

A multistage successive approximation method for Riccati differential equations

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ABSTRACT

Riccati differential equations have played important roles in the theory and practice of control systems engineering. Our goal in this paper is to propose a new multistage successive approximation method for solving Riccati differential equations. The multistage successive approximation method is derived from an existing piecewise variational iteration method for solving Riccati differential equations. The multistage successive approximation method is simpler in terms of computing implementation in comparison with the existing piecewise variational iteration method. Computational tests show that the order of accuracy of the multistage successive approximation method can be made higher by simply taking more number of successive iterations in the multistage evolution. Furthermore, taking small size of each subinterval and taking large number of iterations in the multistage evolution lead that our proposed method produces small error and becomes high order accurate.

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1. INTRODUCTION

Mathematical modelling and simulation have been applied extensively in the areas of telecommunication [1], [2], computing [3]-[5], electronics [6], [7] and control [8], [9]. One of important models in these areas is the Riccati differential equation. Riccati differential equations occur in control systems engineering. Control systems engineering itself has played important roles in electrical engineering and related areas [10]. Riccati differential equations are quadratic with respect to the unknown function. These equations are named after a Venetian mathematician, Jacopo Francesco Riccati (1676–1754) [11]. The term Riccati equation is also used to refer to an analogous matrix equation occurring in quadratic control problems. The non-dynamic steady-state version of them is referred to as the algebraic Riccati equation.

Riccati equations and their properties are applied in recent publications [12]-[15]. A number of studies relating to Riccati equations are also reported in the literature [16]-[21]. Due to the importance of Riccati equations, we focus on proposing a computing method for solving Riccati differential equations.

Amongst available methods in the literature, successive approximation and variational iteration methods are able to provide accurate solutions near the initial point of the domain. Successive approximation methods are successful in solving various problems [22]-[23]. Variational iteration methods are also powerful in solving a wide variety of mathematical models [24]-[27]. Interestingly, these two methods (successive

approximation and variational iteration methods) are identical for a particular setting, as reported by Jafari [28].

Our contribution (our goal) in this paper is to propose a new multistage successive approximation method for solving Riccati differential equations. We recall an existing piecewise variational iteration method due to Geng, *et al.* [29]. We modify the method of Geng, *et al.* [29] to obtain the multistage successive approximation method. The modification leads to a simpler implementation of the resulting method in terms of computation. Riccati differential equations have the following form

$$\frac{dy}{dx} = p(x) + q(x)y(x) + r(x)y^2(x) \quad (1)$$

on $0 < x \leq X$, with initial condition

$$y(0) = y_0 \quad (2)$$

where x is the free variable, $y(x)$ is the unknown function dependent on x , X is a known positive constant, and y_0 is a given constant. We note that if $p(x) = 0$, (1) becomes a Bernoulli equation. If $r(x) = 0$, (1) reduces to a first order linear ordinary differential equation. The Riccati differential equation with the case of $p(x) = 0$ or $r(x) = 0$ can be solved using standard methods for ordinary differential equations. In this paper, we consider that $p(x) \neq 0$ and $r(x) \neq 0$. Due to the important roles of Riccati differential equations in control systems engineering, a simple but accurate solver is desired. Providing a simple and accurate method for solving Riccati differential equations is the aim of this paper.

This paper is organised as follows. We explain the problem that we want to tackle in section 2. We propose a multistage successive approximation method for solving Riccati differential equations in section 3. Results and discussion are provided in section 4. The paper is concluded in section 5.

2. PROBLEM DESCRIPTION

In this section, we recall an existing piecewise variational iteration method for solving Riccati differential equations due to Geng, *et al.* [29]. The variational iteration method itself, for the general case, was originally proposed by He [30]-[32].

Considering Riccati differential (1) with initial condition (2) on domain $0 \leq x \leq X$, Geng, *et al.* [29] took the correction functional

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi) \left[\frac{dy_n(\xi)}{d\xi} - q(\xi)\bar{y}_n(\xi) - r(\xi)\bar{y}_n^2(\xi) - p(\xi) \right] d\xi, \quad (3)$$

where \bar{y}_n is a restricted variation, that is, $\delta\bar{y}_n = 0$; $\lambda(\xi)$ is a Lagrange multiplier, which should be determined optimally. Taking the variation of (3), we obtain

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda(\xi) \left[\frac{dy_n(\xi)}{d\xi} - q(\xi)\bar{y}_n(\xi) - r(\xi)\bar{y}_n^2(\xi) - p(\xi) \right] d\xi, \quad (4)$$

Simplifying (4), we obtain

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda(\xi) \left[\frac{dy_n(\xi)}{d\xi} \right] d\xi, \quad (5)$$

Integrating (5) by parts, we have

$$\delta y_{n+1}(x) = \delta \left[(1 + \lambda(x))y_n(x) \right] + \delta \int_0^x y_n(\xi)\lambda'(\xi)d\xi, \quad (6)$$

Considering (6), we come to the following stationary conditions

$$1 + \lambda(x) = 0, \lambda'(\xi) = 0. \quad (7)$$

Solving stationary conditions (7), we obtain that the optimal Lagrange multiplier is

$$\lambda(\xi) = -1. \quad (8)$$

Therefore, with Lagrange multiplier (8), the variational iteration method due to Geng, *et al.* [29] is

$$y_{n+1}(x) = y_n(x) - \int_0^x \left[\frac{dy_n(\xi)}{d\xi} - q(\xi)y_n(\xi) - r(\xi)y_n^2(\xi) - p(\xi) \right] d\xi, \quad (9)$$

which is for solving Riccati differential (1) with initial condition (2) on domain $0 \leq x \leq X$.

Realising that the variational iteration method (9) produces analytical approximate solutions which are accurate only for points close enough to the initial position, Geng, *et al.* [29] implemented the method (9) piecewisely. By piecewisely, Geng, *et al.* [29] meant that the original interval $I = [0, X]$ was subdivided into a finite number of subintervals $I_j = [x_{j-1}, x_j]$, where $j = 1, 2, 3, \dots, J$ for a positive integer J . The width of each subinterval is assumed to be the same, that is, $\Delta x = x_j - x_{j-1}$ for all j . With this setting, we have $J + 1$ discrete points of the original interval $I = [0, X]$, that is, $x_0 = 0, x_1 = \Delta x, x_2 = 2\Delta x, \dots, x_j = J\Delta x = X$. We denote $y_{j,n}(x)$ the analytical approximate solution on the j th subinterval at the n th variational iteration. Suppose that the maximum number of variational iterations is N , where N is a specified positive integer. This means that $n = 0, 1, 2, \dots, N$.

The piecewise variational iteration method due to Geng, *et al.* [29] works as follows. As the first step, on subinterval $I_1 = [x_0, x_1]$, we take the initialization

$$y_{1,0}(x) = y(x_0) = y(0) = y_0 \quad (10)$$

and iterations

$$y_{1,n+1}(x) = y_{1,n}(x) - \int_{x_0}^x \left[\frac{dy_{1,n}(\xi)}{d\xi} - q(\xi)y_{1,n}(\xi) - r(\xi)y_{1,n}^2(\xi) - p(\xi) \right] d\xi, \quad (11)$$

for $n = 0, 1, 2, \dots, N - 1$. As shown in (10) and (11), as the second step, on subinterval $I_2 = [x_1, x_2]$, we take the initialization

$$y_{2,0}(x) = y_{1,N}(x_1) \quad (12)$$

and iterations

$$y_{2,n+1}(x) = y_{2,n}(x) - \int_{x_1}^x \left[\frac{dy_{2,n}(\xi)}{d\xi} - q(\xi)y_{2,n}(\xi) - r(\xi)y_{2,n}^2(\xi) - p(\xi) \right] d\xi, \quad (13)$$

for $n = 0, 1, 2, \dots, N - 1$. As shown in (12) and (13), as the next steps, on subintervals $I_j = [x_{j-1}, x_j]$, where $j = 3, 4, 5, \dots, J$, we take the initialization

$$y_{j,0}(x) = y_{j-1,N}(x_{j-1}) \quad (14)$$

and iterations

$$y_{j,n+1}(x) = y_{j,n}(x) - \int_{x_{j-1}}^x \left[\frac{dy_{j,n}(\xi)}{d\xi} - q(\xi)y_{j,n}(\xi) - r(\xi)y_{j,n}^2(\xi) - p(\xi) \right] d\xi, \quad (15)$$

for $n = 0, 1, 2, \dots, N - 1$. Using (10)-(15), we obtain the solution on the whole domain.

Now, the most expensive computation with the formulation of Geng, *et al.* [29] above lies on the part where we need to calculate the derivative $dy_{j,n}(\xi)/d\xi$ and integrate the results from x_{j-1} to $x \in I_j$. These tasks are redundant. To obtain a simpler method for solving Riccati differential equations, we need to modify this piecewise variational iteration method of Geng, *et al.* [29]. This is the problem that we aim to solve in this paper.

3. PROPOSED MULTISTAGE SUCCESSIVE APPROXIMATION METHOD

In this section, we propose a modification of the piecewise variational iteration method of Geng, *et al.* [29], so that the modified method is simpler in terms of computing implementation, yet its accuracy does not change. We reconsider the iterative formula of the variational iteration method (9) and do the integration for the term having the derivative of the unknown function, so as shown in (9) becomes

$$y_{n+1}(x) = y_0 + \int_0^x [q(\xi)y_n(\xi) + r(\xi)y_n^2(\xi) + p(\xi)]d\xi, \quad (16)$$

where for $n = 0, 1, 2, \dots, N - 1$. Iterative formula (16) is Picard's successive approximation method for Riccati differential (1). The iterative in (16) of the successive approximation method is equivalent to the iterative formula of the variational iteration method (9), but in (16) is simpler.

Now, we shall implement the iterative (16) of the successive approximation method (SAM) piecewisely. The resulting method is called multistage successive approximation method (MSAM) for solving Riccati differential equations. Our MSAM works as follows. As the first step, on subinterval $I_1 = [x_0; x_1]$, we take the initialization

$$y_{1,0}(x) = y(x_0) = y(0) = y_0 \quad (17)$$

and iterations

$$y_{1,n+1}(x) = y_0 + \int_{x_0}^x [q(\xi)y_{1,n}(\xi) + r(\xi)y_{1,n}^2(\xi) + p(\xi)]d\xi, \quad (18)$$

for $n = 0, 1, 2, \dots, N - 1$. As shown in (17) and (18), as the second step, on subinterval $I_2 = [x_1, x_2]$, we take the initialization

$$y_{2,0}(x) = y_{1,N}(x_1) \quad (19)$$

and iterations

$$y_{2,n+1}(x) = y_{2,0}(x) + \int_{x_1}^x [q(\xi)y_{2,n}(\xi) + r(\xi)y_{2,n}^2(\xi) + p(\xi)]d\xi, \quad (20)$$

for $n = 0, 1, 2, \dots, N - 1$. As shown in (19) and (20), as the second step, on subinterval $I_j = [x_{j-1}, x_j]$, we take the initialization

$$y_{j,0}(x) = y_{j-1,N}(x_{j-1}) \quad (21)$$

and iterations

$$y_{j,n+1}(x) = y_{j,0}(x) + \int_{x_{j-1}}^x [q(\xi)y_{j,n}(\xi) + r(\xi)y_{j,n}^2(\xi) + p(\xi)]d\xi, \quad (22)$$

for $n = 0, 1, 2, \dots, N - 1$. Using (17)-(22), we obtain the solution on the whole domain.

To optimise our MSAM further, in the computer implementation, we compute the successive approximation formula symbolically only once:

$$y_{j,n+1}(x) = y_{j-1,N}(x_{j-1}) + \int_{x_{j-1}}^x [q(\xi)y_{j,n}(\xi) + r(\xi)y_{j,n}^2(\xi) + p(\xi)]d\xi, \quad (23)$$

for $n = 0, 1, 2, \dots, N - 1$. Then, the obtained symbolic formula from (23) is used to solve the Riccati differential equation on each subinterval I_j consecutively for $j = 1, 2, 3, \dots, J$. Here, we specify that

$$y_{0,n}(x_0) = y_0 \quad (24)$$

for all n . In (24) means that the given initial value is used as the starting point of solution.

4. RESULTS AND DISCUSSION

In this section, we provide research results on the performance tests of our proposed method and discuss about them. We take two computational tests, namely, a Riccati differential equation with constant coefficients and a Riccati differential equation involving a variable coefficient. Error on the considered domain is defined as the average of relative errors at all discrete points on the domain.

4.1. Riccati differential equation with constant coefficients

As the first test, we consider the Riccati differential equation with constant coefficients [29]:

$$\frac{dy(x)}{dx} = 1 + 2y(x) - y^2(x), 0 < x \leq 4, \tag{25}$$

having initial condition

$$y(0) = 0 \tag{26}$$

The exact solution to this problem is

$$y(x) = 1 + \sqrt{2} \tanh\left(\sqrt{2}x + \frac{1}{2} \log\left(\frac{-1+\sqrt{2}}{1+\sqrt{2}}\right)\right). \tag{27}$$

Our computational experiments show that the standard SAM is not able to solve the problem on the whole domain. In contrast, our proposed MSAM is able to solve the problem on the whole domain accurately. These phenomena are shown in Figure 1(a), where SAM and MSAM use 3 successive iterations, and in addition, for MSAM we use $\Delta x = 0.1$.

To investigate further about the performance of MSAM, we record the errors and their orders of convergence in Tables 1-4. We obtain that the number of successive iterations in the MSAM evolution determines the order of convergence of the solution. One successive iteration in the MSAM evolution leads that MSAM is a first order method. This is observed as Δx approaches to zero, the order of convergence tends to one, as recorded in Table 1. Two successive iterations in the MSAM evolution leads that MSAM is a second order method, because as Δx approaches to zero, the order of convergence tends to two, as recorded in Table 2. Three successive iterations in the MSAM evolution leads that MSAM is a third order method, because as Δx approaches to zero, the order of convergence tends to three, as recorded in Table 3. Similarly, four successive iterations in the MSAM evolution leads that MSAM is a fourth order method, because as Δx approaches to zero, the order of convergence tends to four, as recorded in Table 4. In general, smaller Δx results in smaller error. Furthermore, more successive iterations results in higher order accurate method.

Table 1. Error and order of convergence of MSAM solution for Riccati differential equation with constant coefficients, in which we use 1 iteration in the MSAM evolution. Error is computed on interval [0,4]

Δx	Average of Relative Error	Order of Convergence
0.25	4.420E-02	-
0.125	2.425E-02	0.87
0.0625	1.274E-02	0.93
0.03125	6.536E-03	0.96
0.015625	3.311E-03	0.98

Table 2. Error and order of convergence of MSAM solution for Riccati differential equation with constant coefficients, in which we use 2 iterations in the MSAM evolution. Error is computed on interval [0,4]

Δx	Average of Relative Error	Order of Convergence
0.25	7.393E-03	-
0.125	2.022E-03	1.87
0.0625	5.395E-04	1.91
0.03125	1.399E-04	1.95
0.015625	3.565E-05	1.97

Table 3. Error and order of convergence of MSAM solution for Riccati differential equation with constant coefficients, in which we use 3 iterations in the MSAM evolution. Error is computed on interval [0,4]

Δx	Average of Relative Error	Order of Convergence
0.25	6.674E-04	-
0.125	9.271E-05	2.85
0.0625	1.252E-05	2.89
0.03125	1.636E-06	2.94
0.015625	2.094E-07	2.97

Table 4. Error and order of convergence of MSAM solution for Riccati differential equation with constant coefficients, in which we use 4 iterations in the MSAM evolution. Error is computed on interval [0,4]

Δx	Average of Relative Error	Order of Convergence
0.25	7.349E-05	-
0.125	5.049E-06	3.86
0.0625	3.388E-07	3.90
0.03125	2.205E-08	3.94
0.015625	1.407E-09	3.97

4.2. Riccati differential equation involving a variable coefficient

As the second test, let us consider the Riccati differential equation with a variable coefficient [29]:

$$\frac{dy(x)}{dx} = 1 + x^2 - y^2(x), 0 < x \leq 4, \tag{28}$$

with initial condition

$$y(0) = 1 \tag{29}$$

The exact solution to this problem is

$$y(x) = x + \frac{e^{-x^2}}{1 + \int_0^x e^{-\xi^2} d\xi} \tag{30}$$

For this second test, the standard SAM is not able to solve the problem, even for the first half [0,2] of the given domain [0,4], as shown in Figure 1(b). In contrast, our proposed MSAM is able to solve the problem on the whole domain [0,4] accurately. For this Figure 1(b), SAM and MSAM use 3 successive iterations, and in addition, for MSAM we take $\Delta x = 0.1$.

To investigate further about the performance of MSAM in solving this second test case, we record the errors and their orders of convergence in Tables 5-8. The behaviour of MSAM in this test case is consistent with that of MSAM in the previous test case. We obtain that the number of successive iterations in the MSAM evolution determines the order of convergence of the solution. One successive iteration in the MSAM evolution leads that MSAM is a first order method. This is observed as Δx approaches to zero, the order of convergence tends to one, as recorded in Table 5. Two successive iterations in the MSAM evolution leads that MSAM is a second order method. This is observed as Δx approaches to zero, the order of convergence tends to two, as recorded in Table 6. Three successive iterations in the MSAM evolution leads that MSAM is a third order method. This is observed as Δx approaches to zero, the order of convergence tends to three, as recorded in Table 7. Similarly, four successive iterations in the MSAM evolution leads that MSAM is a fourth order method, because as Δx approaches to zero, the order of convergence tends to four, as recorded in Table 8. Again, in general, smaller Δx results in a more accurate method. Furthermore, more number of successive iterations makes MSAM to be higher order accurate.

With our accurate results in this paper, we are confident that MSAM is a reliable method to be used for other kinds of initial value problems. The idea of MSAM could be adapted for continuous versions of discrete problems in computing [33], and it could be extended to be analysed using advanced mathematical tools, such as, algebraic geometry and conformal mapping.

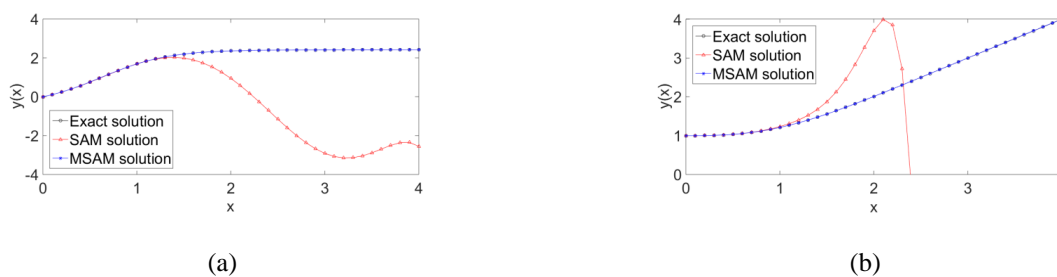


Figure 1. Exact, SAM, and MSAM solutions on interval [0, 4]. SAM solution is accurate only at points close to the initial condition. MSAM solution coincides graphically with the exact solution, (a) Results of the first test case, (b) Results of the second test case

Table 5. Error and order of convergence of MSAM solution for Riccati differential equation involving a variable coefficient, in which we use 1 iteration in the MSAM evolution on interval [0,4]

Δx	Average of Relative Error	Order of Convergence
0.25	3.544E-02	–
0.125	1.764E-02	1.01
0.0625	8.802E-03	1.00
0.03125	4.396E-03	1.00
0.015625	2.197E-03	1.00

Table 6. Error and order of convergence of MSAM solution for Riccati differential equation involving a variable coefficient, in which we use 2 iterations in the MSAM evolution on interval [0,4]

Δx	Average of Relative Error	Order of Convergence
0.25	3.408E-02	–
0.125	4.625E-03	2.88
0.0625	9.436E-04	2.29
0.03125	2.165E-04	2.12
0.015625	5.202E-05	2.06

Table 7. Error and order of convergence of MSAM solution for Riccati differential equation involving a variable coefficient, in which we use 3 iterations in the MSAM evolution on interval [0,4]

Δx	Average of Relative Error	Order of Convergence
0.25	6.406E-03	–
0.125	6.894E-04	3.22
0.0625	7.439E-05	3.21
0.03125	8.555E-06	3.12
0.015625	1.024E-06	3.06

Table 8. Error and order of convergence of MSAM solution for Riccati differential equation involving a variable coefficient, in which we use 4 iterations in the MSAM evolution on interval [0,4]

Δx	Average of Relative Error	Order of Convergence
0.25	2.393E-03	–
0.125	1.019E-04	4.55
0.0625	5.294E-06	4.27
0.03125	3.007E-07	4.14
0.015625	1.790E-08	4.07

5. CONCLUSION

We have proposed successfully a new multistage successive approximation method for solving Riccati differential equations. The main advantage of the proposed method is that it is simpler than the existing variational iteration method for solving Riccati differential equations. The proposed method is analytically equivalent to the existing method, but simpler in terms of computing implementation. We have tested the performance of the multistage successive approximation method in cases that their exact solutions are known. The order of accuracy of the proposed method can be made higher by simply taking more number of successive iterations in the multistage evolution. Obviously, taking smaller size of each subinterval and taking more number of iterations in the multistage evolution lead that our proposed method produces smaller error and it becomes higher order accurate. With these results, in cases that the exact solutions to Riccati differential equations are not known, we are confident to propose the use of the multistage successive approximation method for solving them to obtain accurate approximate solutions.

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