

# **The Significance of Spatial Reconstruction in Finite Volume Methods for the Shallow Water Equations**

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## **Abstract**

We study the significance of the spatial reconstruction when solving the one dimensional shallow water equations using a finite volume method. For that aim, we implement the explicit forward Euler method for temporal integration while the spatial discretization is performed by finite volume method. We compare the results of constant spatial reconstruction with those of linear spatial reconstruction. The numerical tests include the steady state of a lake at rest, the steady state of moving water and an unsteady state of dam break problem. It is shown that the spatial reconstruction has a significant role in the accuracy of the finite volume method.

**Keywords:** spatial reconstruction, finite volume, shallow water equations

## 1 Introduction

The free surface, unsteady water flow is modeled by the well-known Saint-Venant equations. This model is also called the shallow water (wave) equations. Accurately solving these equations is important, because it can help simulations of natural events, such as floods, tsunamis, dam breaks, tides, etc. To get numerical solutions of these equations, there are many numerical methods available in the literatures [8, 9, 12, 15], for examples finite difference and finite volume methods. Finite difference methods are based on the differential form of the equations. They may lead to some difficulties when we want to resolve discontinuities, because differential equations assume that solutions are smooth. In contrast, finite volume methods are based on the integral form of the equations. Integral equations do not assume smoothness of their solutions, and hence finite volume methods are able to resolve smooth and nonsmooth solutions (see [2, 6, 7, 8, 13]). However the accuracy of those numerical methods will be dependent on the integration with respect to both time (temporal) and space (spatial).

In this paper we investigate the significance of spatial reconstruction in finite volume methods when solving the shallow water equations. We show that a higher order reconstruction of the spatial domain can improve the accuracy of the numerical methods. To do so we use one type of temporal integration. We then compare the performance of two types (that is, constant and linear) of spatial reconstructions.

This paper is organized as follows. Shallow water equations are recalled in Section 2. We present the finite volume method that we use to solve the shallow water equations in Section 3. Numerical results are presented in Section 4. We draw some concluding remarks in Section 5.

## 2 Shallow Water Equations

We consider the following one dimensional shallow water equations

$$h_t + (hu)_x = 0 \quad (1)$$

$$(hu)_t + \left( hu^2 + \frac{1}{2} gh^2 \right)_x = -ghB_x \quad (2)$$

where  $t$  denotes the time variable,  $x$  denotes the space variable,  $h = h(x, t)$  is water height or depth,  $u = u(x, t)$  is velocity,  $B = B(x)$  represents the bottom elevation or topography, and  $g$  is the acceleration due to gravity. The absolute water level (stage) is defined as  $w(x, t) := h(x, t) + B(x)$ . Equations (1) and (2) can be written in vector form as

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = \mathbf{s} \quad (3)$$

in which

$$\mathbf{q} = \begin{pmatrix} w \\ hu \end{pmatrix}, \quad \mathbf{f}(\mathbf{q}) = \left( \frac{(hu)^2}{w-B} + \frac{1}{2} g(w-B)^2 \right), \quad \mathbf{s} = \begin{pmatrix} 0 \\ -g(w-B)B_x \end{pmatrix}. \quad (4)$$

We refer to [5] for these forms of shallow water equations.

### 3 Finite Volume Methods

In this section, we recall a finite volume method proposed in [5, 7], which was developed for steady state problems. The finite volume method can then be used to solve steady and unsteady state problems. Here we assume that the space is discretized into a finite number of cells uniformly with cell width  $\Delta x$  and that time is also discretized uniformly with size of time step is  $\Delta t$ . Then equation (3) can be solved using the finite volume method

$$\mathbf{Q}_j^{n+1} = \mathbf{Q}_j^n - \frac{\Delta t}{\Delta x} \left( \mathbf{F}_{j+\frac{1}{2}}^n - \mathbf{F}_{j-\frac{1}{2}}^n \right) + \mathbf{S}_j^n \quad (5)$$

where

$$\mathbf{Q}_j^n \approx \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathbf{q}(x, t^n) dx, \quad (6)$$

$$\mathbf{F}_{j+\frac{1}{2}}^n \approx \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathbf{f} \left( \mathbf{q}(x_{j+\frac{1}{2}}, t) \right) dt. \quad (7)$$

Here subscript  $j$  represents the  $j$ th cell and superscript  $n$  denotes the time level at  $n \Delta t$ . This means  $x_{j+\frac{1}{2}} = x_j + \frac{\Delta x}{2}$  is the right vertex of the  $j$ th cell. The variable  $\mathbf{S}_j^n$  is an approximation of the analytical source  $\mathbf{s}$ .

At a vertex of a cell, we use approximations for both sides

$$\mathbf{Q}_{j+\frac{1}{2}}^\pm = \begin{pmatrix} w \\ hu \end{pmatrix}_{j+\frac{1}{2}}^\pm, \quad \mathbf{f}(\mathbf{U})_{j+\frac{1}{2}}^\pm = \left( \frac{(hu)^2}{w-B} + \frac{1}{2} g(w-B)^2 \right)_{j+\frac{1}{2}}^\pm \quad (8)$$

in which the superscript “-” is for the left side approximation and the superscript “+” is for the right side approximation of that vertex. Both approximations at left and right sides of the vertex is obtained from polynomial reconstructions

$$\mathbf{Q}(x, t^n) = \sum_j \mathbf{p}_j^n(x) \chi_j(x), \quad (9)$$

where  $\mathbf{p}_j^n(x)$  is a polynomial supported on the interval  $\left[ x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}} \right]$  which is centered at the midpoint  $x_j = j\Delta x$ , and is defined at time  $t = t^n$ ,  $\chi_j(x)$  is the characteristic function. Note that for the first order space discretization we only require constant polynomials, while for the second order we need piecewise linear

polynomials. Let the linear functions are

$$p_j^n(x) = Q_j^n + \sigma_j^n(x - x_j), \quad x_{j-\frac{1}{2}} \leq x \leq x_{j+\frac{1}{2}} \tag{10}$$

where  $\sigma_j^n$  is the slope. This slope must be chosen with care so that numerical solutions of the shallow water equations are non oscillatory. This requires that the value of the slope must be limited. A well-known type of that limiter is the minmod slope. The minmod limiter was used by a number of authors for their work, such as in [1, 4, 5, 10, 11, 16]. In this paper, we use the following minmod limiter as in [5]:

$$\sigma_j^n = \text{minmod} \left( \frac{Q_j^n - Q_{j-1}^n}{\Delta x}, \frac{Q_{j+1}^n - Q_j^n}{\Delta x} \right), \tag{11}$$

where  $\text{minmod}(a, b) := \frac{1}{2}(\text{sgn}(a) + \text{sgn}(b)) \min(|a|, |b|)$ . Note that if  $\sigma_j^n = 0$  for all  $j$ , then the space discretization becomes first order.

In order that the finite volume method is able to solve the steady state problems (as well as unsteady state problems), we use the central semi discrete scheme. Here we implement a numerical flux,  $H_{j+\frac{1}{2}}(t)$ , as in [7] and numerical source terms,  $\bar{S}_j^n$ , as given in [5]. This scheme is then

$$\frac{d}{dt} Q_j(t) = -\frac{1}{\Delta x} \left( H_{j+\frac{1}{2}}(t) - H_{j-\frac{1}{2}}(t) \right) + \bar{S}_j^n \tag{12}$$

where

$$H_{j+\frac{1}{2}}(t) := \frac{1}{2} \left( f \left( Q_{j+\frac{1}{2}}^+(t) \right) + f \left( Q_{j+\frac{1}{2}}^-(t) \right) \right) - \frac{1}{2} a_{j+\frac{1}{2}}(t) \left[ Q_{j+\frac{1}{2}}^+(t) - Q_{j+\frac{1}{2}}^-(t) \right]$$

and  $\bar{S}_j^n = \begin{pmatrix} \bar{S}_j^{(1)}(t) \\ \bar{S}_j^{(2)}(t) \end{pmatrix}$ . Here  $\bar{S}_j^{(1)}(t) = 0$  and

$$\bar{S}_j^{(2)}(t) \approx -g \times \frac{B(x_{j+\frac{1}{2}}) - B(x_{j-\frac{1}{2}})}{\Delta x} \times \frac{\left( w_{j+\frac{1}{2}}^- - B(x_{j+\frac{1}{2}}) \right) + \left( w_{j-\frac{1}{2}}^+ - B(x_{j-\frac{1}{2}}) \right)}{2}. \tag{13}$$

For simplicity, we apply the forward Euler method to solve equation (12).

### 4 Numerical Results

In this section we present numerical results for three test cases, namely

(a) the steady state of a lake at rest, (b) the steady state of moving water and (c) an unsteady state of dam break problem. We compare the results of first order discretization in space (Method I) and those of second order spatial discretization (Method II).

Our numerical setting is as follows. We test the finite volume method for the following three cases using the uniform cells. The number of cells ( $N$ ) are chosen to be 100, 200, 400, 800, 1600, 3200. For the time step, we take uniform  $\Delta t = 0.01 \times \Delta x$ . We calculate the numerical error and the convergence rate. To quantify numerical errors ( $E$ ), we use the  $L^1$  absolute error

$$E = \frac{1}{N} \sum_{i=1}^N |q(x_i) - Q_i| \tag{14}$$

where  $q(x_i)$  and  $Q_i$  are the exact and numerical solution at  $x_i$ , respectively. To compute the convergence rate, we use the following formula [14]

$$\text{Rate} = \frac{1}{N-1} \sum_i^{N-1} R_i \tag{15}$$

where

$$R_i = \frac{\log\left(\frac{E_i}{E_{i+1}}\right)}{\log\left(\frac{\Delta x_i}{\Delta x_{i+1}}\right)}. \tag{16}$$

Here  $E_i$  is the error at the  $i$ th cell. All quantities are measured in SI units. Therefore, any omitted units should be noted to have SI units.

**(a). The steady state of a lake at rest**

The test of a lake at rest problem is intended to see if the above finite volume method is able to resolve the steady state of still water. We follow the test presented in [3]. Consider a lake with 1500 m of length. At the downstream boundary the water level is imposed to be 12 m and at the upstream there is no discharge. The initial condition is water at rest at the level of 12 m. The analytical solution is obviously

- water at rest: discharge and flow velocity are zero,
- flat free surface: water level stays at the initial level of 12 m.

We consider the geometry as given in Figure 1 and the complete description of this geometry (see [3]) is given in the following Table 1.

Table 1. Complete description of the geometry. Here  $x$  is abscissa of  $B$  and  $B(x)$  is value of  $B$  function at  $x$  point. Both  $x$  and  $B(x)$  are measured in meters.

$x$	0	50	100	150	250	300	350	400	425	435	450	470	475	500	505
$B(x)$	0	0	2.5	5	5	3	5	5	7.5	8	9	9	9	9.1	9
$x$	530	550	565	575	600	650	700	750	800	820	900	950	1000	1500	
$B(x)$	9	6	5.5	5.5	5	4	3	3	2.3	2	1.2	0.4	0	0	

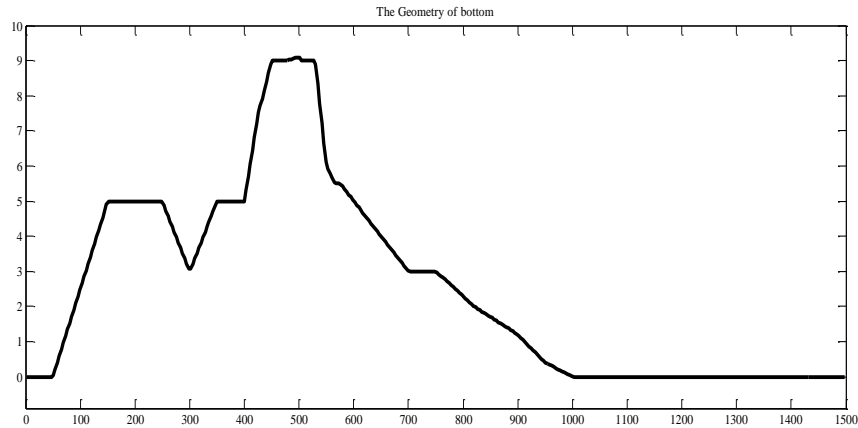


Figure 1. The geometry profile of the lake at rest.

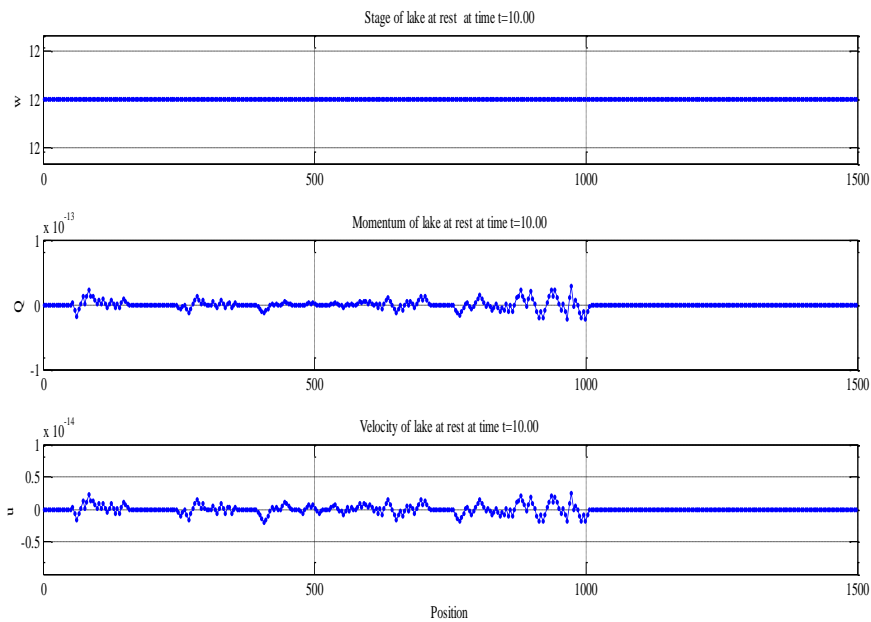


Figure 2. Stage, momentum and velocity of lake at rest problem by Method I.  
Here we use 400 cells and final time 10 seconds.

We confirm that both methods are well-balanced, that is, the steady state of the lake at rest is preserved up to discrete level, see Figure 2 for the stage, momentum and velocity produced by Method I. We note that for that scale, Method II generates the same plots. These results are calculated using 400 cells with final time 10 seconds.

**(b). The steady state of moving water over a bump**

This test of steady flow over a bump is intended to verify if the numerical

method can resolve the steady state of moving water. We use the following data geometry [3]. The channel length is 25 m (meter) and the bottom equation is

$$B(x) = \begin{cases} 0 & \text{if } 0 < x < 8, 12 < x < 25, \\ 0.2 - 0.05(x - 10)^2 & \text{if } 8 < x < 12. \end{cases} \quad (17)$$

The boundary and initial conditions are as follow. At downstream, the water level is imposed to be 2 m. At upstream, the water discharge is imposed to be 4.42 m<sup>3</sup>/s (s= second). Initially, we have a constant water level which is equal to the level imposed downstream with discharge equals to zero. The stage, momentum and velocity obtained by Method II at final time 30 seconds are plotted in Figure 2. It is seen that the numerical solutions agree very well with the exact solution. We note that Method I also generates the same plot. However, detail analysis shows that Method II has much better accuracy as shown in Table 2.

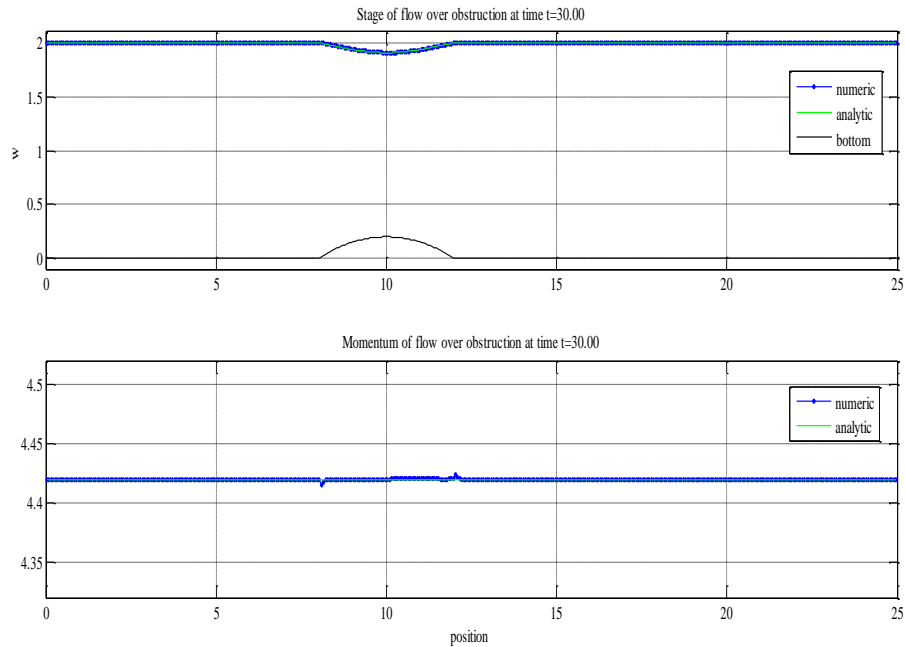


Figure 3. Stage and momentum on flow of obstruction by Method II. Here we use 400 cells and final time 30 seconds.

Table 2. The error of obstruction problems by Method I and II.

N	Method I				Method II			
	Error_w	CR_w	Error_Q	CR_Q	Error_w	CR_w	Error_Q	CR_Q
100	0.0053		0.0102		0.0025		0.0020	
200	0.0030	0.8210	0.0060	0.7655	0.0011	1.1844	5.3031e-004	1.9151
400	0.0015	1	0.0032	0.9069	5.0682e-004	1.1180	1.3496e-004	1.9743
800	7.8534e-004	0.9336	0.0016	1	2.4255e-004	1.0632	3.4005e-005	1.9887
1600	3.9488e-004	0.9919	8.2664e-004	0.9527	1.1855e-004	1.0328	8.5320e-006	1.9948
	<b>Rate</b>	<b>0.9366</b>		<b>0.9063</b>	<b>Rate</b>	<b>1.0996</b>		<b>1.9682</b>

(c). An unsteady state of dam break problem

The dam break problem is intended to test if the numerical method can

resolve unsteady flows. The topography is given by a horizontal bottom  $B(x) = 0$ , where  $-1 \leq x \leq 1$ . The initial water height is given by

$$h(x, 0) = \begin{cases} 10, & x < 0 \\ 4, & x > 0 \end{cases} \quad (18)$$

The analytical solution of this problem has been found in [17] and extended in [13]. The simulation results using 400 cells and final time 0.05 seconds are shown in Table 3 for errors and Figure 4 for stage and momentum. It is seen from Table 3 that Method II gives more accurate results and higher convergence rate.

Table 3. Error of dam-break problems by Method I and II.

N	Method I				Method II			
	Error_w	CR_w	Error_Q	CR_Q	Error_w	CR_w	Error_Q	CR_Q
100	0.1562		1.2520	0.6894	0.0479		0.3738	
200	0.0960	0.7023	0.7764	0.7773	0.0244	0.9731	0.2040	0.8737
400	0.0571	0.7495	0.4530	0.7795	0.0110	1.1494	0.0869	1.2311
800	0.0334	0.7736	0.2639	0.7940	0.0053	1.0534	0.0426	1.0285
1600	0.0194	0.7838	0.1522	0.8036	0.0027	0.9730	0.0215	0.9865
3200	0.0111	0.8055	0.0872	0.7688	0.0014	0.9475	0.0122	0.8175
<b>Rate</b>		<b>0.7629</b>		<b>0.7688</b>		<b>1.0193</b>		<b>0.9875</b>

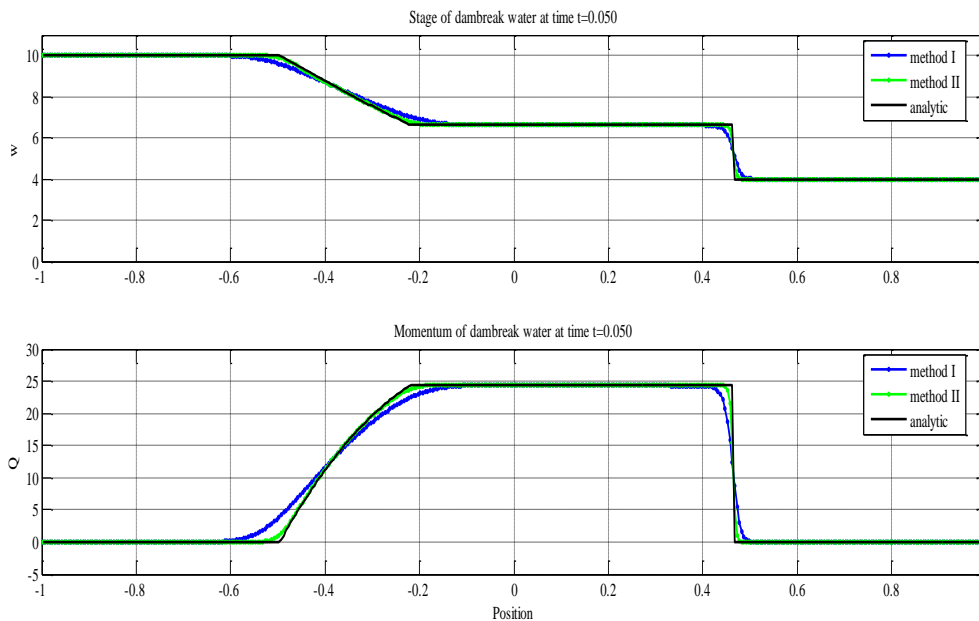


Figure 4. Stage and momentum on simulation of dam-break problem by Method I and II. Here we use 400 cells and final time 0.05 seconds.

## 5 Conclusions

The influence of spatial reconstruction in finite volume methods when solving the shallow water equations has been investigated. The constant reconstruction for the space domain is simple and cheap to compute. However, we



find that linear reconstruction of the space domain has a great improvement to the accuracy of the methods. We conclude that the accuracy of the spatial reconstruction has a significant role in the accuracy of the numerical methods.

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