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Optimizing Range Norm of The Image Set of Matrix over Interval Max-Plus Algebra with Prescribed Components

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Abstract. Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$ whose $\varepsilon = -\infty$. Max-plus algebra is the set $\mathbb{R}_\varepsilon$ that is equipped two operations maximum and addition. It can be formed matrices in the size of $m \times n$ whose elements belong to $\mathbb{R}_\varepsilon$, called matrix over max-plus algebra. Optimizing range norm of the image set of matrix over max-plus algebra with prescribed components has been discussed. Interval Max-Plus Algebra is the set $I(\mathbb{R})_\varepsilon = \{ \mathrm{x} = [\underline{\mathrm{x}}, \overline{\mathrm{x}}] | \underline{\mathrm{x}}, \overline{\mathrm{x}} \in \mathbb{R}, \varepsilon < \underline{\mathrm{x}} \leq \overline{\mathrm{x}} \} \cup \{\varepsilon\}$ with $\varepsilon = [\varepsilon, \varepsilon]$, is equipped with two operations maximum $(\overline{\oplus})$ and addition $(\overline{\otimes})$. The set of all matrices in the size of $m \times n$ whose elements belong to $I(\mathbb{R})_\varepsilon$, called matrix over interval max-plus algebra. Optimizing range norm of the image set of matrix over interval max-plus algebra has been discussed. In this paper, we will discuss optimizing range norm of the image set of matrix over interval max-plus algebra with prescribed components.

1. Introduction

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$ where $\varepsilon = -\infty$. Max-Plus algebra is the set $\mathbb{R}_\varepsilon$ that is equipped two operations maximum $(\oplus)$ and addition $(\otimes)$. It can be formed matrices in the size of $m \times n$ whose elements belong to $\mathbb{R}_\varepsilon$. Max-plus algebra has been used to model and analyze algebraically planning problems, communication, production system, queueing system with finite capacity, parallel computation, and traffic (Bacelli; *et*, *al.* [1]). Min-Plus algebra is the set $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon'\}$ where $\varepsilon' = -\infty$ which is equipped two operations minimum $(\oplus')$ and addition $(\otimes')$. Tam [3] also disccus about complete max-plus algebra that is the $\overline{\mathbb{R}}_\varepsilon = \mathbb{R} \cup \{\varepsilon, \varepsilon'\}$ is equipped with operations $\oplus$ and $\otimes$, while complete min-plus algebra is $\overline{\mathbb{R}}_{\varepsilon'} = \mathbb{R} \cup \{\varepsilon, \varepsilon'\}$ which is equipped operations $\oplus'$ and $\otimes'$. Then $\overline{\mathbb{R}}_\varepsilon$ and $\overline{\mathbb{R}}_{\varepsilon'}$ are written as $\overline{\mathbb{R}}$.

The set of all matrices in the size of $m \times n$ over $\mathbb{R}_\varepsilon$, $\mathbb{R}_{\varepsilon'}$ and $\overline{\mathbb{R}}$ can be formed. It is called the set of all matrices over max-plus algebra if its components belong to $\mathbb{R}_\varepsilon$ and denoted by $\mathbb{R}_\varepsilon^{m\times n}$. Moreover, if $n = 1$ we obtain the set of all vectors over max-plus algebra, namely $\mathbb{R}_\varepsilon^m = \{(x_1, x_2, \dots, x_m)^T | x_1, x_2, \dots, x_m \in \mathbb{R}_\varepsilon\}$. If $m = n$, $\mathbb{R}_\varepsilon^{n\times n}$ is idempotent semiring with operations $\oplus$ and

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$\otimes$ (Farlow [2], Akian, *et. al.* [4], Konigsberg [12]). Tam [3] gives an example of application of max-plus algebra in the production system. The production matrix $A = (A_{ij}) \in \mathbb{R}_{\varepsilon}^{n\times n}$ with A_{ij} shows the time is used of the production process from machine *j* to *i*, while vector $x(k) = \left(x_i(k)\right) \in \mathbb{R}_{\varepsilon}^{n}$ where $x_i(k)$ is the starting time of i machine at the k^{th} stage. In this production process we obtain the equation $x(k+1) = A \otimes x(k)$.

One of the criteria is used by manufacturers is that the production process runs periodically with period λ, such that $x(k+1) = \lambda \otimes x(k)$. From equations $x(k+1) = A \otimes x(k)$ and $x(k+1) = \lambda \otimes x(k)$ are obtained $A \otimes x(k) = \lambda \otimes x(k)$. The eigenvalue and eigenvector problem of matrix A is problem of how to find eigenvalue λ and eigen vector $x(k)$ such that $A \otimes x(k) = \lambda \otimes x(k)$.

In the real life situation, there are several ways for the manufacturer to determine the starting time of each machine. One of the ways is by choosing eigenvector as the starting time, so that the system will immediately reach the steady state; that is the process works periodically with the eigenvalue as its periods.

However, in reality, there are more than one independent eigenvectors for the manufacturer to choose. In that case, a set of linear combination of the independent eigenvectors is constructed, and as such, the manufacturer needs an additional criteria to choose one element of the set. The additional criteria is by considering the difference between the largest and the smallest of the starting time of each machine. The difference is expressed as range norm of the starting time of each machine. Manufacturer can optimize the range norm of the starting time of each machine (Tam [3], Butkovic and Tam [6]). There are some determining factors of a manufacturer to optimize vector of range norm of the starting time of machine, that are the readiness of the raw material, the availability of the resources, and the distribution of the product. In solving the problems of a production system, Tam [3] has discussed optimizing range norm of the image set of matrix over max-plus algebra.

Besides the criteria that have been submitted previously, manufacturers can also determine the start time of the machine began to work on certain machines. In this case, means that there are components that have been determined at the initial time vector machine start working. Furthermore, manufacturers only determine the initial time for the machine left so initial time vector is an eigenvector with optimal range norm.

Based on Kreinovich's idea [11] about the probability to give an estimation of the period of time of a certain process and Rudhito [5] about generalization of max-plus algebra i.e. interval max-plus algebra, we will to expand concept in max-plus algebra which has been discussed by Tam [3] in interval max-plus algebra. Siswanto, *et, al.* [9] has discussed about existence of solution to the system of linear Equations in interval max-Plus algebra. Siswanto, *et, al.* [8] has discussed the optimizing range norm of the image set of matrix over interval max-plus algebra. In this paper, we will discuss how to minimize and maximize range norm of the image set of matrix over interval max-plus algebra with prescribed components. Specially, about optimizing range norm of the image set of matrix over interval max-plus algebra with one component is prescribed and another one is unprescribed.

Before discussing the result of this paper, several concepts which support the discussion are discussed. These are minimizing and maximizing range norm of the image set of a matrix over max-plus algebra with prescribed components. Other than that optimizing range norm of the image set of a matrix over interval max-plus algebra.

2. Preliminaries

2.1. Minimizing Range Norm

The following the concepts about the problem minimizing of range norm of eigenvector of matrices over maks-plus algebra will be presented.

Definition 2.1. *If* $x \in \mathbb{R}^m$ *then we will denote the function* $\delta(x) = \sum_{i\in M}^{\oplus} x_{\mathrm{i}} - \sum_{i\in M}^{\oplus'} x_{\mathrm{i}}$ *and called the range norm of* x*, i.e. range norm is largest value in* x *– smallest value in* x.

Problem 2.1. *Given a matrix* $A \in \mathbb{R}_{\varepsilon}^{m \times n}$, *solve* $\delta(b)$ *minimum, subject to* $b \in Im\,(A)$.

Definition 2.2. *If* $x \in \mathbb{R}_{\varepsilon}^{m}$ *then the function* $\tilde{\delta}(x) = \sum_{\substack{i \in M \\ x_i \neq \varepsilon}}^{\oplus} x_i - \sum_{\substack{i \in M \\ x_i \neq \varepsilon}}^{\oplus'} x_i$ *is the range norm of x after only considering the finite components.*

Problem 2.2. *Given a matrix* $A \in \mathbb{R}_{\varepsilon}^{m \times n}$, *solve* $\tilde{\delta}(b)$ *minimum, subject to* $b \in Im\,(A)$.

Futhermore, the minimizing of a norm range eigenvector with prescribed components are presented.

Definition 2.3. *Let* $A \in \mathbb{R}_{\varepsilon}^{m\times n}$ *and* $1 \leq p \leq m-1$. *We will denote* A_1^p *to be the matrix that consist of the first p row(s) of A and* A_2^p *to be the matrix generated by deleting the first p row(s) of A.*

Problem 2.3. *Given* $A \in \mathbb{R}_{\varepsilon}^{m\times n}$ *and* $c \in Im(A_1^p) \cap \mathbb{R}^p$, $1 \leq p \leq m-1$. *Find* $d \in \mathbb{R}^{m-p}$ *such that* $\delta(b)$ *minimum, subject to* $b \in Im(A)$ *and* $b^T = (c^T, d^T)$.

Before we discussed the problem for all case of p, the two special cases for this problem when $p = 1$ and $p = m-1$ are presented.

Theorem 2.4. *Let* $A \in \mathbb{R}_{\varepsilon}^{m\times n}$ *be doubly* $\mathbb{R}$*-astic and* $c \in Im(A_1^p) \cap \mathbb{R}^p$, $1 \leq p \leq m-1$. *Suppose that* $\bar{b} \in \mathbb{R}^m$ *is an optimal solution to Problem 1.1 for A and* $d \in \mathbb{R}^{m-p}$ *is an optimal solution for Problem 2.3 for A and* c, *then* $\delta(\bar{b}) \leq \delta(b)$
where $b^T = (c^T, d^T)$.

Theorem 2.4 guarantee lower bound of range norm which is obtained from optimal solution Problem 2.3. Using this result of the Problem 2.3, be obtained the solution of $p = 1$. Suppose that only the starting time of one machine is prescribed, then by Theorem 2.4 can be obtained the result as the following.

Theorema 2.5. *Let* $A \in \mathbb{R}_{\varepsilon}^{m\times n}$ *be doubly* $\mathbb{R}$*-astic and* $c = (c_1) \in \mathbb{R}$ *be the vector of prescribed component. Suppose that* $b = A \otimes (A^* \otimes' 0)$ *and* $d = (c_1 \otimes b_1^{-1}) \otimes$ b where b_1 *is the first component of b. Then the vector* $(d_2, \ldots, d_m)^T$ *is an optimal solution to Problem 2.3.*

Futhermore, the case of $p = m-1$, used the following theorem.

Theorema 2.6. *Let* $A \in \mathbb{R}_{\varepsilon}^{m\times n}$ *be doubly* $\mathbb{R}$*-astic,* $c \in Im(A_1^p) \cap \mathbb{R}^p$ *and* $\tilde{d}$ *be an optimal solution for Problem 2.3. Suppose* $\bar{x} = \left(A_1^p\right)^* \otimes' c$ *and* $\bar{d} = A_2^p \otimes \bar{x}$ *then* $\tilde{d} \leq \bar{d}$.

2.2. Maximizing Range Norm

The following the concepts be related with the problem maximizing of eigenvector range norm of matrices over maks-plus algebra will be presented.

Problem 2.4. *Given* $A \in \mathbb{R}_{\varepsilon}^{m\times n}$, *solve* $\delta(b)$ *maximum, subject to* $b \in Im\,(A)$.

The maximizing range norm of the image set of a matrix over max-plus algebra with prescribed components will be presented.

Problem 2.5. *Given* $A \in \mathbb{R}_{\varepsilon}^{m\times n}$ *and* $c \in Im(A_1^p) \cap \mathbb{R}^p$, $1 \leq p \leq m-1$, *find* $d \in \mathbb{R}^{m-p}$ *such that* $\delta(b)$ *maximum, subject to* $b \in Im(A)$ *and* $b^T = (c^T, d^T)$.

Before discussing the problem for all case of p, the two special cases for this problem when $p = 1$ and $p = m-1$ are presented. First discussed the cases where only one component prescribed that is $p = 1$. In the case of this special, Problem 2.5 can be considered the same as it counterpart Problems 2.4. This can be seen in the following theorem.

Theorem 2.7. *Let* $A \in \mathbb{R}^{m\times n}$ *and* $c = (c_1) \in \mathbb{R}$ *be the prescribed component. Suppose that* $b \in Im(A)$ *such that b is an optimal solution to Problem 2.4 and* $d = (c_1 \otimes b_1^{-1}) \otimes b$ *where* b_1 *is the first component of b, then vector* $(d_2, \ldots, d_m)^T$ *is an optimal solution to Problem 2.5.*

Teorema 2.8. *Let* $A \in \mathbb{R}^{m\times n}$ *be doubly* $\mathbb{R}$*-astic and non finite, i.e.* $\exists i \in M,\ j \in N$ *such that* $a_{ij} = \varepsilon$. *Suppose that* $c = (c_1) \in \mathbb{R}$ *is the vector of prescribed component then Problem 2.5 is unbounded.*

Futhermore, we will consider the case when $p = m-1$.

Theorem 2.9. *Let* $A \in \mathbb{R}_{\varepsilon}^{m\times n}$ *be doubly* $\mathbb{R}$*-astic,* $c \in Im\left(A_1^p\right) \cap \mathbb{R}^p$ *and* $\tilde{d}$ *be an optimal solution to Problem 2.5. Suppose that* $\bar{x} = \left(A_1^p\right)^* \otimes' c$ *and* $\bar{d} = A_2^p \otimes \bar{x}$ *then* $\tilde{d} \leq \bar{d}$.

Besides the concepts above namely about max-plus algebra, interval min-plus algebra, complete interval max-plus algebra, and complete interval algebra min-plus (Rudhito [5]; Siswanto [7]), also are used the concepts about optimizing a range norm the set of image matrix over interval algebra max-plus (Siswanto [10]).

2.3. Minimizing Range Norm of The Image Set of Matrix over Interval Max Plus Algebra

Definition 2.10. *Suppose that* $\mathrm{x} \in I(\mathbb{R})^m$ *where* $\mathrm{x} \approx [\underline{\mathrm{x}}, \overline{\mathrm{x}}]$; $\underline{\mathrm{x}}, \overline{\mathrm{x}} \in \mathbb{R}^m$. *The function* $\delta(\mathbf{x}) = \left[min\left(\delta(\underline{\mathrm{x}}), \delta(\overline{\mathrm{x}})\right), \delta(\overline{\mathrm{x}})\right]$ *is called the range norm of* x.

Definition 2.11. *Given that matrix* $\mathrm{A} \in I(\mathbb{R})_\varepsilon^{m\times n}$ *with* $\mathrm{A} \approx [\underline{\mathrm{A}}, \overline{\mathrm{A}}]$; $\underline{\mathrm{A}}, \overline{\mathrm{A}} \in \mathbb{R}_\varepsilon^{m\times n}$, *then* $Im(\mathrm{A}) = \{\mathrm{A} \overline{\otimes} \mathrm{p} \mid \mathrm{p} \in I(\mathbb{R})_\varepsilon^n\}$.

Problem 2.6 *Given that matrix* $\mathrm{A} \in I(\mathbb{R})_\varepsilon^{m\times n}$, *solve* $\delta(\mathrm{b})$ *minimize, subject to* $b \in \mathrm{Im}\,(\mathrm{A}), \mathrm{b} \approx [\underline{\mathrm{b}}, \overline{\mathrm{b}}]$.

Definisi 2.12. *Given that matrix* $\mathrm{b} \in I(\mathbb{R})_\varepsilon^m$ *with* $\mathrm{b} \approx [\underline{\mathrm{b}}, \overline{\mathrm{b}}]$. *The function* $\delta(\mathrm{b})$ *is said minimum if and only if* $\delta(\overline{\mathrm{b}})$ *and* $\delta(\underline{\mathrm{b}})$ *are minimum.*

Definition 2.13. *Suppose that* $\mathrm{x} \in I(\mathbb{R})^m$ *with* $\mathrm{x} \approx [\underline{\mathrm{x}}, \overline{\mathrm{x}}]$. *The function* $\tilde{\delta}(\mathrm{x}) = \left[\min\left(\tilde{\delta}(\underline{\mathrm{x}}), \tilde{\delta}(\overline{\mathrm{x}})\right), \tilde{\delta}(\overline{\mathrm{x}})\right]$ *is the range norm of* x, *after only considering the finite components.*

Problem 2.7. *Given that matrix* $\mathrm{A} \in I(\mathbb{R})_\varepsilon^{m\times n}$, $\mathrm{A} \approx [\underline{\mathrm{A}}, \overline{\mathrm{A}}]$. *Solve* $\tilde{\delta}(\mathrm{b})$ *minimize, subject to* $\mathrm{b} \in Im\,(A), \mathrm{b} \approx [\underline{\mathrm{b}}, \overline{\mathrm{b}}]$.

Definition 2.14. *Given that matrix* $\mathrm{b} \in I(\mathbb{R})_\varepsilon^m$ *with* $\mathrm{b} \approx [\underline{\mathrm{b}}, \overline{\mathrm{b}}]$. *The function* $\tilde{\delta}(b)$ *is said minimum if and only if* $\tilde{\delta}(\underline{b})$ *and* $\tilde{\delta}(\overline{\mathrm{b}})$ *are minimum.*

Definition 2.15. *Suppose that* $\mathrm{A} \in I(\mathbb{R})_\varepsilon^{m\times n}$, $\mathrm{A} \approx [\underline{\mathrm{A}}, \overline{\mathrm{A}}]$ *is called double* $I(\mathbb{R})$*- astic if* A *is double* $\mathbb{R}$*-astic for every* $\mathrm{A} \in [\underline{\mathrm{A}}, \overline{\mathrm{A}}]$.

Theorem 2.16. *The matrix* $\mathrm{A} \in \mathrm{I}(\mathbb{R})_\varepsilon^{m\times n}$ *where* $\mathrm{A} \approx [\underline{\mathrm{A}}, \overline{\mathrm{A}}]$ *is double* $I(\mathbb{R})$*- astic if and only if* $\underline{\mathrm{A}}$ *is double* $\mathbb{R}$*-astic .*

Lemma 2.17. *Suppose that* $A \in I(\mathbb{R})_\varepsilon^{m\times n}$ *where* $\mathrm{A} \approx [\underline{\mathrm{A}}, \overline{\mathrm{A}}]$. *If* $\mathrm{A} = [\mathrm{A}_{ij}]$ *be doubly* $I(\mathbb{R})$-astic, $\mathrm{b} \approx [\underline{\mathrm{b}}, \overline{\mathrm{b}}]$, $\underline{\mathrm{b}} = \underline{\mathrm{A}} \otimes (\underline{\mathrm{A}}^* \otimes' v_0)$, $\overline{\mathrm{b}} = \alpha \otimes \left(\overline{\mathrm{A}} \otimes (\overline{\mathrm{A}}^* \otimes' v_0)\right)$ *where* $v_0 \in \mathbb{R}^m$ *be a vector whose every components is equal to a component* 0 *and* $\alpha = max_i\left(\left(\underline{\mathrm{A}} \otimes (\underline{\mathrm{A}}^* \otimes' \mathrm{v}_0)\right)_\mathrm{i} - \left(\overline{\mathrm{A}} \otimes (\overline{\mathrm{A}}^* \otimes' \mathrm{v}_0)\right)_\mathrm{i}\right) \geq 0$ *then*

i. $\mathrm{b} \leq ([0, \alpha], [0, \alpha], \ldots., [0, \alpha])^T$,

ii. $\underline{\mathrm{b}}_i = 0$ *for some* $i \in M$,

iii. $\overline{\mathrm{b}}_j = \alpha$ *for some* $j \in M$.

Lemma 2.18. *If* $\mathrm{x}, \mathrm{y} \in I(\mathbb{R})^m$ *and* $\alpha \in I(\mathbb{R})$ *where* $\mathrm{x} \approx [\underline{\mathrm{x}}, \overline{\mathrm{x}}]$, $\mathrm{y} \approx [\underline{\mathrm{y}}, \overline{\mathrm{y}}]$ *and* $\mathrm{a} \approx [\underline{\mathrm{a}}, \overline{\mathrm{a}}]$ *then* $\delta(\mathrm{x}) = \delta(\mathrm{a} \overline{\otimes} \mathrm{x})$.

Theorem 2.19. *Let* $\mathrm{A} \in I(\mathbb{R})_\varepsilon^{m\times n}$ *be double* $I(\mathbb{R})$*-astic and* $v_0 \in \mathbb{R}^m$ *be a vector whose every components is equal to a constant 0 then* $\mathrm{b} \approx [\underline{\mathrm{b}}, \overline{\mathrm{b}}]$ *is a solution for Problem 2.6 where* $\underline{\mathrm{b}} = \underline{\mathrm{A}} \otimes (\underline{\mathrm{A}}^* \otimes' v_0)$, $\overline{\mathrm{b}} = \alpha \otimes \left(\overline{\mathrm{A}} \otimes \left(\overline{\mathrm{A}}^* \otimes' v_0\right)\right)$ *and* $\alpha = max_\mathrm{i}\left(\left(\underline{\mathrm{A}} \otimes (\underline{\mathrm{A}}^* \otimes' v_0)\right)_i - \left(\overline{\mathrm{A}} \otimes (\overline{\mathrm{A}}^* \otimes' v_0)\right)_i\right)$.

To determine solution of Problem 2.7 the next theorem is needed :

Theorem 2.20. *Suppose that* $\mathrm{A} \in I(\mathbb{R})_\varepsilon^{m\times n}$, $\mathrm{A} \approx [\underline{\mathrm{A}}, \overline{\mathrm{A}}]$ *double* $I(\mathbb{R})$*-astic and* $\mathrm{v}_\mathrm{a} \approx [v_{\underline{\mathrm{a}}}, v_{\overline{\mathrm{a}}}] \in I(\mathbb{R})^m$ *be a vector whose every components is equal to an interval* $\mathrm{a} \approx [\underline{\mathrm{a}}, \overline{\mathrm{a}}]$. *If* $\mathrm{x} \in I(\mathbb{R})_\varepsilon^n$ *such that* $\mathrm{b} = \mathrm{A} \overline{\otimes} \mathrm{x}, \mathrm{b} \approx [\underline{\mathrm{b}}, \overline{\mathrm{b}}] \in Im(\mathrm{A})$ *is solution of Problem 2.7 then* $\forall j \in N$ *satisfy the conditions* :

a. $\mathrm{x}_j = \varepsilon$ *or*

b. $\mathrm{x}_j = \left[\left(\underline{\mathrm{A}}^* \otimes' v_{\underline{\mathrm{a}}}\right)_j, \left(\overline{\mathrm{A}}^* \otimes' v_{\overline{\mathrm{a}}}\right)_j\right]$.

2.4. Maximizing Range Norm of The Image Set of Matrix over Interval Max-Plus Algebra

Problem 2.8 *Given that matrix $A \in I(\mathbb{R})_\varepsilon^{m\times n}$, solve : maximize $\delta(b)$ subject to $b \in Im\,(A)$, $b \approx [\underline{b}, \overline{b}]$.*

Definition 2.22. *The function* $\delta(\mathrm{b})$ *is called maximum if and only if* $\delta(\overline{\mathrm{b}})$ *and* $\delta(\underline{\mathrm{b}})$ *are maximum.*

Theorem 2.23. *If* $\mathrm{A} \in I(\mathbb{R})^{m\times n}$ *then*

$\delta(\mathrm{b}) \leq \left[min\left(\underset{j=1,2,\ldots n}{maks}\ \delta(\underline{\mathrm{A}}_{\mathrm{j}}), max_{j=1,2,\ldots n}\ \delta(\overline{A}_j)\right), max_{j=1,2,\ldots n}\ \delta(\overline{\mathrm{A}}_{\mathrm{j}})\right]$ *for every* $\mathrm{b} \in Im(\mathrm{A})$.

Theorem 2.24. *Suppose that $M = \{1,2,\ldots,m\}$ and $N = \{1,2,\ldots,n\}$. If $A \in I(\mathbb{R})_\varepsilon^{m\times n}$ be doubly $I(\mathbb{R})$-astic and there are $i \in M, j \in N$ such that $a_{ij} = \varepsilon$ then Problem 2.8 unbounded.*

3. Results and Discussion

This section will discuss the results of this research about optimizing range norm of the image set of matrix over interval max-plus algebra with some components are prescribed.

3.1. Minimizing Range Norm

Some definitions and theorems also problems the minimizing of a norm range the image set of matrix where one component is prescribed and one component is unprescribed, given as follows.

Definition 3.1. *Let* $\mathrm{A} \in I(\mathbb{R})_\varepsilon^{m\times n}$ *and* $1 \leq p \leq m-1$. *We will denote* A_1^p *to be the matrix that consists of the first p row (rows) of A and* A_2^p *to be the matrix generated by deleting the first p row (rows) of A.*

Problem 3.1. *Given* $\mathrm{A} \in I(\mathbb{R})_\varepsilon^{m\times n}$ *and* $c \in Im\left(\mathrm{A}_1^p\right) \cap I(\mathbb{R})^p$, $1 \leq p \leq m-1$. *Find* $\mathrm{d} \in I(\mathbb{R})^{m-p}$ *such that* $\delta(\mathrm{b})$ *minimum subject to* $\mathrm{b} \in Im(\mathrm{A})$ *and* $\mathrm{b}^T = (\mathrm{c}^T, \mathrm{d}^T)$.

To solve this problem, started by the two special cases for $p = 1$ and $p = m-1$. Let us consider the following theorem.

Theorem 3.2 *Let* $\mathrm{A} \in I(\mathbb{R})_\varepsilon^{m\times n}$ *be doubly* $I(\mathbb{R})$*-astic and* $c \in Im\left(A_1^p\right) \cap I(\mathbb{R})^p$, $1 \leq p \leq m-1$. *Suppose that* e *is an optimal solution to Problem 2.6 for* A *and* $\mathrm{d} \in I(\mathbb{R})^{m-p}$ *is an optimal solution to Problem 3.1 for* A *and* c *then* $\delta(\mathrm{e}) \leq \delta(\mathrm{b})$ *where* $\mathrm{b}^T = (\mathrm{c}^T, \mathrm{d}^T)$.

Proof. Suppose that $\mathrm{A} \approx [\underline{\mathrm{A}}, \overline{\mathrm{A}}]$, $\mathrm{b} \approx [\underline{\mathrm{b}}, \overline{\mathrm{b}}]$, $\mathrm{c} \approx [\underline{\mathrm{c}}, \overline{\mathrm{c}}]$, $\mathrm{d} \approx [\underline{\mathrm{d}}, \overline{\mathrm{d}}]$ and $\mathrm{e} \approx [\underline{\mathrm{e}}, \overline{\mathrm{e}}]$. Let $\mathrm{A} \in I(\mathbb{R})_\varepsilon^{m\times n}$ be doubly $I(\mathbb{R})$-astic and $\mathrm{c} \in Im\left(\mathrm{A}_1^{\mathrm{p}}\right) \cap I(\mathbb{R})^p$, $1 \leq p \leq m-1$. Therefore $\underline{\mathrm{A}}$, $\overline{\mathrm{A}} \in \mathbb{R}_\varepsilon^{m\times n}$ be doubly $\mathbb{R}$-astic and $\underline{\mathrm{c}} \in Im\left(\underline{\mathrm{A}}_1^{\mathrm{p}}\right) \cap \mathbb{R}^p$, $\overline{\mathrm{c}} \in Im\left(\overline{\mathrm{A}}_1^{\mathrm{p}}\right) \cap \mathbb{R}^p$. Suppose e is an optimal solution to Problem 2.6 for A and $\mathrm{d} \in I(\mathbb{R})^{m-p}$ is an optimal solution to Problem 3.1 for A and c. Since $\mathrm{e} \in Im\,(\mathrm{A})$, so $\underline{\mathrm{e}} \in Im\left(\underline{\mathrm{A}}\right)$, $\overline{\mathrm{e}} \in Im\,(\underline{\mathrm{A}})$. with $\underline{\mathrm{e}}$ is an optimal solution to Problem 2.1 for $\underline{\mathrm{A}}$ and $\overline{\mathrm{e}}$ is an optimal solution to Problem 2.1 for $\overline{\mathrm{A}}$. While $\underline{\mathrm{d}} \in \mathbb{R}^{m-p}$ is an optimal solution to Problem 2.3 for $\underline{\mathrm{A}}$ and $\underline{\mathrm{c}}$ and $\overline{\mathrm{d}} \in \mathbb{R}^{m-p}$ is an optimal solution to Problem 2.3 for $\overline{\mathrm{A}}$ and $\overline{\mathrm{c}}$. Acording to Theorem 2.4, $\delta(\underline{\mathrm{e}}) \leq \delta(\underline{\mathrm{b}})$ and $\delta(\overline{\mathrm{e}}) \leq \delta(\overline{\mathrm{b}})$ where $\underline{\mathrm{b}}^T = (\underline{\mathrm{c}}^T, \underline{\mathrm{d}}^T)$ and $\overline{\mathrm{b}}^T = (\overline{\mathrm{c}}^T, \overline{\mathrm{d}}^T)$. As a result $\delta(\mathrm{e}) \leq \delta(\mathrm{b})$ where $\mathrm{b}^T = (\mathrm{c}^T, \mathrm{d}^T)$. ∎

Theorem 3.2 provides a lower bound for the range norm of the optimal solution to Problem 3.1. Suppose that only the starting time of one machine is prescribed then by Theorem 2.20 and Theorem 3.2 are obtained results as follows.

Theorem 3.3 *Let* $\mathrm{A} \approx [\underline{\mathrm{A}}, \overline{\mathrm{A}}] \in I(\mathbb{R})_\varepsilon^{m\times n}$ *be doubly* $I(\mathbb{R})$*-astic*, $\mathrm{c} = \left([\underline{\mathrm{c}}_1, \overline{\mathrm{c}}_1]\right) \in I(\mathbb{R})$ *be the vector of prescribed components,* $v_0 \in \mathbb{R}^m$ *be a vector whose every component is equal to a constant* 0. *If* $\underline{\mathrm{b}} = \underline{\mathrm{A}} \otimes \left(\underline{\mathrm{A}}^* \otimes' v_0\right)$, $\overline{\mathrm{b}} = \alpha \otimes \left(\overline{\mathrm{A}} \otimes \left(\overline{\mathrm{A}}^* \otimes' v_0\right)\right)$ *where* $\alpha = max_{\mathrm{i}}\left(\left(\underline{\mathrm{A}} \otimes \left(\underline{\mathrm{A}}^* \otimes' v_0\right)\right)_i - \left(\overline{\mathrm{A}} \otimes \left(\overline{\mathrm{A}}^* \otimes' v_0\right)\right)_i\right) \geq 0$, $\underline{\mathrm{d}} = (\underline{\mathrm{c}}_1 \otimes \underline{\mathrm{b}}_1^{-1}) \otimes \underline{\mathrm{b}}$, $\overline{\mathrm{d}} = (\overline{\mathrm{c}}_1 \otimes \overline{\mathrm{b}}_1^{-1}) \otimes \overline{\mathrm{b}}$ *when* $\underline{\mathrm{d}} \leq \overline{\mathrm{d}}$ *then* $\mathrm{d}_1^1 = (\mathrm{d}_2, \mathrm{d}_3, \ldots, \mathrm{d}_m)$ *is an optimal solution to Problem 3.1.*

Proof. Since $\mathrm{A} \in I(\mathbb{R})_\varepsilon^{m\times n}$ be doubly $I(\mathbb{R})$-astic then $\underline{\mathrm{A}}$, $\overline{\mathrm{A}}$ be doubly $\mathbb{R}$-astic. Since $\mathrm{c} = (\mathrm{c}_1) = \left([\underline{\mathrm{c}}_1, \overline{\mathrm{c}}_1]\right) \in I(\mathbb{R})$ then $\underline{\mathrm{c}} = (\underline{\mathrm{c}}_1)$, $\overline{\mathrm{c}} = (\overline{\mathrm{c}}_1) \in \mathbb{R}$ the vector of prescribed components. If $\underline{\mathrm{b}} = \underline{\mathrm{A}} \otimes$

$(\underline{A}^* \otimes' v_0)$ and $\overline{b} = \alpha \otimes \left(\overline{A} \otimes \left(\overline{A}^* \otimes' v_0\right)\right)$ where $\alpha = max_i \left(\left(\underline{A} \otimes (\underline{A}^* \otimes' v_0)\right)_i - \left(\overline{A} \otimes (\overline{A}^* \otimes' v_0)\right)_i\right) \geq 0$, $\underline{d} = (\underline{c}_1 \otimes \underline{b}_1^{-1}) \otimes \underline{b}$, $\overline{d} = (\overline{c}_1 \otimes \overline{b}_1^{-1}) \otimes \overline{b}$ when $\underline{d} \leq \overline{d}$. According to Theorem 2.5, the vector $\left(\underline{d}_2, \ldots, \underline{d}_m\right)^T$ is an optimal solution to Problem 2.3 for $\underline{c}_1$ and $\underline{A}$, while $\left(\overline{d}_2, \ldots, \overline{d}_m\right)^T$ is an optimal solution to Problem 2.3 for $\overline{c}_1$ and $\overline{A}$. Therefore vector $(d_2, \ldots, d_m)^T$ where $d_2 \approx \left[\underline{d}_2, \overline{d}_2\right], \ldots, d_m \approx [\underline{d}_m, \overline{d}_m]$ is an optimal solution to Problem 3.1 for c and A. ■

Next, in the case of $p = m - 1$ we needed to find the fisibel solution for d. Given $A \in I(\mathbb{R})_\varepsilon^{m\times n}$ be doubly $I(\mathbb{R})$-astic and $c \in Im(A^p) \cap I(\mathbb{R})^p$ be the vector of prescribed components that $c \approx \left[\underline{c}, \overline{c}\right]$, then there are $\hat{x} \in I(\mathbb{R})_\varepsilon^n$ where $\hat{x} = (A_1^{m-1})^* \overline{\otimes}' c$ such that $c = A^{m-1} \overline{\otimes}\, \hat{x}$. By using $\hat{x}$ is obtained $g = A_2^{m-1} \overline{\otimes}\, \hat{x}$. According to equation,

$$A \overline{\otimes}\, \hat{x} = \begin{pmatrix} A_1^{m-1} \\ A_2^{m-1} \end{pmatrix} \overline{\otimes}\, \hat{x} = \begin{pmatrix} A_1^{m-1} \overline{\otimes}\, \hat{x} \\ A_2^{m-1} \overline{\otimes}\, \hat{x} \end{pmatrix} = \begin{pmatrix} c \\ g \end{pmatrix} = b,$$

that g is the optimal solution to Problem 3.1.

Suppose that $\underline{L}$ and $\overline{L}$ represent the minimum value of $\underline{c}$ and $\overline{c}$, while $\underline{U}$ dan $\overline{U}$ represent the maximum value of $\underline{c}$ dan $\overline{c}$, that is

a. $\underline{L} = \min\limits_{i=1,..,m-1} \underline{c}_i$ and $\overline{L} = \min\limits_{i=1,..,m-1} \overline{c}_i$,

b. $\underline{U} = \operatorname*{maks}\limits_{i=1,..,m-1} \underline{c}_i$ and $\overline{U} = \operatorname*{maks}\limits_{i=1,..,m-1} \overline{c}_i$.

Since $g \approx \left[\underline{g}, \overline{g}\right]$, therefore there are three cases for $\underline{g}$, i.e.

1. $\underline{L} \leq \underline{g} \leq \underline{U}$ where $\delta\left(\underline{b}\right) = \delta\left(\underline{c}\right) = \underline{U} - \underline{L}$,
2. $\underline{g} < \underline{L}$ where $\delta\left(\underline{b}\right) = \underline{L} - \underline{g}$,
3. $\underline{U} < \underline{g}$ where $\delta\left(\underline{b}\right) = \underline{g} - \underline{U}$.

Also there are three cases for $\overline{g}$, i.e.

1'. $\overline{L} \leq \overline{g} \leq \overline{U}$ where $\delta\left(\overline{b}\right) = \delta(\overline{c}) = \overline{U} - \overline{L}$,

2'. $\overline{g} \leq \overline{L}$ where $\delta\left(\overline{b}\right) = \overline{L} - \overline{g}$,

3'. $\overline{U} < \overline{g}$ where $\delta\left(\overline{b}\right) = \overline{g} - \overline{U}$.

According to the disscusion in max-plus algebra, casses 1 and 2 and also casses 1' and 2' are solution to Problem 2.3 for $p = m - 1$. While casses 3 and 3', may be there are $\underline{g}$ and $\overline{g}$ such that $\delta\left(\underline{b}\right)$ and $\delta\left(\overline{b}\right)$ are optimal. The probability there exis $\underline{g}$ and $\overline{g}$ are determined the same way in max-plus algebra.

Next, from combination casses 1 and 1', cases 1 dan 2', cases 2 and 1', and also cases 2 and 2' can be obtained an optimal solution to Problem 3.1. The next theorem which is indicated upper bound of optimal solution to Problem 3.1.

Theorem 3.4. *Let* $A \in I(\mathbb{R})_\varepsilon^{m\times n}$ *be doubly* $I(\mathbb{R})$*-astic,* $c \in Im(A^p) \cap \mathbb{R}^p$ *and* $f \in I(\mathbb{R})^{m-p}$ *is an optimal solution to Problem 3.1. Suppose* $\underline{x} = \left(\underline{A}_1^p\right)^* \otimes' \underline{c}$ *and* $\overline{x} = \left(\overline{A}_1^p\right)^* \otimes' \underline{c}$*. If* $\underline{g} = \underline{A}_2^p \otimes \underline{x}$ *and* $\overline{g} = \alpha \otimes \left(\overline{A}_2^p \otimes \overline{x}\right)$ *with* $\alpha = max\left(\left(\underline{A}_2^p \otimes' \underline{x}\right)_i - \left(\left(\overline{A}_2^p \otimes' \overline{x}\right)\right)_i\right)$ *then* $g \approx [\underline{g}, \overline{g}]$ *and* $f \leq g$.

Proof. Suppose $A \approx \left[\underline{A}, \overline{A}\right]$, $c \approx \left[\underline{c}, \overline{c}\right]$ dan $f \approx \left[\underline{f}, \overline{f}\right]$. Let $A \in I(\mathbb{R})_\varepsilon^{m\times n}$ be doubly $I(\mathbb{R})$-astic, $c \in Im(A^p) \cap \mathbb{R}^p$ and $f \in I(\mathbb{R})^{m-p}$ is an optimal solution to Problem 3.1. Therefore $\underline{A}$, $\overline{A} \in \mathbb{R}_\varepsilon^{m\times n}$, $\underline{c} \in Im\left(\underline{A}_1^p\right) \cap \mathbb{R}^p$, $\overline{c} \in Im\left(\overline{A}_1^p\right) \cap \mathbb{R}^p$ with $\underline{f}$ is an optimal solution to Problem 2.3 for $\underline{A}$ and $\underline{c}$, while $\overline{f}$ is optimal solution to Problem 2.3 for $\overline{A}$ and $\overline{c}$. Suposse $\underline{x} = \left(\underline{A}_1^p\right)^* \otimes' \underline{c}$, $\overline{x} = \left(\overline{A}_1^p\right)^* \otimes' \overline{c}$, Acording to

Theorem 2.9, if $\underline{g} = \underline{A}_2^p \otimes \underline{x}$, $\overline{g'} = \overline{A}_2^p \otimes \overline{x}$ then $\underline{f} \leq \underline{g}$ and $\overline{f} \leq \overline{g'}$. Since $\overline{g} = \alpha \otimes \left(\overline{A}_2^p \otimes \overline{x}\right) = \alpha \otimes \overline{g'}$ where $\alpha = \max_i \left(\left(\underline{A}_2^p \otimes' \underline{x}\right)_i - \left(\left(\overline{A}_2^p \otimes' \overline{x}\right)\right)_i \right)$ such that $g \approx \left[\underline{g}, \overline{g}\right]$ and $f \leq g$. ■

3.2. Maximizing range norm

In the section will discuss about maximize a norm range of the image set of a matrix with some elements vector had been determined. Used an assumption that same as minimize a norm range that elements a vector that determined finite, image of the matrix the top namely A_1^p is $c \in \mathrm{Im}(A_1^p) \cap I(\mathbb{R})^p$ and the elements a vector that will be determined also finite.

Problem 3.2. *Given* $A \in I(\mathbb{R})_\varepsilon^{m\times n}$ *and* $c \in Im(A_1^p) \cap I(\mathbb{R})^p$, $1 \leq p \leq m-1$. *Find* $d \in I(\mathbb{R})^{m-p}$ *such that* $\delta(b)$ *maximum subject to* $b \in \mathrm{Im}(A)$ *and* $b^T = (c^T, d^T)$.

As in minimizing problem, to solve this problem, started by the two special cases $p = 1$ and $p = m-1$. Based on a case for $p = m-1$, it can be discussed in the case of common. Given the following theorem.

Theorem 3.5. *Let* $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R})_\varepsilon^{m\times n}$ *be doubly* $I(\mathbb{R})$*-astic,* $c = ([\underline{c}_1, \overline{c}_1]) \in I(\mathbb{R})$ *be a vector of prescribed components. If* $b \in \mathrm{Im}(A)$ *is an optimal solution to Problem 2.8 and* $\underline{d} = (\underline{c}_1 \otimes \underline{b}_1^{-1}) \otimes \underline{b}$, $\overline{d} = (\overline{c}_1 \otimes \overline{b}_1^{-1}) \otimes \overline{b}$ *when* $\underline{d} \leq \overline{d}$ *then* $e = (c, d_1^1)^T \in \mathrm{Im}(A)$ *is an solution to Problem 3.2 with* d_1^1 *is the vector* $d = (d_1, d_2, \dots, d_m) \approx [\underline{d}, \overline{d}]$ *by deleting the first component.*

Proof. Let $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R})_\varepsilon^{m\times n}$ be doubly $I(\mathbb{R})$-astic, so $\underline{A}$, $\overline{A}$ be doubly $\mathbb{R}$-astic. Then, $c = ([\underline{c}_1, \overline{c}_1]) \in I(\mathbb{R})$ be the vector of prescribed components. So $\underline{c} = (\underline{c}_1)$, $\overline{c} = (\overline{c}_1) \in \mathbb{R}$ be the vector of prescribed components. Since $b \in \mathrm{Im}(A)$ is an optimal solution to Problem 2.8, so $\underline{b} \in Im(\underline{A})$ is an optimal solution to Problem 2.4 for $\underline{A}$, while $\overline{b} \in Im(\overline{A})$ is an optimal solution to Problem 2.4 for $\underline{A}$. Since, $\underline{d} = (\underline{c}_1 \otimes \underline{b}_1^{-1}) \otimes \underline{b}$, $\overline{d} = (\overline{c}_1 \otimes \overline{b}_1^{-1}) \otimes \overline{b}$ and $\underline{d} \leq \overline{d}$. According to Theorem 2.7, $\underline{e} = (\underline{c}_1, \underline{d}_1^1)^T \in Im(\underline{A})$ is a optimal solution to Problem 2.5 for $\underline{c}_1$ and $\underline{A}$ with $\underline{d}_1^1$ is the vector $\underline{d} = (\underline{d}_1, \underline{d}_2, \dots, \underline{d}_m) \approx [\underline{d}, \overline{d}]$ by deleting the first component and $\overline{e} = \left(\overline{c}_1, \overline{d}_1^{-1}\right)^T \in Im(\overline{A})$ is a solution to Problem 2.5 for $\overline{c}_1$ and $\overline{A}$ with $\overline{d}_1^{-1}$ is the vector $d = (\overline{d}_1, \overline{d}_2, \dots, \overline{d}_m) \approx [\underline{d}, \overline{d}]$ by deleting the first component. Therefore, $e = (c, d_1^1)^T \in \mathrm{Im}(A)$ is an solution to Problem 3.2 for c_1 and A with d_1^1 is the vector $d = (d_1, d_2, \dots, d_m) \approx [\underline{d}, \overline{d}]$ by deleting the first component. ■

Theorem 3.6. *Let* $A \in I(\mathbb{R})_\varepsilon^{m\times n}$ *be doubly* $I(\mathbb{R})$*-astic and* $\exists i \in M, j \in N$ *such that* $a_{ij} = \varepsilon$. *If* $c = (c_1) \in I(\mathbb{R})$ *is the vector of prescribed then Problem 3.2 has unbounded solution.*

Proof. Suppose $A \approx [\underline{A}, \overline{A}]$ and $a_{ij} = [\underline{a}_{ij}, \overline{a}_{ij}]$. Let $A \in I(\mathbb{R})_\varepsilon^{m\times n}$ be doubly $I(\mathbb{R})$-astic and there are $i \in M, j \in N$ such that $a_{ij} = \varepsilon$. Therefore $\underline{A}$, $\overline{A} \in \mathbb{R}_\varepsilon^{m\times n}$ be doubly $\mathbb{R}$-astic and there are $i \in M, j \in N$ such that $\underline{a}_{ij} = \varepsilon$ and $\overline{a}_{ij} = \varepsilon$. Because $c = (c_1) \in I(\mathbb{R})$ is the vector of prescribed. Suppose $c \approx [\underline{c}, \overline{c}]$ so $\underline{c} = \underline{c}_1$, $\overline{c} = \overline{c}_1$. According to Theorem 2.8 there is $\underline{d} \in \mathbb{R}^{m-1}$ so the Problems 2.5 for $\underline{c}$ and $\underline{A}$ have unbounded solution and there is $\overline{d} \in \mathbb{R}^{m-1}$ so the Problems 2.5 for $\overline{c}$ and $\overline{A}$ have unbounded solution. As a result Problems 3.2 have unbounded solution. ■

Next, in the case of $p = m-1$ we need to find the fisibel solution for d. Given matrix $A \in I(\mathbb{R})_\varepsilon^{m\times n}$ be doubly $I(\mathbb{R})$-astic and $c \in \mathrm{Im}(A^p) \cap I(\mathbb{R})^p$ as costrain vector that $c \approx [\underline{c}, \overline{c}]$, then $\exists \hat{x} \in I(\mathbb{R})_\varepsilon^n$ where $\hat{x} = (A_1^{m-1})^* \overline{\otimes}' c$ such that $c = A^{m-1} \overline{\otimes} \hat{x}$. Using the vector $\hat{x}$ will be obtained, $g = A_2^{m-1} \overline{\otimes} \hat{x}$.

It appears that g is a fisibel solution for Problem 2.2 from the following equation,

$$A \overline{\otimes} \hat{x} = \begin{pmatrix} A_1^{m-1} \\ A_2^{m-1} \end{pmatrix} \overline{\otimes} \hat{x} = \begin{pmatrix} A_1^{m-1} \overline{\otimes} \hat{x} \\ A_2^{m-1} \overline{\otimes} \hat{x} \end{pmatrix} = \begin{pmatrix} c \\ g \end{pmatrix} = b.$$

Suppose that $\underline{\mathrm{L}}$ and $\overline{\mathrm{L}}$ represent the minimum value of $\underline{\mathrm{c}}$ and $\overline{\mathrm{c}}$, while $\underline{\mathrm{U}}$ dan $\overline{\mathrm{U}}$ represent the maximum value of $\underline{\mathrm{c}}$ dan $\overline{\mathrm{c}}$, that is

a. $\underline{\mathrm{L}} = \min_{i=1,..,m-1} \underline{\mathrm{c}}_i$ and $\overline{\mathrm{L}} = \min_{i=1,..,m-1} \overline{\mathrm{c}}_i$,

b. $\underline{\mathrm{U}} = \max_{i=1,..,m-1} \underline{\mathrm{c}}_i$ and $\overline{\mathrm{U}} = \max_{i=1,..,m-1} \overline{\mathrm{c}}_i$.

Since $\mathrm{g} \approx [\underline{\mathrm{g}}, \overline{\mathrm{g}}]$, therefore there are three cases for $\underline{\mathrm{g}}$, i.e.

1. $\underline{\mathrm{L}} \le \underline{\mathrm{g}} \le \underline{\mathrm{U}}$ where $\delta(\underline{\mathrm{b}}) = \delta(\underline{\mathrm{c}}) = \underline{\mathrm{U}} - \underline{\mathrm{L}}$,
2. $\underline{\mathrm{g}} < \underline{\mathrm{L}}$ where $\delta(\underline{\mathrm{b}}) = \underline{\mathrm{L}} - \underline{\mathrm{g}}$,
3. $\underline{\mathrm{U}} < \underline{\mathrm{g}}$ where $\delta(\underline{\mathrm{b}}) = \underline{\mathrm{g}} - \underline{\mathrm{U}}$.

Also there are three cases for $\overline{\mathrm{g}}$, i.e.

1'. $\overline{\mathrm{L}} \le \overline{\mathrm{g}} \le \overline{\mathrm{U}}$ where $\delta(\overline{\mathrm{b}}) = \delta(\overline{c}) = \overline{\mathrm{U}} - \overline{\mathrm{L}}$,
2'. $\overline{\mathrm{g}} \le \overline{\mathrm{L}}$ where $\delta(\overline{\mathrm{b}}) = \overline{\mathrm{L}} - \overline{\mathrm{g}}$,
3'. $\overline{\mathrm{U}} < \overline{\mathrm{g}}$ where $\delta(\overline{\mathrm{b}}) = \overline{\mathrm{g}} - \overline{\mathrm{U}}$.

According to the disscution in max-plus algebra, casses 1 and 3 and also casses 1' and 3' are solution of Problem 2.5 for $p = m - 1$. While casses 2 and 2', may be there are $\underline{\mathrm{g}}$ and $\overline{\mathrm{g}}$ such that $\delta(\underline{\mathrm{b}})$ and $\delta(\overline{\mathrm{b}})$ are optimal. The probability there exis $\underline{\mathrm{g}}$ and $\overline{\mathrm{g}}$ are determined the same way in max-plus algebra.

Next, from combination casses 1 and 1', cases 1 dan 3', cases 3 and 1', and also cases 3 and 3' can be obtained an optimal solution for Problem 3.2. Next, the theorem which is indicated upper bound of optimal solution for Problem 3.2.

Theorem 2.7. *Let* $\mathrm{A} \in I(\mathbb{R})_\varepsilon^{m\times n}$ *be double* $I(\mathbb{R})$*-astic* $\mathrm{c} \in Im(\mathrm{A}^p) \cap I(\mathbb{R})^p$ *and* f *is an optimal solution to problem 2.2. Suppose* $\underline{\mathrm{y}} = \left(\underline{\mathrm{A}}_1^p\right)^* \otimes' \underline{\mathrm{c}}$ *and* $\overline{\mathrm{y}} = \left(\overline{\mathrm{A}}_1^p\right)^* \otimes' \underline{\mathrm{c}}$. *If* $\underline{\mathrm{g}} = \underline{\mathrm{A}}_2^p \otimes \underline{\mathrm{y}}$ *and* $\overline{\mathrm{g}} = \alpha \otimes \left(\overline{\mathrm{A}}_2^p \otimes \overline{\mathrm{y}}\right)$ *with* $\alpha = max \left(\left(\underline{\mathrm{A}}_2^p \otimes' \underline{\mathrm{y}}\right)_i - \left(\left(\overline{\mathrm{A}}_2^p \otimes' \overline{y}\right)\right)_i\right)$ *then* $\mathrm{g} \approx [\underline{\mathrm{g}}, \overline{\mathrm{g}}]$ *and* $\mathrm{f} \le \mathrm{g}$.

Proof. Let $\mathrm{A} \approx [\underline{\mathrm{A}}, \overline{\mathrm{A}}], \mathrm{c} \approx [\underline{\mathrm{c}}, \overline{\mathrm{c}}]$ dan $\mathrm{f} \approx [\underline{\mathrm{f}}, \overline{\mathrm{f}}]$. Since $\mathrm{A} \in Ie(\mathbb{R})_\varepsilon^{m\times n}$ be doubly $I(\mathbb{R})$-astic so $\underline{\mathrm{A}}$, $\overline{\mathrm{A}} \in \mathbb{R}_\varepsilon^{m\times n}$ be doubly $\mathbb{R}$-astic, $\mathrm{c} \in Im(\mathrm{A}^p) \cap \mathbb{R}^p$ so $\underline{\mathrm{c}} \in Im(\underline{\mathrm{A}}_1^p) \cap \mathbb{R}^p$, $\overline{\mathrm{c}} \in Im\left(\overline{\mathrm{A}}_1^p\right) \cap \mathbb{R}^p$. Since $\mathrm{f} \in I(\mathbb{R})^{m-p}$ is an optimal solution to Problem 3.2 so $\underline{\mathrm{f}} \in \mathbb{R}^{m-p}$ is an optimal solution to Problem 2.5 for $\underline{\mathrm{c}}$ and $\underline{\mathrm{A}}$, $\overline{\mathrm{f}} \in \mathbb{R}^{m-p}$ is an optimal solution to Problem 2.5 for $\overline{\mathrm{c}}$ and $\overline{\mathrm{A}}$. Let $\underline{\mathrm{y}} = \left(\underline{\mathrm{A}}_1^p\right)^* \otimes' \underline{\mathrm{c}}$ and $\overline{\mathrm{y}} = \left(\overline{\mathrm{A}}_1^p\right)^* \otimes' \overline{\mathrm{c}}$. According to Theorem 2.9 , if $\underline{\mathrm{g}} = \underline{\mathrm{A}}_2^p \otimes \underline{\mathrm{y}}$, $\overline{\mathrm{g}'} = \overline{\mathrm{A}}_2^p \otimes \overline{\mathrm{y}}$ then $\underline{\mathrm{f}} \le \underline{\mathrm{g}}$ and $\overline{\mathrm{f}} \le \overline{\mathrm{g}'}$. Because $\overline{\mathrm{g}} = \alpha \otimes \left(\overline{\mathrm{A}}_2^p \otimes \overline{\mathrm{y}}\right) = \alpha \otimes \overline{\mathrm{g}'}$ with $\alpha = \max_i \left(\left(\underline{\mathrm{A}}_2^p \otimes' \underline{\mathrm{y}}\right)_i - \left(\left(\overline{\mathrm{A}}_2^p \otimes' \overline{\mathrm{y}}\right)\right)_i\right)$ then $\mathrm{g} \approx [\underline{\mathrm{g}}, \overline{\mathrm{g}}]$ and $\mathrm{f} \le \mathrm{g}$. ■

4. Conclusion

Based on the results of discussion, we obtain conclusion how to,

a. Minimizing range norm of the set of matrix over interval max-plus algebra with component prescribed.
b. Maximizing range norm of the set of matrix over interval max-plus algebra with component prescribed.

References

[1] Bacelli, F., G. Cohen, G. J. Older, and J. P. Quadrat, *Synchronization and Linearity, An Algebra for Discrete Events System*, John Wiley and Sons, 2001.

[2] K. G. Farlow, *Max-Plus Algebra*, Master's Thesis submitted to the Faculty of the Virginia Polytechnic Institute and State University in partial fulfillment of the requirements for the degree of Masters in Mathematics, 2009.

[3] K. P. Tam, *Optimizing and Approximating Eigenvectors in Max Algebra*, A Thesis Submitted to The University of Birmingham for The Degree of Doctor of Philosophy (Ph.D), (2010).

[4] M. Akian, G. Cohen, S. Gaubert, J. P. Quadrat, and M. Viot, *Max-Plus Algebra and Applications to System Theory and Optimal Control*, Proceedings of the International Congress of Mathematicians, Zurich, Switzerland, (1994).

[5] M. A. Rudhito, *Fuzzy NumberMax-Plus Algebra and Its Application to Fuzzy Scheduling and Queuing Network Problems*, The Study Program S3 Mathematics Faculty of Mathematics and Natural Sciences : Gadjah Mada University, 2011. (Indonesian).

[6] P. Butkovic, and K. P. Tam, *On Some Properties of The Image of a Max Linear Mapping*, Contemporary Mathematics, (495) (2009), pp. 115 – 126.

[7] Siswanto, A. Suparwanto, M. A. Rudhito, *Minimizing Range Norm Of The Image Set Of Matrix Over Interval Max-Plus Algebra*, Proceedings of The National Seminar on Mathematics and Applications, Department of Mathematics, Faculty of Science and Technology, Airlangga University; ISBN : 978-602-14413-0-5, (2013). (Indonesian).

[8] Siswanto, A. Suparwanto, M. A. Rudhito. *Eigenvector Space of a Matrix of Interval Max-Plus Algebra*, Jurnal Matematika & Sains (JMS). FMIPA ITB, (19) (1) (2014), pp. 8 – 15. (Indonesian).

[9] Siswanto, A. Suparwanto, M. A. Rudhito, *Existence of Solution to the System of Linear Equations in Interval Max-Plus Algebra*, Proceedings of the National Conference of Mathematics XVII, Institute of Technology Surabaya, ISBN : 978-602-96426-3-6, (2014). (Indonesian).

[10] Siswanto, A. Suparwanto, M. A. Rudhito, *Optimizing Range Norm of The Image Set of Matrix Over Interval Max-Plus Algebra*, Far East Journal of Mathematical Sciences, (19) (2016), pp. 17 – 32.

[11] V. Kreinovich, *Why Intervals* ? *Why Fuzzy Numbers* ? *Towards a New Justification*, International Journal of Information Technology and Intelligent Computing. (2) (1) (2007).

[12] Z. R. Konigsberg, *A Generalized Eigenmode Algorithm for Reducible Regular Matrices over the Max-Plus Algebra*, International Mathematical Forum. (4) (24) (2009), pp. 1157 – 1171.