstrated the necessary and sufficient conditions of the elements of the image set is presented.

Theorem 2.2. *Given* $A \in I(\mathbb{R})_{\varepsilon}^{m \times n}$ *and* $b \in I(\mathbb{R})_{\varepsilon}^{m}$. *Vector* $b \in IIm(A)$ *if and only if* $IS(A, b) \neq \emptyset$.

Proof. Suppose that $A \approx [\underline{A}, \overline{A}]$ and $b \approx [\underline{b}, \overline{b}]$ with $\underline{A}, \overline{A} \in \mathbb{R}_{\varepsilon}^{m \times n}$ and $\underline{b}, \overline{b} \in \mathbb{R}_{\varepsilon}^{m}$. Since $b \in IIm(A)$, it means that $\underline{b} \in Im(\underline{A})$ and $\overline{b} \in Im(\overline{A})$. Therefore, based on Theorem 1.2, i.e., $S(\underline{A}, \underline{b}) \neq \emptyset$ and $S(\overline{A}, \overline{b}) \neq \emptyset$, then $IS(A, b) \neq \emptyset$. On the other hand, since $IS(A, b) \neq \emptyset$, $S(\underline{A}, \underline{b}) \neq \emptyset$ and $S(\overline{A}, \overline{b}) \neq \emptyset$. Thus, based on Theorem 1.2 that is $\underline{b} \in Im(\underline{A})$ and $\overline{b} \in Im(\overline{A})$, then $b \in IIm(A)$ with $b \approx [\underline{b}, \overline{b}]$. □

With regard to Theorem 2.2, the following corollary is obtained:

Corollary 2.3. *Given* $A \in I(\mathbb{R})_{\varepsilon}^{m\times n}$ *and* $b \in I(\mathbb{R})_{\varepsilon}^{m}$. *Vector* $b \in IIm(A)$ *if and only if* $\underline{b} = \underline{A} \otimes (\underline{A}^{*} \otimes' \underline{b})$ *and* $\overline{b} = \overline{A} \otimes (\overline{A}^{*} \otimes' \overline{b})$.

Proof. Suppose that $A \approx [\underline{A}, \overline{A}]$ and $b \approx [\underline{b}, \overline{b}]$. Since $b \in IIm(A)$, it means that $\underline{b} \in Im(\underline{A})$ and $\overline{b} \in Im(\overline{A})$. Based on Corollary 1.3, $\underline{b} = \underline{A} \otimes (\underline{A}^{*} \otimes' \underline{b})$ and $\overline{b} = \overline{A} \otimes (\overline{A}^{*} \otimes' \overline{b})$. On the other hand, since $\underline{b} = \underline{A} \otimes (\underline{A}^{*} \otimes' \underline{b})$ and $\overline{b} = \overline{A} \otimes (\overline{A}^{*} \otimes' \overline{b})$, based on Corollary 1.3, $\underline{b} \in Im(\underline{A})$ and $\overline{b} \in Im(\overline{A})$ are obtained. Therefore, $b \approx [\underline{b}, \overline{b}] \in IIm(A)$. □

The following theorem presents that $IIm(A)$ is a subspace of $I(\mathbb{R})_{\varepsilon}^{m}$.

Theorem 2.4. *Given* $A \in I(\mathbb{R})_{\varepsilon}^{m\times n}$, $IIm(A)$ *is a subspace of* $I(\mathbb{R})_{\varepsilon}^{m}$.

Proof. Since $IIm(A) = \{A \overline{\otimes} x \mid x \in I(\mathbb{R})_{\varepsilon}^{n}\}$, $IIm(A) \subseteq I(\mathbb{R})_{\varepsilon}^{m}$ and $Im(A) \neq \varnothing$. Taking $p, q \in IIm(A)$ and $\alpha, \beta \in I(\mathbb{R})$, accordingly, $\alpha \overline{\otimes} p \overline{\otimes} \beta \overline{\otimes} q \approx [\underline{\alpha} \otimes \underline{p} \oplus \underline{\beta} \otimes \underline{q}, \overline{\alpha} \otimes \overline{p} \oplus \overline{\beta} \otimes \overline{q}]$ with $\underline{\alpha} \otimes \underline{p} \oplus \underline{\beta} \otimes \underline{q} \in Im(\underline{A})$ and $\overline{\alpha} \otimes \overline{p} \oplus \overline{\beta} \otimes \overline{q} \in Im(\overline{A})$. It means that $\alpha \overline{\otimes} p \overline{\oplus} \beta \overline{\otimes} q \in IIm(A)$. In other words, $\alpha \overline{\otimes} p \overline{\oplus} \beta \overline{\otimes} q \in IIm(A)$ and hence $IIm(A)$ is a subspace of $I(\mathbb{R})_{\varepsilon}^{m}$. □

Several definitions and theorems are presented to support the discussion on strongly regular matrix.

Definition 2.5. The vectors $A_1, ..., A_n \in I(\mathbb{R})_{\varepsilon}^{m}$ is said to be *linearly dependent* if one of those vectors can be said to be as *linear combination* of other vectors. Otherwise then, $A_1, ..., A_n$ is said to be *linearly independent*.

Theorem 2.6. *If the vectors* $A_1, ..., A_n \in I(\mathbb{R})_{\varepsilon}^{m}$ *with* $A_1 \approx [\underline{A}_1, \overline{A}_1]$, $A_2 \approx [\underline{A}_2, \overline{A}_2], ..., A_n \approx [\underline{A}_n, \overline{A}_n]$ *are linearly dependent, then* $\underline{A}_1, \underline{A}_2, ..., \underline{A}_n$ *and* $\overline{A}_1, \overline{A}_2, ..., \overline{A}_n$ *are linearly independent*.

Proof. It is known that $A_1, ..., A_n \in I(\mathbb{R})_\varepsilon^m$ is linearly dependent. It means that one of those vectors can be stated as a linear combination of the others. Without losing the generality of the proof, for example $A_1 = \overline{\oplus}_{i=2}^n(\alpha_i \overline{\otimes} A_i)$ with $\alpha_i \in I(\mathbb{R})_m$, since $A_1 \approx [\underline{A}_1, \overline{A}_1]$, $A_2 \approx [\underline{A}_2, \overline{A}_2], ..., A_n \approx [\underline{A}_n, \overline{A}_n]$, $\underline{A}_1 = \oplus_{i=2}^n(\underline{\alpha}_i \otimes \underline{A}_i)$ and $\overline{A}_1 = \oplus_{i=2}^n(\overline{\alpha}_i \otimes \overline{A}_i)$. In other words, $\underline{A}_1, \underline{A}_2, ..., \underline{A}_n$ and $\overline{A}_1, \overline{A}_2, ..., \overline{A}_n$ are linearly dependent.

Definition 2.7. The vectors $A_1, ..., A_n \in I(\mathbb{R})_\varepsilon^m$ is said to be *strongly linearly independent* if there is $b \in I(\mathbb{R})^m$, b can be uniquely expressed as a linear combination of $A_1, ..., A_n$, that is $b = \overline{\oplus}_{i=1}^n(\alpha_i \overline{\otimes} A_i)$, $\alpha_i \in I(\mathbb{R})$.

The following is the definition of strongly regular matrix.

Definition 2.8. Matrix $A = (A_{1,} A_2 \ldots A_n) \in I(\mathbb{R})_\varepsilon^{n\times n}$ is said to be *strongly regular* if the vectors $A_1, ..., A_n \in I(\mathbb{R})_\varepsilon^n$ are strongly linearly independent.

Based on the definition of strongly linearly independent in the max-plus algebra, the following theorem is formulated.

Theorem 2.9. *If the vectors* $A_{1,}, ..., A_n \in I(\mathbb{R})_\varepsilon^m$ *with* $A_1 \approx [\underline{A}_1, \overline{A}_1]$, $A_2 \approx [\underline{A}_2, \overline{A}_2], ..., A_n \approx [\underline{A}_n, \overline{A}_n]$ *are strongly linearly independent, then* $\underline{A}_1, \underline{A}_2, ..., \underline{A}_n$ *and* $\overline{A}_1, \overline{A}_2, ..., \overline{A}_n$ *are strongly linearly independent.*

Proof. It is known that $A_1 \approx [\underline{A}_1, \overline{A}_1], A_2 \approx [\underline{A}_2, \overline{A}_2], ..., A_n \approx [\underline{A}_n, \overline{A}_n]$ are strongly linearly independent, then there is $b \approx [\underline{b}, \overline{b}] \in I(\mathbb{R})^m$ so that b can be uniquely expressed as linear combination of $A_1, ..., A_n$. Suppose that $b = \overline{\oplus}_{i=1}^n(\alpha_i \overline{\otimes} A_i)$; since

$$\overline{\oplus}_{i=1}^n (\alpha_i \overline{\otimes} A_i) \approx [\oplus_{i=1}^n (\underline{\alpha}_i \otimes \underline{A}_i),$$

$\oplus_{i=1}^{n}(\overline{\alpha}_i \otimes \overline{A}_i)]$, $\underline{b} = \oplus_{i=1}^{n}(\underline{\alpha}_i \otimes \underline{A}_i)$ and $\overline{b} = \oplus_{i=1}^{n}(\overline{\alpha}_i \otimes \overline{A}_i)$, It means that $\underline{b}$ can be uniquely expressed as a linear combination of $\underline{A}_1, \underline{A}_2, ..., \underline{A}_n$ and $\overline{b}$ can be uniquely expressed as a linear combination $\overline{A}_1, \overline{A}_2, ..., \overline{A}_n$. In other words, $\underline{A}_1, \underline{A}_2, ..., \underline{A}_n$ and $\overline{A}_1, \overline{A}_2, ..., \overline{A}_n$ are strongly linearly independent. □

The following is the definition of simple image set.

Definition 2.10. Given $A \in I(\mathbb{R})_{\varepsilon}^{m\times n}$. The set $IS_A = \{b \in I(\mathbb{R})^m \mid A \overline{\otimes} x = b$ has a unique solution$\}$ is defined as a simple image set of matrix A.

The regularity properties of the matrix are closely related to the number of the solutions of the system of linear equation. It can be stated in the following definitions and theorems.

Theorem 2.11. *Suppose that* $A \in I(\mathbb{R})_{\varepsilon}^{m\times n}$ *is a double* $I(\mathbb{R})$*-astic and* $b \in I(\mathbb{R})^m$ *is the number of the elements of* $IS(A, b)$ *is* $|IS(A, b)| \in \{0\ 1, \infty\}$.

Proof. Suppose that $A \in I(\mathbb{R})_{\varepsilon}^{m\times n}$ is double $I(\mathbb{R})$-astic and $b \in I(\mathbb{R})^m$. Suppose that $A \approx [\underline{A}, \overline{A}]$ and $b \approx [\underline{b}, \overline{b}]$ with each of $\underline{A}$ and $\overline{A}$ is double $\mathbb{R}$-astic, while $\underline{b}, \overline{b} \in \mathbb{R}^m$. Based on the Theorem 1.10, $|S(\underline{A}, \underline{b})| \in \{0, 1, \infty\}$ and $|S(\overline{A}, \overline{b})| \in \{0, 1, \infty\}$. Therefore, following the Definition 1.18, it is confirmed that $|IS(A, b)| \in \{0, 1, \infty\}$. □

Definition 2.12. Suppose that $A \in I(\mathbb{R})_{\varepsilon}^{m\times n}$, $T(A)$ is defined as $\{|IS(A, b)| \mid b \in \mathbb{R}^m\}$.

Theorem 2.13. *If* $A \in I(\mathbb{R})_{\varepsilon}^{m\times n}$ *is double* $I(\mathbb{R})$*-astic, then* $T(A) = \{0, \infty\}$ *or* $T(A) = \{0, 1, \infty\}$.

Proof. Suppose that $A \in I(\mathbb{R})_{\varepsilon}^{m\times n}$, $A \approx [\underline{A}, \overline{A}]$ with $\underline{A} \in \mathbb{R}_{\varepsilon}^{m\times n}$ and $\overline{A} \in \mathbb{R}_{\varepsilon}^{m\times n}$. Based on the theorem for the case of max-plus algebra, $T(\underline{A}) = \{0, 1, \infty\}$ or $T(\underline{A}) = \{0, \infty\}$ and $T(\overline{A}) = \{0, \infty\}$ or $T(\overline{A}) = \{0, 1, \infty\}$. Therefore, $T(A) = \{0, \infty\}$ or $(A) = \{0, 1, \infty\}$. □

Based on Theorem 2.13, A is a strongly regular matrix if and only if $1 \in T(A)$. Then, the criteria of strongly regular matrix is discussed.

Definition 2.14. Given $A = (a_{ij}) \in I(\mathbb{R})_{\varepsilon}^{n\times n}$ and $\pi \in P_n$, with P_n is the set of all permutations of $\{1, 2, ..., n\}$. $\omega(A, \pi)$ is defined as $\overline{\otimes}_{j\in n}\, a_{j,\pi(j)}$.

Theorem 2.15. *If* $A \in I(\mathbb{R})_{\varepsilon}^{n\times n}$ *double* $I(\mathbb{R})$*-astic and* $\pi \in P_n$ *with* $A \approx [\underline{A}, \overline{A}]$, *then* $\omega(A, \pi) = [\omega(\underline{A}, x)\ \omega(\overline{A}, \pi)]$.

Proof. If $A \in I(\mathbb{R})_{\varepsilon}^{m\times n}$ and $\pi \in P_n$ with $A \approx [\underline{A}, \overline{A}]$, it means that

$$\omega(A, \pi) = \overline{\otimes}_{j\in n} a_{j,\pi(j)} = \overline{\otimes}_{j\in n}[\underline{a}_{j,\pi(j)}, \overline{a}_{j,\pi(j)}]$$

$$= [\otimes_{j\in n}\, \underline{a}_{j,\pi(j)}, \otimes_{j\in n}\, \overline{a}_{j,\pi(j)}] = [\omega(\underline{A}, \pi), \omega(\overline{A}, \pi)].$$ □

Definition 2.16. The permanent of matrix A is defined as maper $\overline{\otimes}_{\pi\in P_n}\ \omega(A, \pi)$.

Next discussion is matrix over interval max-plus algebra which has weak permanent and strong permanent.

Definition 2.17. Suppose that $A \in I(\mathbb{R})_{\varepsilon}^{n\times n}$ double $I(\mathbb{R})$-astic with $A \approx [\underline{A}, \overline{A}]$. The set of all optimal permutations is $ap(A)$ with $ap(A) = \{\pi \in P_n \mid \text{maper}(\underline{A}) = \omega(\underline{A}, \pi) \text{ or maper}(\overline{A}) = \omega(\overline{A}, \pi)\}$. Matrix A is said to have *strong permanent* if $|\,ap(A)\,| = 1$ and it is said to have *weak permanent* if $|\,ap(\underline{A})\,| = 1$ and $|ap(\overline{A})| = 1$.

Theorem 2.18. *Suppose that* $A \in I(\mathbb{R})_{\varepsilon}^{n\times n}$ *is double* $I(\mathbb{R})$*-astic. If matrix A has strong permanent, then matrix A has weak permanent.*

Proof. Suppose that $A \in I(\mathbb{R})_{\varepsilon}^{n\times n}$, $A \approx [\underline{A}, \overline{A}]$ with $\underline{A} \in (\mathbb{R})_{\varepsilon}^{n\times n}$ and $\overline{A} \in \mathbb{R}_{\varepsilon}^{n\times n}$. Since A has strong permanent, it means that $|\,ap(A)\,| = 1$. Suppose that $\pi_1 \in ap(A)$, then $\text{maper}(\underline{A}) = \omega(\underline{A}, \pi_1)$ and $\text{maper}(\overline{A}) = \omega(\overline{A}, \pi_1)$. Therefore, $|\,ap(\underline{A})\,| = 1$ and $|ap(\overline{A})| = 1$. In other words, A has weak permanent. □

Theorem 2.19. *Suppose that $A \in I(\mathbb{R})_{\varepsilon}^{n\times n}$ is double $I(\mathbb{R})$-astic with $A \approx [\underline{A}, \overline{A}]$. Matrix A has weak permanent if and only if $\underline{A}$ and $\overline{A}$ have strong permanent.*

Proof. Suppose that $A \in I(\mathbb{R})_{\varepsilon}^{n\times n}$, $A \approx [\underline{A}, \overline{A}]$ with $\underline{A} \in \mathbb{R}_{\varepsilon}^{n\times n}$ and $\overline{A} \in \mathbb{R}_{\varepsilon}^{n\times n}$. It is known that matrix A has weak permanent. It means that $|\,ap(\underline{A})\,| = 1$ and $|ap(\overline{A})| = 1$. Therefore, matrix $\underline{A}$ and $\overline{A}$ have strong permanent. On the other hand, suppose that $\underline{A}$ and $\overline{A}$ have strong permanent, it means that $|\,ap(\underline{A})\,| = 1$ and $|ap(\overline{A})| = A$. Therefore, matrix A with $A \approx [\underline{A}, \overline{A}]$ has weak permanent. □

Theorem 2.20. *Suppose that $A \in I(\mathbb{R})_{\varepsilon}^{n\times n}$ is double $I(\mathbb{R})$-astic with $A \approx [\underline{A}, \overline{A}]$. Matrix A has strong permanent if and only if $\underline{A}$ and $\overline{A}$ have strong permanent as such that if $\text{maper}(\underline{A}) = \omega(\underline{A}, \pi_1)$ and $\text{maper}(\overline{A}) = \omega(\overline{A}, \pi_2)$, then $\pi_1 = \pi_2$.*

Proof. Suppose that $A \in I(\mathbb{R})_{\varepsilon}^{n\times n}$ with $A \approx [\underline{A}, \overline{A}]$. It is known that matrix A has strong permanent. Since matrix A has strong permanent, it means that $|\,ap(A)\,| = 1$. Suppose that $\pi \in ap(A)$ it means that $\pi \in P_n$ so that $\text{maper}(\underline{A}) = \omega(\underline{A}, \pi)$ and $\text{maper}(\overline{A}) = \omega(\overline{A}, \pi)$. Thus, $|\,ap(\underline{A})\,| = 1$ and $|ap(\overline{A})| = 1$. Therefore, $\underline{A}$ and $\overline{A}$ have strong permanent. On the other hand, it is known that matrix $\underline{A}$ and $\overline{A}$ have strong permanent. If $\text{maper}(\underline{A}) =$

$\omega(\underline{A}, \pi_1)$ and $\text{maper}(\overline{A}) = \omega(\overline{A}, \pi_2)$, then $\pi_1 = \pi_2$. Therefore, $|ap(\underline{A})| = 1$ and $|ap(\overline{A})| = 1$ with $\pi_1 \in |ap(\underline{A})|$, $\pi_2 \in |ap(\overline{A})|$ and $\pi_1 = \pi_2$. Since $ap(A) = \{\pi \in P_n \,|\, \text{maper}(\underline{A}) = \omega(\underline{A}, \pi) \text{ or } \text{maper}(\overline{A}) = \omega(\overline{A}, \pi)\}$, $|ap(A)| = 1$. Therefore, matrix A has strong permanent.

The following is theorem which has the characteristics of strongly regular matrix.

Theorem 2.21. *Suppose that* $A \in I(\mathbb{R})_\varepsilon^{n\times n}$ *is double* $I(\mathbb{R})$*-astic. Matrix A is strongly regular if and only if matrix A has strong permanent.*

Proof. Suppose that $A \in I(\mathbb{R})_\varepsilon^{n\times n}$ is double $I(\mathbb{R})$-astic, with $A \approx [\underline{A}, \overline{A}]$. It is known that A is strongly regular matrix. Suppose that $A_1, \ldots, A_n$ are the columns of matrix A. Since A is strongly regular matrix, there is $b \in I(\mathbb{R})^n$ so that b can be uniquely expressed as linear combination of $A_1, \ldots, A_n$, that is $b = \overline{\oplus}_{i=1}^n (x_i \,\overline{\otimes}\, A_i)$, $x_i \in I(\mathbb{R})$. The equation $b = \overline{\oplus}_{i=1}^n (x_i \,\overline{\otimes}\, A_i) \Leftrightarrow A \,\overline{\otimes}\, x = b$, is the system of linear equation in interval max-plus algebra which has a unique solution. Based on Corollary 1.23, $|ap(A)| = 1$. Therefore, matrix A has strong permanent. The converse, it is known that matrix A has strong permanent. Based on Theorem 2.20, matrix $\underline{A}$ and $\overline{A}$ have such strong permanent, so that if $\text{maper}(\underline{A}) = \omega(\underline{A}, \pi_1)$ and $\text{maper}(\overline{A}) = \omega(\overline{A}, \pi_2)$, $\pi_1 = \pi_2$. Therefore, matrix $\underline{A}$ and $\overline{A}$ are both strongly regular matrix, and thus, there are $\underline{b}$ and $\overline{b}$ which can be uniquely written as the linear combinations $\underline{b} = \oplus_{i=1}^n (\underline{x}_i \otimes \underline{A}_i)$ and $\overline{b} = \oplus_{i=1}^n (\overline{x}_i \otimes \overline{A}_i)$, in which $\underline{A}_1, \underline{A}_2, \ldots, \underline{A}_n$ are the columns of matrix $\underline{A}$, while $\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_n$ are the columns of matrix $\overline{A}$. That $\underline{b} = \oplus_{i=1}^n (\underline{x}_i \otimes \underline{A}_i) \Leftrightarrow \underline{A} \otimes \underline{x} = \underline{b}$ and $\overline{b} = \oplus_{i=1}^n (\overline{x}_i \otimes \overline{A}_i)$, $\Leftrightarrow \overline{A} \otimes \overline{x} = \overline{b}$. Thus, the system of linear equation $\underline{A} \otimes \underline{x} = \underline{b}$ and $\overline{A} \otimes \overline{x} = \overline{b}$ have unique solution. Since $\operatorname{maper}(\underline{A}) = \omega(\underline{A}, \pi_1)$ and $\operatorname{maper}(\overline{A}) = \omega(\overline{A}, \pi_2)$, $\pi_1 = \pi_2$ so that $[\underline{b}, \overline{b}] \approx b$. As the result, $[\underline{A} \otimes \underline{x}, \overline{A} \otimes \overline{x}] = [\underline{b}, \overline{b}]$ and since $A \overline{\otimes} x \approx [\underline{A} \otimes \underline{x}, \overline{A} \otimes \overline{x}]$, $A \overline{\otimes} x = b$ or $b = \overline{\oplus}_{i=1}^{n} (x_i \overline{\otimes} A_i)$ with $A_1, \ldots, A_n$ as the columns of matrix A. It means that there is b which can be uniquely expressed as linear combination of $A_1, \ldots, A_n$. This matrix A is known as strongly regular matrix. □

3. Concluding Remarks

Based on the discussion, the following conclusions are obtained:

1. The necessary and sufficient condition of matrix $A \in I(\mathbb{R})_{\varepsilon}^{n \times n}$, which is double $I(\mathbb{R})$-astic is strongly regular matrix.

2. The system of linear equation in the interval max-plus algebra $A \overline{\otimes} x = b$, $A \in I(\mathbb{R})_{\varepsilon}^{n \times n}$ which is double $I(\mathbb{R})$-astic and $b \in I(\mathbb{R})^n$ has unique solution if and only if matrix A has strong permanent.

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