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Order of Accuracy of Numerical Methods for Fluid Dynamics

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Abstract

This paper presents research results about order of accuracy of numerical methods for fluid dynamics. High order accurate numerical methods are often desired. One could think that higher order accurate numerical methods would always lead to smaller numerical errors. However, this is misleading. A high order accurate method does not mean that it is always more accurate than a lower order accurate method for any discretized domain. The truth of this claim is demonstrated in this paper. We consider finite volume methods used to solve the shallow water equations. These equations form a mathematical model of fluid dynamics governing shallow water waves or flows. Two types of finite volume methods are implemented. The first is a finite volume method which is second order accurate in space but first order accurate in time. The second is a finite volume method which is second order accurate in space as well as in time. One would hope that the second finite volume method should always produce smaller errors than the first. However, that is untrue. The first finite volume method is sometimes more accurate than the second regardless of the quality of the numerical solution.

Keywords: *finite volume methods, fluid dynamics, order of accuracy, domain discretization*

1. Introduction

Some mathematical models for fluid flows are available in the literatures, such as the Saint-Venant model, the Boussinesq model, the Kortewig de Vries model, etc. To the author's knowledge, the most common in use for simulations of shallow water flows is the first, that is, the Saint-Venant model which was developed since 1892. This model is named after A.J.C. Barre de Saint-Venant (see the References [1-7]).

The Saint-Venant model or as known as the shallow water equations can be used to model and simulate flows in open channels. For example, we can simulate tsunami, flood, dam breaks, etc. using this model, as long as the flow or the wave is relatively shallow with respect to the wave length. In practice, researchers can now use either some software packages like ANUGA or Delf3D as an aid in the simulation, or code the numerical solver themselves.

A well-known numerical solver for the shallow water equations is finite volume method. This method is derived based on the integral equation rather than the differential equation of the model. Because integral equations do not need the assumption that solutions must be smooth, the finite volume method is able to handle smooth and nonsmooth solutions of the shallow water equations. This motivates the choice of finite volume method to be used in the present paper, as a numerical method in interest. We use the Matlab programming language to code the finite volume method of our work.

This paper shows that a lower order accurate method does not mean that it is always less accurate than a higher order one in any condition. This claim was stated by LeVeque [8, 9] without proof. We consider in this paper two types of finite volume methods to support the truth of this claim numerically. The first method is second order accurate in space but first order accurate in time. The second method is second order accurate in space and in time. We shall see that that the second method is actually less accurate in our simulations presented in this paper.

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The rest of this paper is organized as follows. Section 2 provides the numerical method that we implement for simulations of water flows. Section 3 contains numerical results. Finally some concluding remarks are drawn in Section 4.

2. Research Method

In this section we consider the one-dimensional shallow water equations that preserve the steady state solutions to water motion. We refer to the work of Kurganov and Levy [5] and others [10, 11, 14-16] for these equations. Assume that we are given a domain with space variable x and time variable t. Both are free variables. We consider a bottom elevation given by $B(x)$. Water depth above the bottom is denoted by $h(x,t)$. Water velocity is represented by $u(x,t)$. We denote the free surface of water by $w := h + B$. In addition the acceleration due to gravity is g. The one-dimensional shallow water equations that preserve the steady state solutions to water motion are then given by the following system of two simultaneous equations

$$w_t + (hu)_x = 0, \tag{1}$$

$$(hu)_t + \left[\frac{(hu)^2}{w-B} + \frac{1}{2}g(w-B)^2\right]_x = -g(w-B)B_x. \tag{2}$$

Again we refer to the work of Kurganov and Levy [KL2002] for the discretization of equations (1) and (2).

It is described as follows. We first consider the following general system of balance laws

$$q_t + \nabla_x \cdot f(q) = S(q,x,t) \tag{3}$$

where $x \in R^d$ and $q \in R^N$ subject to the initial condition $q(x,0) = q_0(x)$.We then take a uniform cell-grid $x_j = j\Delta x$ where Δx is the cell width. The cell average of $q(\cdot,t)$ over the j th cell is denoted by $\bar{q}_j(t)$ and is

$$\bar{q}_j(t) := \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} q(x,t)\, dx. \tag{4}$$

The system of balance laws can now be written as

$$\frac{d}{dt}\bar{q}_j(t) + \frac{f(q(x_{j+\frac{1}{2}},t)) - f(q(x_{j-\frac{1}{2}},t))}{\Delta x} = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} S(q(x,t),x,t)\, dx. \tag{5}$$

Further, the corresponding *central-upwind* semi-discrete scheme for the system of balance laws is

$$\frac{d}{dt}\bar{q}_j(t) = -\frac{H_{j+\frac{1}{2}}(t) - H_{j-\frac{1}{2}}(t)}{\Delta x} + \bar{S}_j(t), \tag{6}$$

with the numerical fluxes follows the formulation of Kurganov, Noelle and Petrova [KNP2001] and are given by

$$H_{j+\frac{1}{2}}(t) := \frac{a^+_{j+\frac{1}{2}} f(q^-_{j-\frac{1}{2}}) - a^-_{j-\frac{1}{2}} f(a^+_{j+\frac{1}{2}})}{a^+_{j+\frac{1}{2}} - a^-_{j-\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}}\, a^-_{j-\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j-\frac{1}{2}}}(q^+_{j+\frac{1}{2}} - q^-_{j-\frac{1}{2}}). \tag{7}$$

Here $q^+_{j+\frac{1}{2}} := p_{j+1}(x_{j+\frac{1}{2}},t)$ and $q^-_{j+\frac{1}{2}} := p_j(x_{j+\frac{1}{2}},t)$ are the right and the left values at the cell interface $x = x_{j+\frac{1}{2}}$ of a conservative non-oscillatory piecewise polynomial interpolant

$$\tilde{q}(x,t) = \sum_j p_j(x,t)\, \chi_j. \tag{8}$$

This polynomial interpolant is reconstructed at each time step from the previously computed cell averages, $\{\overline{q}_j(t)\}$. The notation $p_j(\cdot,t)$ represents a polynomial of a specified degree. The notation χ_j denotes the characteristic function over the jth cell. In addition, variables $a^{\pm}_{j+\frac{1}{2}}$ are the one-sided local speeds of wave propagation, which are determined by

$$a^{+}_{j+\frac{1}{2}} = \max\left\{\lambda_N\left(\frac{\partial f}{\partial q}(q^{-}_{j+\frac{1}{2}})\right), \lambda_N\left(\frac{\partial f}{\partial q}(q^{+}_{j+\frac{1}{2}})\right), 0\right\}, \tag{9}$$

$$a^{-}_{j+\frac{1}{2}} = \min\left\{\lambda_1\left(\frac{\partial f}{\partial q}(q^{-}_{j+\frac{1}{2}})\right), \lambda_1\left(\frac{\partial f}{\partial q}(q^{+}_{j+\frac{1}{2}})\right), 0\right\} \tag{10}$$

with $\lambda_1 < \ldots < \lambda_N$ are the N eigenvalues of the Jacobian $\partial f/\partial q$. Finally $\overline{S}_j(t)$ are some discretizations of the source term.

Now coming back to the shallow water equations (1) and (2), we have

$$q = \begin{bmatrix} w \\ hu \end{bmatrix}, \quad f(q) = \begin{bmatrix} hu \\ \dfrac{(hu)^2}{w-B} + \dfrac{1}{2}g(w-B)^2 \end{bmatrix}, \quad S(q,x,t) = \begin{bmatrix} 0 \\ -g(w-B)B_x \end{bmatrix}. \tag{11}$$

Therefore, the one-sided local speeds of wave propagation are

$$a^{+}_{j+\frac{1}{2}} = \max\left\{ u^{+}_{j+\frac{1}{2}} + \sqrt{gh^{+}_{j+\frac{1}{2}}}, u^{-}_{j+\frac{1}{2}} + \sqrt{gh^{-}_{j+\frac{1}{2}}}, 0 \right\}, \tag{12}$$

$$a^{-}_{j+\frac{1}{2}} = \min\left\{ u^{+}_{j+\frac{1}{2}} - \sqrt{gh^{+}_{j+\frac{1}{2}}}, u^{-}_{j+\frac{1}{2}} - \sqrt{gh^{-}_{j+\frac{1}{2}}}, 0 \right\}. \tag{13}$$

Note that for our shallow water equations, a second order discretization of the source term is given by $\overline{S}^{(1)}_j(t) = 0$ for the first component of the source vector and

$$\overline{S}^{(2)}_j(t) \approx -g \times \frac{(w^{-}_{j+\frac{1}{2}} - B(x_{j+\frac{1}{2}})) + (w^{+}_{j-\frac{1}{2}} - B(x_{j-\frac{1}{2}}))}{2} \times \frac{B(x_{j+\frac{1}{2}}) - B(x_{j-\frac{1}{2}})}{\Delta x}. \tag{14}$$

for the second component of the source vector.

As in this paper we implement a second order finite volume method, a limiter is used to suppress artificial oscillation in numerical solution. We use the minmod limiter

$$\sigma^n_j = \text{minmod}\left(\frac{q^n_j - q^n_{j-1}}{\Delta x}, \frac{q^n_{j+1} - q^n_j}{\Delta x}\right), \tag{15}$$

where

$$\text{minmod (a,b)} := \frac{1}{2}\left(\text{sgn}(a) + \text{sgn}(b)\right)\min(|a|, |b|). \tag{16}$$

This minmod limiter is implemented at every time step of the numerical evolution of finite volume methods.

3. Numerical Results

In this section we provide a numerical demonstration for our claim that the method which is second order accurate in space and in time may actually less accurate than the one which is second order accurate in space but first order in time.

As a test case, we consider a dam break problem with the spatial domain $-1 < x < 1$. Initially there is dam wall at the point $x = 0$. Note that in this paper all quantities are measured in Systeme Intenational (SI) units, so we omit the writings of units for simplicity. The initial water depth on the left of the wall is 10, while on the right of the wall is 4. The acceleration due to gravity is 9.81.

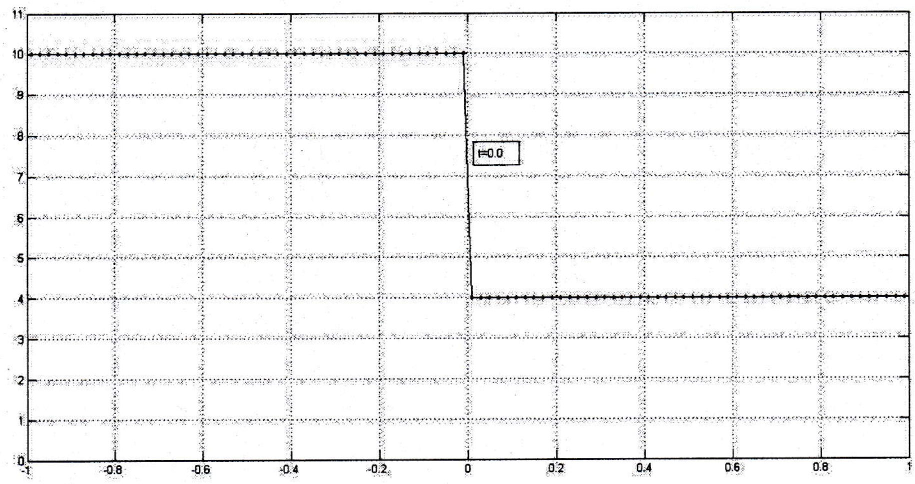

Figure 1. Initial condition of water surface for the dam break problem at time $t = 0.0$. Here the solid line shows the analytical solution, whereas the dotted line shows the numerical solution using 100 cells.

Tabel 1. Numerical absolute L^1 errors for different numbers of cells with second order in space and *first order in time*.

Number of cells	Error of stage	Error of discharge	Error of velocity
100	0.0478594	0.3738300	0.0600871
200	0.0243614	0.2039731	0.0296580
400	0.0109585	0.0868961	0.0135623
800	0.0053363	0.0425737	0.0065963
1600	0.0026519	0.0215230	0.0032709

Tabel 2. Numerical absolute L^1 errors for different numbers of cells with second order in space and *second order in time*.

Number of cells	Error of stage	Error of discharge	Error of velocity
100	0.0569322	0.4526225	0.0705843
200	0.0296662	0.2345232	0.0373933
400	0.0140063	0.1138966	0.0171898
800	0.0070051	0.0571872	0.0085543
1600	0.0035362	0.0291737	0.0042862

As we run the simulations, absolute L^1 errors are quantified at time $t = 0.05$. Those errors are summarized in Table 1 and Table 2 for various numbers of cells. Table 1 contains errors for different numbers of cells with second order in space and *first order in time*. In addition Table 2 presents errors for numbers of cells with second order in space and *second order in time*. Note that the analytical solution of this dam break problem has been derived by Stoker [17] and extended by Mungkasi [12, 13]. This test case is also used by Goutal and Maurel [3].

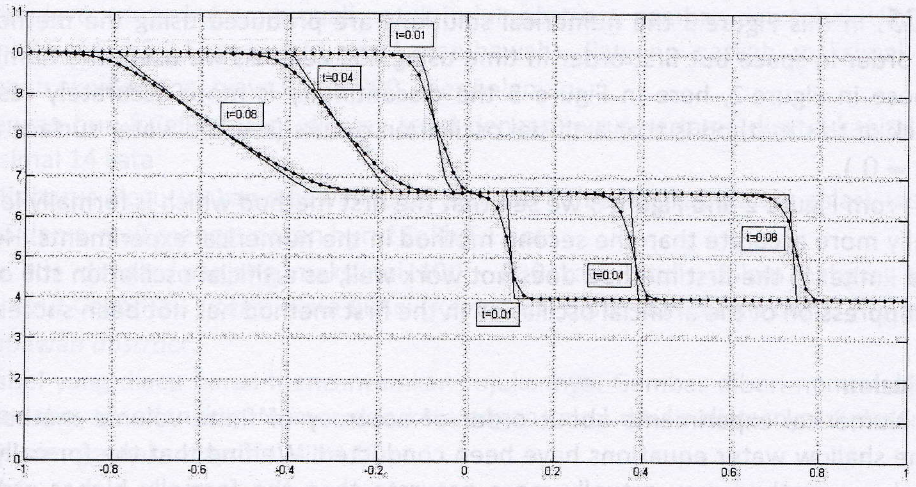

Figure 2. Water surface for the breach of the dam at time $t = 0.01, t = 0.04, t = 0.08$. Here solid lines show analytical solutions, whereas dotted lines show numerical solutions. The numerical solutions are produced using the method which is second order in space and second order in time using 100 cells.

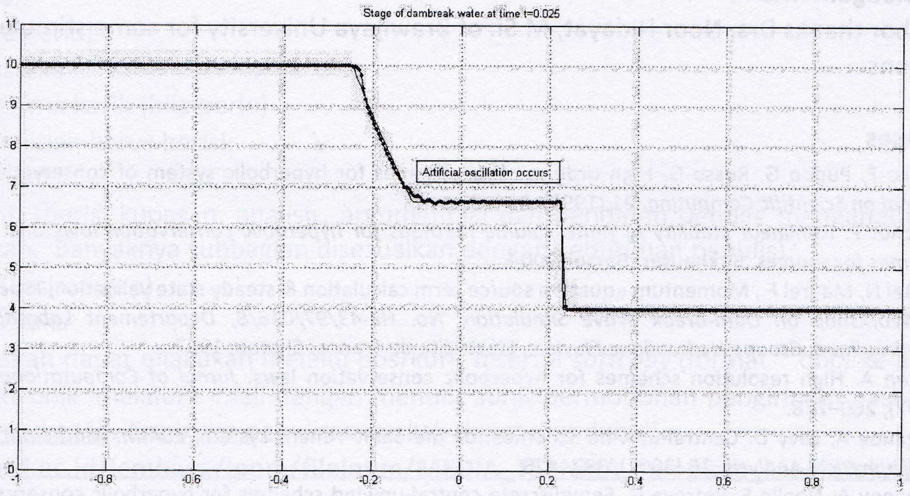

Figure 3. Water surface for the breach of the dam at time $t = 0.025$. Here solid lines show analytical solutions, whereas dotted lines show numerical solutions. The numerical solutions are produced using the method which is second order in space but first order in time using 400 cells. Here the discontinuity is accurately resolved, but artificial oscillation occurs.

Representatives of numerical results of the second method are shown in Figure 1 and Figure 2. Here Figure 1 illustrates the initial condition both for analytical and numerical solution. Then as time evolves, the water surface is shown in Figure 2. In this Figure 2 we see the propagation of water waves or flows at time $t = 0.01, t = 0.04, t = 0.08$. In this Figure 2 we have used the second finite volume method which is second order accurate in space as well as in time with 100 cells. We can see here some diffusion occurs around the corners of the solutions. This diffusion can be minimized using more number of cells involved in the computation.

In comparison Figure 3 shows water surface for the breach of the dam at time $t = 0.025$. In this Figure 3 the numerical solutions are produced using the method which is second order in space but first order in time using 400 cells. As we use more number of cells than those in Figure 2, here in Figure 3 the discontinuity is more accurately resolved. The drawback of this method is that artificial oscillation occurs (see the water surface around the point $x = 0$).

From Figure 2 and Figure 3 we see that the first method which is formally less accurate is actually more accurate than the second method in the numerical experiments. However the minmod limiter in the first method does not work well, as artificial oscillation still occurs. That is, the suppression of the artificial oscillation in the first method has not been successful.

4. Conclusion

Numerical experiments about order of accuracy of finite volume methods used to solve the shallow water equations have been conducted. We find that the formally low order finite volume method may actually more accurate than the formally higher order method regardless of the quality of the numerical solution. Further research could be taken in the direction of the numerical analysis of these results.

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