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DAFTAR ISI

EDITORIAL

Salam sejahtera,

Puji syukur ke hadirat Tuhan Yang Maha Kuasa atas perkenanannya Jurnal Teknologi Media Teknika pada tahun 2015 ini dapat hadir kembali di tengah-tengah Bapak/Ibu/Sdr semuanya. Merupakan kebanggaan tersendiri bagi kami tim redaksi dapat melalui tahapan yang begitu rumit dan terjal, untuk mewujudkan agar jurnal ini bisa hadir di tengah-tengah perkembangan ipteks yang ada sekarang ini.

Jurnal Teknologi Media Teknika Vol. 10, No. 2, Desember 2015 ini memuat 8 tulisan yang mencakup bidang ilmu matematika terapan 1 paper, teknik elektro 2 paper, teknik informatika 4 paper dan teknik mesin 1 paper. Redaksi mengucapkan terima kasih kepada para penulis yang telah rela meluangkan waktu dan pikiran untuk menulis paper-paper tersebut dalam berbagi kepada yang lain. Semoga atas jerih payahnya mendapatkan balasan yang sepadan.

Selanjutnya kami mengundang kepada Bapak/Ibu/Sdr untuk turut juga berpartisipasi dalam jurnal Media Teknika dengan mengirimkan naskah/paper hasil penelitian atau kajian ilmiah. Kiranya tulisan Bapak/Ibu/Sdr akan membantu perkembangan ipteks ke arah yang lebih baik dan berguna bagi masyarakat.

Akhirnya tim redaksi berharap semoga kehadiran jurnal ini dapat bermanfaat dalam menyebarkan ipteks untuk membantu kepada masyarakat yang membutuhkannya. Tiada gading yang tak retak, kritik dan saran yang membangun sangat kami harapkan demi perbaikan dikemudian hari.

Salam.

Redaksi

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Weak Local Residual in Relation to the Accuracy of Numerical Solutions to Conservation Laws

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Abstract

As the exact solutions to differential equations are generally very difficult to find, numerical solutions are often desired. Numerical solutions are approximations to the exact solutions, so they have errors. Because we do not know the exact solutions, a tool for checking the accuracy of numerical solutions is needed. In this paper, we present a formula as the tool for investigating the accuracy of numerical solutions to conservation laws. The formula is derived from the weak local residual of the numerical solution. The residual is zero if the solution is exact. The larger the residual means the less accurate the approximate solution. We consider two specific conservation laws, namely the advection equation and the acoustics equations. With these two problems, our results show that the weak local residual behaves correctly as an accuracy-checking formula of numerical solutions to conservation laws.

Keywords: *accuracy-checking formula, conservation laws, finite volume methods, weak local residual*

1. Introduction

Differential equations have important roles in mathematical modelling of real problems, such as fluid flows, wave propagation, weather prediction, etc. Differential equations need to be solved to find the solution to the real problems. Solving exactly the equations is generally difficult. Therefore numerically solving the equations is an option that can be considered.

Numerical solutions are approximations of the exact solutions. We are interested in a way to check the accuracy of numerical solutions to conservation laws, where the exact solutions are not known due to their difficulty to find. Conservation laws themselves have many applications in fluid and solid dynamics modelling. Therefore, they are important to study.

Some numerical techniques for solving conservation laws have been available in the literature for years. One of them is the finite difference method, which is powerful for smooth solutions. Conservation laws can be hyperbolic, so they admit discontinuous solutions. This means that conservative methods are needed for solving conservation laws accurately. One of conservative methods is the finite volume method which is implemented in this paper to solve conservation laws. The resulting solutions still have errors, but we do not know the magnitude of the errors, because once again we do not know the exact solution.

To know the magnitude of the errors, a formula is needed. We propose the use of the weak local residual formula in order to investigate the accuracy of numerical solutions. Our formula follows from the work of Constantin and Kurganov [1] as well as Mungkasi *et al.* [2]. The formula is explicit. It is simple to compute at all time, but the residual formula is valid only for conservative numerical methods [3].

This paper is structured as follows. We present the formulation of the weak local residual in the next section. After that we test the performance of the formula for the advection equation and the acoustics equations. Concluding remarks are drawn at the end.

2. Weak Local Residual Formulation

Consider the scalar conservation laws in the form

$$\frac{\partial q}{\partial t}+\frac{\partial f(q)}{\partial x}=0, \quad -\infty<x<\infty \tag{1}$$

with initial condition

$$q(x,t)=q_0(x), \quad t=0. \tag{2}$$

Here variable x represents the space and variable t denotes the time. The quantity $q=q(x,t)$ is conserved. The function $q(x,0)=q_0(x)$ is given. The function $f=f(q(x,t))$ is the flux.

We consider two problems, namely the advection equation and the acoustics equations. The advection equation is

$$\frac{\partial q}{\partial t}+\frac{\partial q}{\partial x}=0, \tag{3}$$

where the conserved quantity $q=q(x,t)$ is transported to the right direction with the unit velocity. The (constant-coefficient) acoustics equations are

$$\frac{\partial p}{\partial t}+\rho c^2\frac{\partial u}{\partial x}=0, \tag{4}$$

$$\frac{\partial u}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0. \tag{5}$$

Here in the acoustics equations:

- $p=p(x,t)$ represents the pressure,
- $u=u(x,t)$ is the velocity variable,
- ρ is the density which is assumed to be constant, and
- c is the pressure wave propagation speed, which is also assumed to be constant.

Let us consider a conservative numerical method to solve the conservation laws. In particular, let us take a standard finite volume method. In the standard finite volume method, the space domain is discretised into a finite number of cells with the cell width Δx. The time domain is dicretised into a finite number of time steps Δt, where the value of Δt is chosen such that the method is stable. The centroids of cells are denoted x_i, with $x_{i+1}:=x_i+\Delta x$. The vertices of cells are denoted $x_{i+1/2}:=x_i+\Delta x/2$. The discrete time is denoted $t^{n+1}:=t^n+\Delta t$. The notation $t^{n+1/2}$ means $t^{n+1/2}:=t^n+\Delta t/2$.

The weak local residual for the conservation laws (1) with initial condition (2) has been formulated by Constantin and Kurganov [1] and is given by

$$R_{i+1/2}^{n-1/2}=\frac{\Delta x}{2}\left[q_i^n-q_i^{n-1}+q_{i+1}^n-q_{i+1}^{n-1}\right]+\frac{\Delta t}{2}\left[f(q_{i+1}^{n-1})-f(q_i^{n-1})+f(q_{i+1}^n)-f(q_i^n)\right]. \tag{6}$$

This formula is derived from the weak formulation of the conservation laws as follows. First we write the initial value problem (1)-(2) in the weak form

$$\int_0^\infty\int_{-\infty}^\infty\left[q(x,t)\frac{\partial T(x,t)}{\partial t}+f(q(x,t))\frac{\partial T(x,t)}{\partial x}\right]dx\,dt+\int_{-\infty}^\infty q_0(x)T(x,0)\,dx=0, \tag{7}$$

with $T(x,t)$ is a test function. In this work we take the test function at every point $(x_{i+1/2},t^{n-1/2})$ as

$$T(x,t):=T_{i+1/2}^{n-1/2}(x,t)=B_{i+1/2}(x)B^{n-1/2}(t), \tag{8}$$

where

$$B_{i+1/2}(x)=\begin{cases}\dfrac{x-x_{i-1/2}}{\Delta x} & \text{if } x_{i-1/2}\le x\le x_{i+1/2}\,,\\[2ex] \dfrac{x_{i+3/2}-x}{\Delta x} & \text{if } x_{i+1/2}\le x\le x_{i+3/2}\,,\\[2ex] 0 & \text{otherwise.}\end{cases} \qquad (9)$$

and

$$B^{n-1/2}(t)=\begin{cases}\dfrac{t-t^{n-3/2}}{\Delta t} & \text{if } t^{n-3/2}\le t\le t^{n-1/2}\,,\\[2ex] \dfrac{t^{n+1/2}-t}{\Delta t} & \text{if } t^{n-1/2}\le t\le t^{n+1/2}\,,\\[2ex] 0 & \text{otherwise.}\end{cases} \qquad (10)$$

Substituting this test function into the weak form (7) of conservation laws, we obtain

$$R^{n-1/2}_{i+1/2}=-\int_{t^{n-3/2}}^{t^{n+3/2}}\int_{x_{j-3/2}}^{x_{j+3/2}}\left[q(x,t)\frac{\partial T^{n-1/2}_{i+1/2}(x,t)}{\partial t}+f(q(x,t))\frac{\partial T^{n-1/2}_{i+1/2}(x,t)}{\partial x}\right]dx\,dt\,, \qquad (11)$$

which results in the Constantin-Kurganov residual formulation (6), as also discussed by Mungkasi *et al.* [2]. Note that if the right hand side of equation (1) is not zero, the formulation of the weak local residual needs to be well-balanced [4].

3. Numerical Tests

In this section we present our numerical results. We assume that all quantities are measured in SI units. We consider three initial conditions as follows for our test cases:

- a non-smooth initial condition

$$q(x,0)=\begin{cases}0.5(1+\cos(x)) & \text{if } 0\le x\le 2\pi,\\ 0 & \text{if } 2\pi< x\le 10,\\ 1 & \text{if } 10< x\le 15,\\ 0 & \text{if } 15< x\le 40,\end{cases} \qquad (12)$$

  for the advection equation,

- a smooth initial condition

$$q(x,0)=\begin{cases}0.5(1+\cos(x)) & \text{if } 0\le x\le 2\pi,\\ 0 & \text{if } 2\pi< x\le 40,\end{cases} \qquad (13)$$

  for the advection equation,

- an initial condition together with

$$u(x,0)=0\,, \qquad (14)$$

$$p(x,0)=\begin{cases}1+\cos(x-50) & \text{if } 50-\pi\le x\le 50+\pi\,,\\ 0 & \text{if } 0\le x<50-\pi \cup 50+\pi<x\le 100.\end{cases} \qquad (15)$$

  for the acoustics equations.

The numerical methods used for our simulations are first order finite volume methods [5]. Next, we report four simulations to achieve our goal.

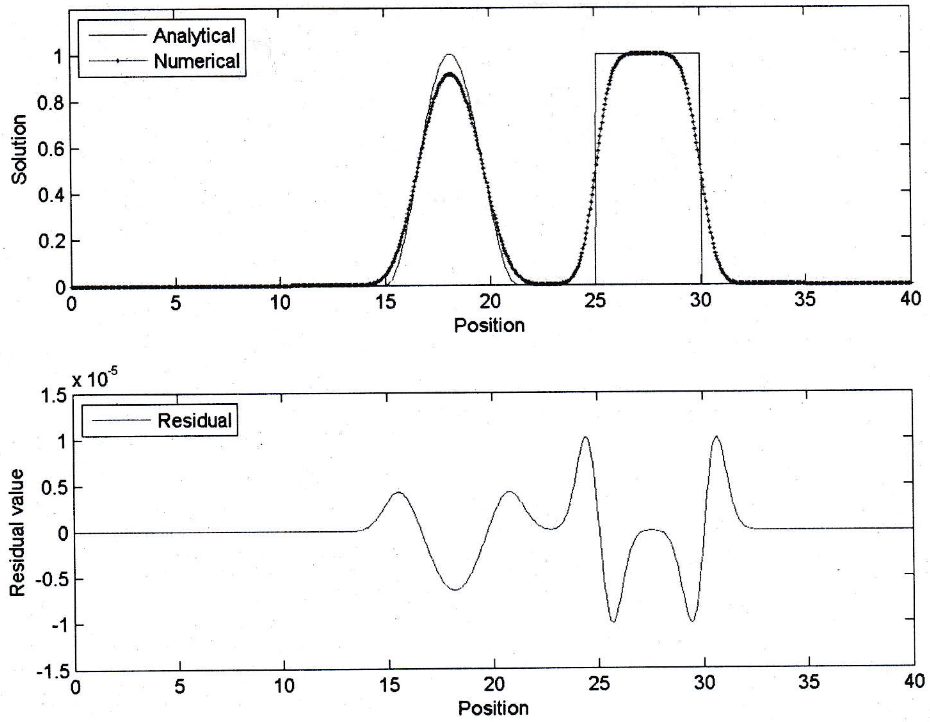

Figure 1. Results of the advection equation using the upwind finite volume method with the time step is a half of the cell width. The residual detects the positions of where errors occur.

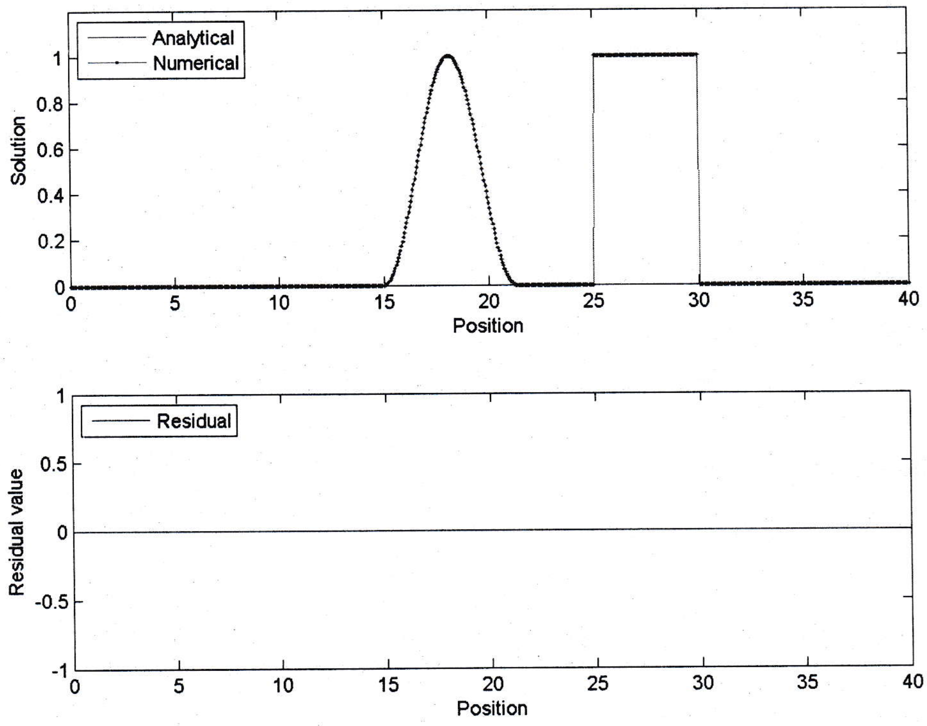

Figure 2. Results of the advection equation using the upwind finite volume method with the time step equals to the cell width. The residual is zero, as the error is also zero everywhere.

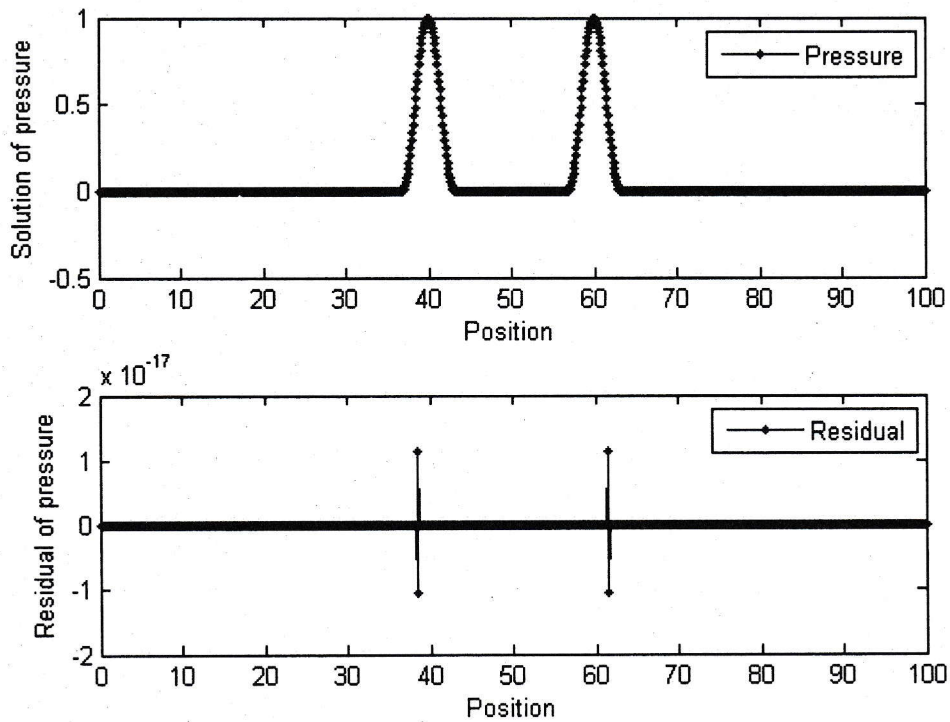

Figure 3. Pressure solution of the acoustics equations obtained using the Lax-Friedrichs finite volume method with the time step equals to the cell width. As the numerical solution is exact, the residual is zero everywhere up to the machine precision. The magnitude of the residual is in the scale of $2.0\mathrm{e}-17$.

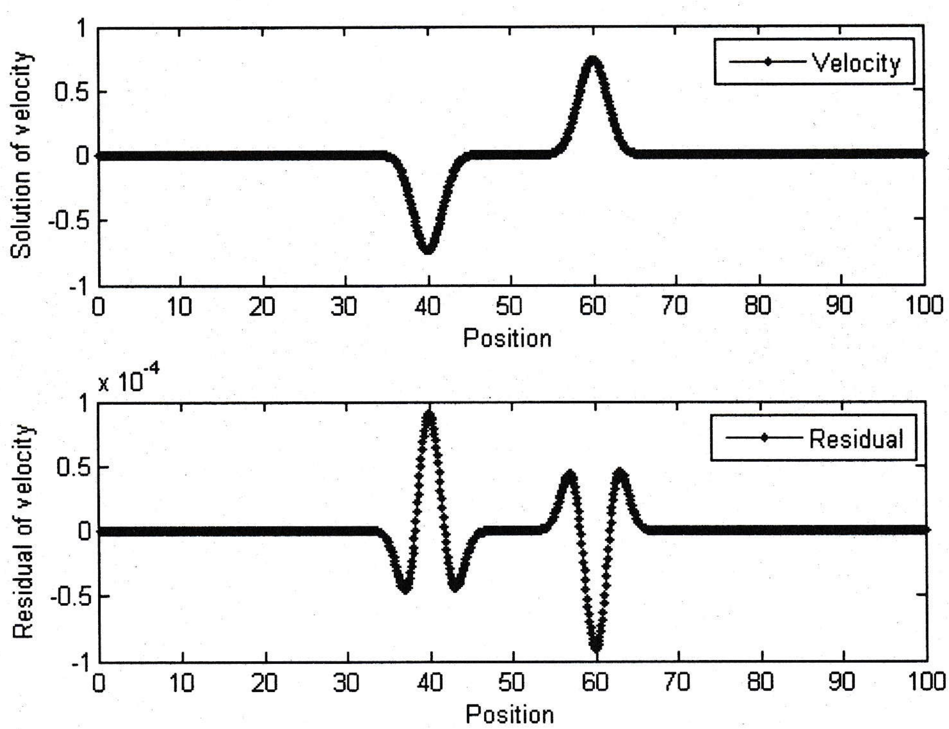

Figure 4. Velocity solution of the acoustics equations found using the Lax-Friedrichs finite volume method with the time step equals to a half of the cell width. The residual finds the places where large errors occur. The magnitude of the residual is in the scale of $1.0\mathrm{e}-4$.

Tabel 1. Numerical absolute L^1 errors and averaged absolute residual for different numbers of cells with the method of first order in space and first order in time for the advection equation with *discontinuous* initial condition (12).

Number of cells	Absolute L^1 errors	Order of L^1 errors	Averaged absolute residual	Order of absolute residual
100	0.1108	-	3.1503e-004	-
200	0.0754	0.5553	5.8533e-005	2.4282
400	0.0501	0.5898	1.0098e-005	2.5352
800	0.0329	0.6067	1.6639e-006	2.6014
1600	0.0217	0.6004	2.6866e-007	2.6307
3200	0.0145	0.5816	4.3441e-008	2.6287

Tabel 2. Numerical absolute L^1 errors and averaged absolute residual for different numbers of cells with the method of first order in space and first order in time for the advection equation with *smooth* initial condition (13).

Number of cells	Absolute L^1 errors	Order of L^1 errors	Averaged absolute residual	Order of absolute residual
100	0.0423	-	1.3844e-004	-
200	0.0269	0.6531	2.6189e-005	2.4022
400	0.0157	0.7768	4.3589e-006	2.5869
800	0.0085	0.8852	6.4767e-007	2.7506
1600	0.0045	0.9175	8.8863e-008	2.8656
3200	0.0023	0.9683	1.1644e-008	2.9320

The first simulation is solving the advection equation with the upwind flux. The initial condition is as given by (12). We consider the space domain $[0, 40]$. We take uniform cell-width $\Delta x = 0.05$ and the time step $\Delta t = 0.5\,\Delta x$. The simulation is stopped at time $t = 15$. The analytical solution of this problem can be found from the work of LeVeque [5]. We find that the largest errors occur at around discontinuities, as shown in Figure 1. In Figure 1 we can also observe that the residual values are at around discontinuities. This means that the residual concludes the same behaviour as the error.

In the second simulation, we modify the time step of the first one. Now we take the time step to be $\Delta t = \Delta x$. Based on the characteristics method, the finite volume method with the upwind flux formulation results in the exact solution. Indeed, we find the exact solution. That is, our numerical solution matches exactly with the analytical solution, as plotted in Figure 2. As shown in Figure 2, we also observe that the residual values are zero everywhere. This means that the residual behaves the same as the error.

The third simulation is about solving the acoustics equations. We consider the initial condition (14)-(15).We consider the space domain $[0, 100]$. We take uniform cell-width $\Delta x = 0.1$ and the time step $\Delta t = \Delta x$. We use the finite volume method with the Lax-Friedrichs flux. Based on the characteristics theory [5], the method results in the exact (analytical) solution. However, we do not know the explicit form of the analytical solution. This is a good test case if the residual formula can give the correct indication of the exact solution. At time $t = 10$, the simulation results are given in Figure 3. In this figure there are two waves, that is, one moves to the left and one moves to the right. The residual values are below the machine precision (less than 2×10^{-17}), as shown in Figure 3. This means that our numerical solution is actually the exact solution up to the machine precision.

The fourth simulation is similar to the third one, but in this fourth simulation we change the time step to $\Delta t = 0.5\,\Delta x$. Of course we shall not obtain the exact solution in this case. The point of this simulation is to make sure that the residual can still detect where the positions have errors in the numerical solution. As shown in Figure 4, the residual indicates that the largest errors occur at positions around large wave amplitudes. The error gets larger as time evolves. This is because the amplitudes of waves, both moving to the left and right, dampen.

To complete our work, we investigate the behaviour of the residual as the grids are refined. Firstly we consider the advection equation with initial condition (12) solved using the upwind finite volume method with $\Delta t = 0.5\,\Delta x$. The order of accuracy (order of error) is about 0.6, whereas the order of the residual is about 2.6, as recorded in Table 1. Secondly we consider the advection equation with initial condition (13) solved using the upwind finite volume method with $\Delta t = 0.5\,\Delta x$, the same time step value as before. The order of accuracy (order of error) is about 1, whereas the order of the residual is about 3, as recorded in Table 2. The order of error is larger for Table 2 than for Table 1, because of the difference in their initial conditions. The smoother the solution gives the larger the order of error. This phenomena is also reflected in the residual results, shown in Tables 1 and 2.

4. Conclusion

Weak local residual has been shown to be powerful in checking the accuracy of numerical solutions where the exact solutions are not known. The behaviour of the residual mimics that of the error. These results may help in the construction of smoothness indicator or discontinuity detector of numerical solutions. Regions of where numerical solutions are accurate and not accurate can be identified using a smoothness indicator or discontinuity detector.

References

[1] Constantin LA, Kurganov A. Adaptive central-upwind schemes for hyperbolic systems of conservation laws. In F. Asakura et al., eds., *Hyperbolic Problems: Theory, Numerics, and Applications*, Vol. 1, pages 95–103. Yokohama Publishers, Yokohama, 2006.

[2] Mungkasi S, Li Z, Roberts SG. Weak local residuals as smoothness indicators for the shallow water equations. *Applied Mathematics Letters*. 2014; 30: 51–55.

[3] Dewar J, Kurganov A, Leopold M. Pressure-based adaption indicator for compressible Euler equations. *Numerical Methods for Partial Differential Equations*. 2015; 31: 1844–1874.

[4] Mungkasi S, Roberts SG. Well-balanced computations of weak local residuals for the shallow water equations. *ANZIAM Journal*. 2015; 56: C128–C147.

[5] LeVeque RJ. *Finite-volume methods for hyperbolic problems.* Cambridge: Cambridge University Press. 2004.