

## ON A STUDY OF STAGGERED LEAPFROG SCHEME FOR LINEAR SHALLOW WATER EQUATIONS

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**ABSTRACT.** In this paper, we present an analytical and numerical study of staggered leapfrog scheme for linear shallow water equation. It is shown that the scheme is stable when Courant number  $< 1$ , has second order accurate in both time and space, and there is no damping error in this scheme. We implement the scheme to simulate standing wave in a closed basin to show that the surface motions stay zero in a node and have constant amplitude at the antinode. For an external force given into the basin, it will induce a resonance, which cause the wave amplitude is getting bigger at the position of antinode. Moreover, we simulate a wave in a tidal basin, and show that the model has infinite spin up time. For a linear shallow water equation with linear friction, it is shown that the model has finite spin up time.

### 1. INTRODUCTION

Mesinger and Arakawa in 1976 applied the leapfrog scheme to the shallow water equations, [3]. The scheme is stable for Courant number  $< 0.5$  in the staggered C-grids, [2]. Thereafter, some researchers introduced a modified leapfrog schemes for shallow water equations, which is a semi-implicit scheme, but more stable than the previous leapfrog scheme. Zhou proposed the 1D leapfrog scheme, which is at every time step, both pressure and momentum fields are computed, [8]. Sun and Sun, using the time-averaged heights in the

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pressure gradient force, has modified the leapfrog scheme and showed that their scheme is stable when Courant number  $< 1$ , [6]. Although the scheme is relatively stable, however, the semi-implicit scheme may indicate a low convergence, [7].

To design an efficient numerical scheme, in this study we implement an explicit leapfrog scheme for linear shallow water equation in staggered grid scheme proposed by Stelling and Duinmeijer, [5]. The staggered leapfrog scheme was also considered in [4]. The continuity and momentum equation are approximated on different cells, that are around full grid points and half grid points, respectively in [1]. We show that the scheme is stable when Courant number  $< 1$ . Moreover, we investigate the ability of numerical scheme to simulate standing wave in a closed basin and tidal basin.

## 2. GOVERNING EQUATIONS

Consider an ideal fluid which is incompressible and has no viscosity over a flat bottom. The fluid domain is bounded by the sea bed at  $z = -d_0$ , where  $d_0$  is constant, and  $z = \eta(x, t)$  is the free surface, where  $t$  represents time. The free surface flow of the fluid in shallow areas can be modelled with the 1D shallow water equations (SWE) for flat bottom, without friction:

$$\eta_t + ((\eta + d_0)u)_x = 0$$

$$u_t + uu_x + g\eta_x = 0$$

with  $u(x, t)$  denote the horizontal component of fluid particle velocity, and  $g$  is the gravity constant. For small deformation of the steady situation, the model can be simplified in the linear form, when the non-linear terms are negligible, which are expressed as follows:

$$(2.1) \quad \eta_t = -d_0 u_x$$

$$(2.2) \quad u_t = -g\eta_x.$$

Derivating equations (2.1) and (2.2) with respect to  $t$  and  $x$  yields, respectively,  $\eta_{tt} = -d_0 u_{xt}$  and  $u_{tx} = -g\eta_{xx}$ . Then we have

$$(2.3) \quad \eta_{tt} - gd_0\eta_{xx} = 0.$$

Similarly, if we derivate (2.1) and (2.2) with respect to  $x$  and  $t$  yields, respectively,  $\eta_{tx} = -d_0 u_{xx}$  and  $u_{tt} = -g\eta_{xt}$ . It results

$$(2.4) \quad u_{tt} - gd_0 u_{xx} = 0$$

Equations (2.3) and (2.4) are two standard wave equations. Therefore, the linear SWE (2.1) and (2.2) is equivalent with a pair of wave equations (2.3) and (2.4). They have the same left and right running characteristics.

An analytical solution of the linear SWE (2.3) and (2.4) with its boundary conditions can be derived by separating variable method. Consider equation (2.4), with boundary condition  $u(0, t) = 0$  and  $u(L, t) = 0$ ; it represent hard wall boundaries. Suppose that  $u(x, t) = X(x)T(t)$  is a solution of the equation. Substituting the solution to equation (2.4) will result  $XT'' = c_0^2 X''T$ , with  $c_0^2 = gd_0$ . If we divide the equation by  $-c_0^2 XT$ , we obtain

$$-\frac{T''}{c_0^2 T} = -\frac{X''}{X} = \lambda$$

where  $\lambda$  is a constant. Thus, it results a pair of ordinary differential equations

$$T'' + c_0^2 \lambda T = 0; \quad X'' + \lambda X = 0.$$

The system will have non trivial solutions when  $\lambda > 0$ . Suppose  $\lambda = \beta^2$ . Then the solutions are:

$$(2.5) \quad T(t) = A \cos(\beta c_0 t) + B \sin(\beta c_0 t)$$

$$(2.6) \quad X(x) = C \cos \beta x + D \sin \beta x.$$

From the boundary condition  $u(0, t) = X(0)T(t) = X(L)T(t) = 0$  for every  $t$ , we have  $X(0) = X(L) = 0$ . Substitute the results into equation (2.6), we obtain:

$$X(0) = C = 0; \quad X(L) = D \sin \beta L = 0.$$

If  $C = D = 0$ , then  $u = 0$ , i.e. a trivial solution. In order to get a non-trivial solution, it should be  $\sin \beta L = 0$  or  $\beta L = n\pi$ . Therefore,  $\lambda_n = \beta^2 = \left(\frac{n\pi}{L}\right)^2$ ,  $n = 1, 2, 3, \dots$ , which are the eigenvalues with eigen function  $X_n(x) = \sin \frac{n\pi}{L}x$ . Hence, the solution is

$$(2.7) \quad u_n(x, t) = \left( A_n \cos c_0 \frac{n\pi}{L} t + B_n \sin c_0 \frac{n\pi}{L} t \right) \sin \frac{n\pi}{L} x, \quad n = 1, 2, 3, \dots$$

We can do the same way in order to get the exact solution for  $\eta(x, t)$ .

### 3. THE STAGGERED LEAPFROG SCHEME

In this paper, we formulate the staggered leapfrog method to solve the linear SWE (2.1) and (2.2). We define the partition points as  $x_{1/2} = 0, x_1, \dots, x_{j-1/2}, x_j, x_{j+1/2}, \dots, x_{N+1/2} = L$ . The equations are discretized around grid points  $(x_j, t_n)$  and  $(x_{j+1/2}, t_{n+1/2})$ , respectively, as follows [4]:

$$(3.1) \quad \frac{\eta_j^n - \eta_j^{n-1}}{\Delta t} + d_0 \frac{u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n}{\Delta x} = 0$$

$$(3.2) \quad \frac{u_{j+\frac{1}{2}}^{n+1} - u_{j+\frac{1}{2}}^n}{\Delta t} + g \frac{\eta_{j+1}^{n+\frac{1}{2}} - \eta_j^{n+\frac{1}{2}}}{\Delta x} = 0$$

**3.1. Von-Neumann stability analysis.** The stability condition of the scheme is obtained using Von-Neumann method, by substituting  $\eta_j^n = \rho^n e^{iaj\Delta x}$  and  $u_j^n = r^n e^{iaj\Delta x}$  to Eqs. (3.1) and (3.2). Then, if we divide the equations by  $e^{iaj\Delta x}$  and  $e^{ia(j+\frac{1}{2})\Delta x}$ , respectively, we obtain

$$(3.3) \quad \rho^{n+1} = \rho^n - d_0 \frac{\Delta t}{\Delta x} 2i \sin\left(a \frac{\Delta x}{2}\right) r^n$$

and

$$(3.4) \quad r^{n+1} = r^n - g \frac{\Delta t}{\Delta x} 2i \sin\left(a \frac{\Delta x}{2}\right) \rho^{n+1}.$$

Equations (3.3) and (3.4) can be written in a matrix form as follows

$$\begin{bmatrix} \rho^{n+1} \\ r^{n+1} \end{bmatrix} = A \begin{bmatrix} \rho^n \\ r^n \end{bmatrix}$$

with

$$A = \begin{bmatrix} 1 & -d_0 \frac{\Delta t}{\Delta x} 2i \sin\left(a \frac{\Delta x}{2}\right) \\ -g \frac{\Delta t}{\Delta x} 2i \sin\left(a \frac{\Delta x}{2}\right) & -4gd_0 \left(\frac{\Delta t}{\Delta x}\right)^2 \sin^2\left(a \frac{\Delta x}{2}\right) + 1 \end{bmatrix}.$$

The eigenvalue of matrix  $A$ , that is  $\lambda$ , satisfies  $|A - \lambda I| = 0$ , i.e.

$$1 + \left(4 C_0^2 \sin^2\left(a \frac{\Delta x}{2}\right) - 2\right) \lambda + \lambda^2 = 0$$

with  $C_0^2 = gd_0 \left(\frac{\Delta t}{\Delta x}\right)^2$ . Therefore, the eigenvalues of  $A$  are

$$\lambda_{1,2} = 1 - 2C_0^2 \sin^2\left(a \frac{\Delta x}{2}\right) \pm 2\sqrt{C_0^4 \sin^4\left(a \frac{\Delta x}{2}\right) - C_0^2 \sin^2\left(a \frac{\Delta x}{2}\right)}.$$

The stability of the scheme is fulfilled if  $\|\lambda\| = \max |\lambda_i| \leq 1$ , that is

$$(3.5) \quad \left| 1 - 2C_0^2 \sin^2 \left( a \frac{\Delta x}{2} \right) - 2\sqrt{C_0^4 \sin^4 \left( a \frac{\Delta x}{2} \right) - C_0^2 \sin^2 \left( a \frac{\Delta x}{2} \right)} \right| \leq 1.$$

Suppose  $y = C_0 \sin \left( a \frac{\Delta x}{2} \right)$ . Eq. (3.5) can be written as follows

$$\left| 1 - 2y^2 - 2\sqrt{y^4 - y^2} \right| \leq 1 \iff -1 \leq C_0 \sin \left( a \frac{\Delta x}{2} \right) \leq 1.$$

Hence, as  $C_0$  always be positive, the scheme is stable if  $C_0 = \sqrt{gd_0} \frac{\Delta t}{\Delta x} \leq 1$ . Here  $C_0$  is a Courant number.

The eigenvalue of matrix  $A$  can also be stated as  $\lambda_{1,2} = 1 - 2y^2 \pm 2\sqrt{y^4 - y^2}$ . As the scheme is stable when  $y^2 \leq 1$ , so the eigenvalue are complex, and they can be written as

$$\lambda_{1,2} = 1 - 2y^2 \pm i2y\sqrt{1 - y^2}$$

Thus,  $|\lambda_{1,2}| = \sqrt{(1 - 2y^2)^2 + 4y^2(1 - 2y^2)^2} = \sqrt{1 - 4y^2 + 4y^4 + 4y^2 - 4y^4} = 1$ . Therefore, there is no damping error in this scheme.

**3.2. Order of accuracy.** The accuracy of the leapfrog scheme can be determined using Taylor expansion of  $\eta_{j+1}^{n+1}$ :

$$\begin{aligned} \eta_{j+1}^{n+1} &= \eta_j^{n+1} + \eta_x|_j^{n+1} \Delta x + \eta_{xx}|_j^{n+1} \cdot \frac{1}{2} \Delta x^2 + O(\Delta x^3) \\ \iff \eta_{j+1}^{n+1} - \eta_j^{n+1} &= \eta_x|_j^n \Delta x + \eta_{xt}|_j^n \Delta t \Delta x + \eta_{xx}|_j^n \cdot \frac{1}{2} \Delta x^2 + O(\Delta x^3) \end{aligned}$$

Substituting the Taylor expansion to equation (3.2), we obtain:

$$\begin{aligned} u_t|_j^n + u_{tt}|_j^n \cdot \frac{1}{2} \Delta t + g \left[ \eta_x|_j^n + \eta_{xt}|_j^n \Delta t + \eta_{xx}|_j^n \cdot \frac{1}{2} \Delta x \right] + O(\Delta t^2, \Delta x^2) &= 0 \\ \iff u_t|_j^n + g\eta_x|_j^n - u_{tt}|_j^n \cdot \frac{1}{2} \Delta t - u_{tx}|_j^n \Delta x + O(\Delta t^2, \Delta x^2) &= 0. \end{aligned}$$

Thus, the *error* of the scheme is

$$error = -u_{tt}|_j^n \cdot \frac{1}{2} \Delta t - u_{tx}|_j^n \Delta x + O(\Delta t^2, \Delta x^2)$$

Hence, the scheme is of second order for time and space, that is  $O(\Delta t^2, \Delta x^2)$ .

## 4. NUMERICAL SIMULATIONS

In this section, the leapfrog scheme (3.1) and (3.2) are implemented for simulating some cases. First, simulation of standing wave in a closed basin, to show that the motion will continue for infinitely long time without any damping. We expand the case by introduce an external force into the closed basin, to show that there is resonance in surface motion. Moreover, we simulate wave in a tidal basin, using linear SWE with and without linear friction.

**4.1. Standing wave in a closed basin.** Consider standing wave in a closed basin with length  $L = 20\text{ m}$ , and depth  $d_0 = 10\text{ m}$ , with hard wall boundaries at the left and right ends. The initial surface elevation is  $\eta(x, 0) = 0.1 \cos(\pi x/L)$ .

The standing wave motion for  $t = 0, 2, 5, 15$  and  $t = 18\text{ s}$  are shown in Figure 1. For the computation we use  $g = 9.81$ ,  $\Delta x = 0.5$  and  $\Delta t = 0.05$ . It can be seen that the surface motion  $\eta(L/2, t)$  and  $\eta(L, t)$  will stays zero and have constant amplitude, respectively.

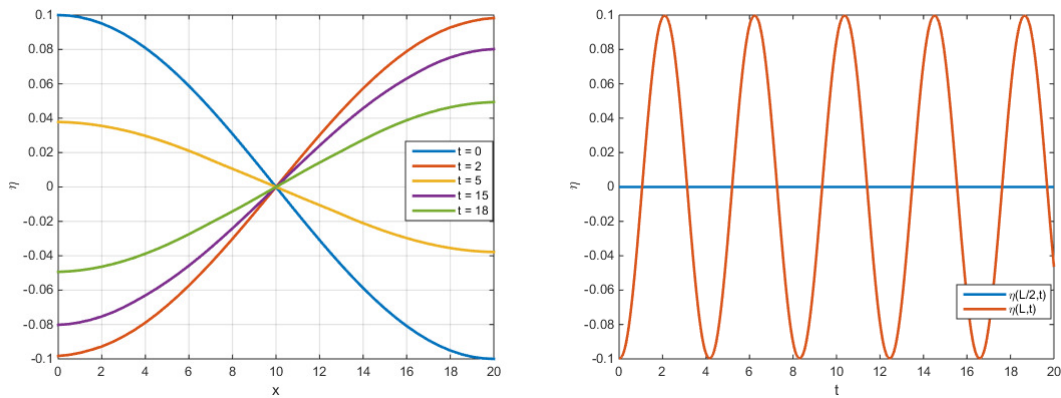


FIGURE 1. Standing wave motion for  $t = 0, 2, 5, 15$  and  $t = 18\text{ s}$  (left); Surface motion at  $x = L/2$  and  $x = L$  (right).

Suppose that we introduce a harmonic external force  $f(t) = 0.01 \sin \omega t$  with certain frequency  $\omega = \pi \sqrt{gd_0}/L$  into the closed basin. The force will induce resonance in the closed basin. Figure 2 shows that resonance resulted in surface motion at the position  $x = L/2$  no longer staying at zero. Likewise, the wave amplitude at the position  $x = L$  is not constant, it is getting bigger.

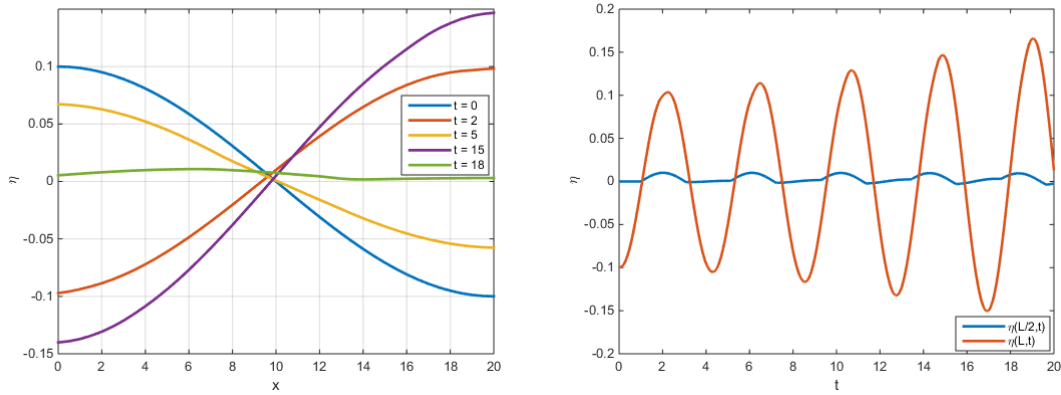


FIGURE 2. Resonance in a closed basin at  $t = 0, 2, 5, 15$  and  $t = 18s$  (left); Surface motion due to resonance at  $x = L/2$  and  $x = L$  (right).

**4.2. Wave in a tidal basin.** Consider a tidal basin, with length  $L = 20\text{ m}$ , depth  $d_0 = 10\text{ m}$ , and with hard wall at the right side  $x = L$ . It is assumed that at the initial state, the water surface is  $\eta(x, 0) = 0$ , with velocity  $u(x, 0) = 0$ , and a monochromatic wave  $\eta(0, t) = 0.2 \sin \pi t$  enters from the left side. The wave propagates to the right, hits the hard wall and produces a reflected wave, which travels to the left. Superposition of the wave which travels to the right and left will produce a wave which has twice the amplitude of the previous wave, as seen in Figure 3. In this simulation, an incoming wave has amplitude 0.2 and after

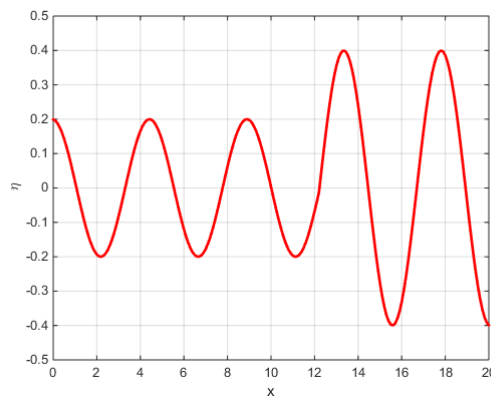


FIGURE 3. Reflected wave in a tidal basin at  $t = 12.5\text{ s}$ .

hits the right wall it results a reflected wave with amplitude 0.4, which travels to the left.

Consider an incoming monochromatic wave is  $\eta(0, t) = 0.2 \cos 2\pi t/T$ , with  $T = 80$ . The surface elevation at position  $x = 0$  and  $x = L$  are shown in Figure 4 (left). From the figure we can explore the spin up time, that is the time for initial

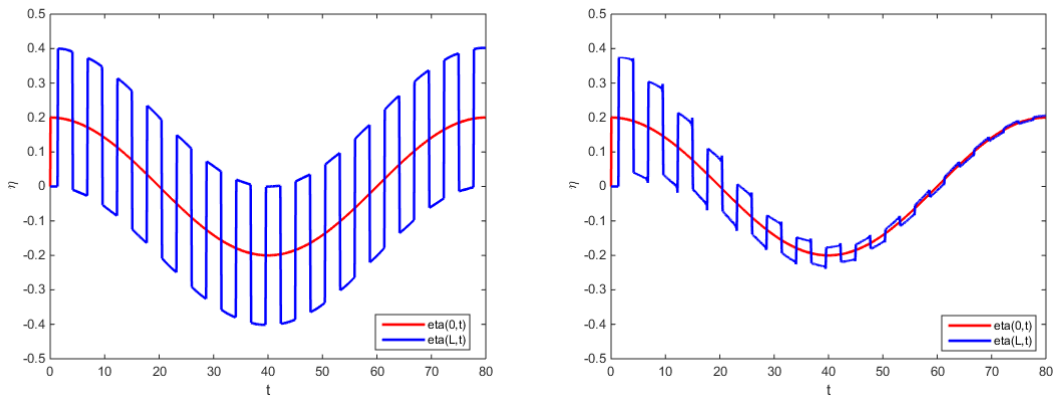


FIGURE 4. Infinite spin up time (left); Finite spin up time (right).

state to vanish in the solution. It is shown that for a long time, the solution are still influenced by the initial state. Therefore, we say that this model has infinite spin up time.

Consider the linear SWE with linear friction:

$$\eta_t = -d_0 u_x; \quad u_t = -g\eta_x - C_f u.$$

Using the same information from the previous part, the surface elevation at  $x = 0$  and  $x = L$ , that are  $\eta(0, t)$  and  $\eta(L, t)$ , respectively, are shown in Figure 4 (right). It can be seen that the initial state influence the solution only in a certain period of time. Therefore, we say this model has finite spin up time. It is because there is an additional term which is a function of  $u$ .

### 5. CONCLUSION

The staggered leapfrog scheme is applied to linear shallow water equation. The von-Neumann stability analysis shows that the scheme is stable if Courant number  $< 1$ . The scheme is explicit and second order accurate in both time and



space. The numerical simulation for standing wave in a closed basin show that the scheme has no damping error.

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