NUMERICAL ENTROPY PRODUCTION OF SHALLOW WATER FLOWS ALONG CHANNELS WITH VARYING WIDTH

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Abstract – Our aim is to indicate the smoothness of solutions to the shallow water equations involving varying width. These equations form a system of balance laws. To achieve the aim, we extend the application of numerical entropy production, as the smoothness indicator, from conservation laws to balance laws. For a numerical test, we consider a radial dam break with the whole spatial domain is wet. The resulting dam break flow is a solution having shock discontinuities. We are interested in finding the positions of discontinuities. These positions are where numerical solutions, such as those generated by a finite volume method, are less accurate. Detecting the positions of discontinuities can be an aid for the improvement of the numerical solution in terms of its accuracy. The numerical entropy production is found to be accurate to detect discontinuities. The discontinuity of the water surface is clearly indicated by large values of the numerical entropy production as the smoothness indicator. In short, the numerical entropy production is simple to implement, but gives an accurate detection of the smoothness of solutions.

Keywords: shallow water equations, finite volume method, numerical entropy production, smoothness indicator

INTRODUCTION

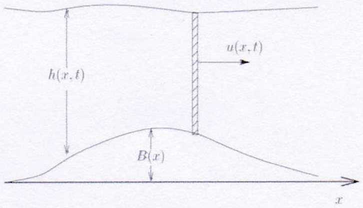
Water flows are well-modelled by shallow water equations. These flows often involve discontinuities or rough regions on its free surface [1]. A smoothness indicator is needed, when we want to know the positions of smooth and rough regions.

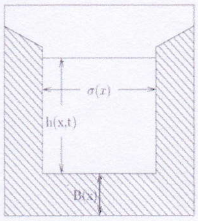
Puppo and Semplice [2] proposed the use of numerical entropy production to measure the smoothness of numerical solutions to conservation laws. The order of accuracy of the numerical entropy production at discontinuities is lower than that at smooth regions.

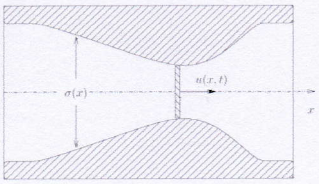
In this work, we extend the application of numerical entropy production from conservation laws to balance laws [3]. A conservation law is a balance law without source terms. The balance law that we consider is the one-dimensional shallow water equations involving varying width. The varying width contributes in the source term of the balance law [4]. To our knowledge, identifying discontinuities along channels with varying width has never been investigated by other authors.

THE CONSIDERED PROBLEM

We consider water flows along channels with varying width.







From the top left with the clockwise direction, the figures show a schematic setting of water flows viewed from (a). aside, (b). front, and (c). above, respectively. Here the figures are from Balbas and Karni [4].

MATHEMATICAL MODEL

We assume a horizontal topography to simplify the problem. The bottom of the channel is the horizontal reference. The mathematical model for water flows is

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0,$$

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\frac{Q^2}{A}+\frac{1}{2}gh^2\sigma\right)=\frac{1}{2}gh^2\frac{d\sigma}{dx}.$$

In this model:

x	is the space variable
t	is the time variable,
$h=h(x,t)$	represents the height of water above the bottom of the channel,
$\sigma=\sigma(x)$	is the channel width,
$Q=Au$	is the discharge,
$A=\sigma h$	is the wet cross-section,
$u=u(x,t)$	represents the depth averaged water velocity, and
g	is the acceleration due to gravity.

NUMERICAL METHOD

To solve the mathematical model, we use a finite volume method. The smoothness indicator of the solution is the numerical entropy production.

The entropy and the entropy inequality are respectively given by

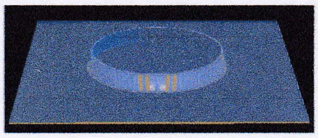
$$\eta(x,t)=\sigma h\left(\frac{1}{2}u^2+\frac{1}{2}gh\right)$$

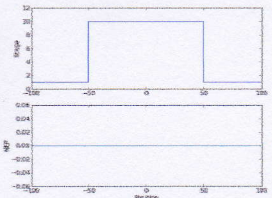
$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}\left[u\left(\eta+\frac{1}{2}gh^2\sigma\right)\right]\leq 0.$$

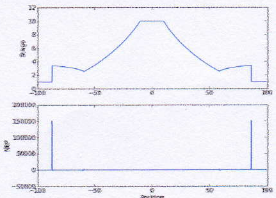
The numerical entropy production is the local truncation error of the entropy. Note that the entropy inequality above is correct in the weak sense.

NUMERICAL RESULTS

Consider the circular dam break problem, where water depth of 10 m is inside a drum with radius of 50 m. Outside the drum, the water depth is 1 m. At time zero, the drum is removed. For positive time, we solve the problem using the finite volume method. We measure the smoothness of the solution using Numerical Entropy Production (NEP). Three figures below show the initial water surface (in 3D view), the initial water surface and its NEP (in 2D view), and the results of water surface and its NEP at time 4 s (in 2D view).







DISCUSSION

We obtain that the numerical entropy production behaves excellently as the smoothness indicator for the radial (circular) dam break problem. We observe that large values of numerical entropy production are generated at the shock discontinuities of the stage (free water surface). The values at smooth regions are very small compared to the values at the discontinuities.

Puppo and Semplice [2] proved that for conservation laws, the numerical entropy production is large at rough regions and small at smooth regions. In fact this is also true for balance laws as we see in these numerical results. This gives the flexibility of the use of the numerical entropy production as smoothness indicator. Note that the results that we obtain here is very accurate.

Conclusion

Our concluding remarks are as follows:

- Numerical entropy production has been found to be accurate as a smoothness indicator for solutions to the shallow water equations involving varying width.
- The computer programming work for numerical entropy production is also simple, as it is merely the local truncation error of the entropy.
- For future work, we recommend the use of numerical entropy production as the refinement and/or coarsening indicator when an adaptive numerical method is desired to solve the shallow water equations.

References

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