Document type
Article • *Gold Open Access*

Source type
Journal

ISSN
11100168

DOI
10.1016/j.aej.2021.04.068

View more

Alexandria Engineering Journal • *Open Access* • Volume 61, Issue 1, Pages 175 - 184 • January 2022

An order verification method for truncated asymptotic expansion solutions to initial value problems

Mungkasi S.

Save all to author list

Department of Mathematics, Faculty of Science and Technology, Sanata Dharma University, Yogyakarta, Indonesia

Abstract

The focus of this paper is to obtain explicit solutions to initial value problems, where numerical methods cannot provide one, and to verify the accuracy orders of the explicit solutions. One of available methods to obtain an explicit solution is the asymptotic (formal) expansion method. However, we must be sure with the accuracy order of the explicit solution. In this paper, an order verification method is proposed for truncated asymptotic formal expansion solutions to initial value problems. A least-squares fit of error data is used in the existing order verification method. The method that we propose does not involve any application of least-squares fit of error data, so is simpler, yet produces accurate expected accuracy orders of solutions of explicit truncated asymptotic formal expansions. With our proposed method, we are successful in verifying the accuracy orders of solutions of truncated asymptotic formal expansions to the linear and nonlinear initial value problems accurately.

Author keywords

Initial value problem; Order verification method; Truncated asymptotic expansion

Indexed keywords

SciVal Topics

Metrics

Funding details

References (30)

View in search results format >

All

RIS export Print E-mail Save to PDF

Create bibliography

1 Meftah, S.

A New Approach to Approximate Solutions for Nonlinear Differential Equation (Open Access)

(2018) *International Journal of Mathematics and Mathematical Sciences*, 2018, art. no. 5129502.
http://www.hindawi.com/journals/ijmms/contents/
doi: 10.1155/2018/5129502

View at Publisher

2 Verhulst, F.
(1996) *Nonlinear Differential Equations and Dynamical Systems*. Cited 1076 times.
Springer Berlin

3 Abu Arqub, O., Al-Smadi, M.

Atangana–Baleanu fractional approach to the solutions of Bagley–Torvik and Painlevé equations in Hilbert space

(2018) *Chaos, Solitons and Fractals*, 117, pp. 161-167. Cited 134 times.
doi: 10.1016/j.chaos.2018.10.013

View at Publisher

4 Arqub, O.A., Maayah, B.

Numerical solutions of integrodifferential equations of Fredholm operator type in the sense of the Atangana–Baleanu fractional operator

(2018) *Chaos, Solitons and Fractals*, 117, pp. 117-124. Cited 108 times.
doi: 10.1016/j.chaos.2018.10.007

View at Publisher

30 Mohammadi, F., Moradi, L., Baleanu, D., Jajarmi, A.

A hybrid functions numerical scheme for fractional optimal control problems: Application to nonanalytic dynamic systems

(2018) *JVC/Journal of Vibration and Control*, 24 (21), pp. 5030-5043. Cited 84 times.
doi: 10.1177/1077546317741769

View at Publisher

SJR Scimago Journal & Country Rank

Enter Journal Title, ISSN or Publisher Name

Home Journal Rankings Country Rankings Viz Tools Help About Us

AEJ - Alexandria Engineering Journal

COUNTRY

Egypt

Universities and research institutions in Egypt

SUBJECT AREA AND CATEGORY

Engineering
Engineering (miscellaneous)

PUBLISHER

Alexandria University

Alexandria University in Scimago Institutions Rankings

H-INDEX

58

PUBLICATION TYPE

Journals

ISSN

11100168

COVERAGE

2000-2020

INFORMA

Homepag

How to publish in this journ

magdy@
u.edu.eg

SCOPE

The Alexandria Engineering Journal (AEJ) is an international journal devoted to publishing high quality papers in the field of engineerin and applied science. AEJ is cited in the Engineering Information Services (EIS) and the Chemical Abstracts (CA). The papers published AEJ are grouped into five sections, according to the following classification: - Mechanical, Production, Marine, and Textile Engineering - Electrical Engineering, Computer Science, and Nuclear Engineering - Civil and Architecture Engineering - Chemical Engineering and App Sciences - Environmental Engineering Alexandria Engineering Journal publishes original papers, critical reviews, technical papers, tech data, short notes, and letters to the editor. Papers covering experimental, theoretical, and computational aspects which contribute to th understanding of engineering and applied sciences or give an insight into engineering practices and processes are welcome. Authors f all over the world are invited to submit manuscripts for possible publications in Alexandria Engineering Journal.

Join the conversation about this journal

Quartiles

FIND SIMILAR JOURNALS

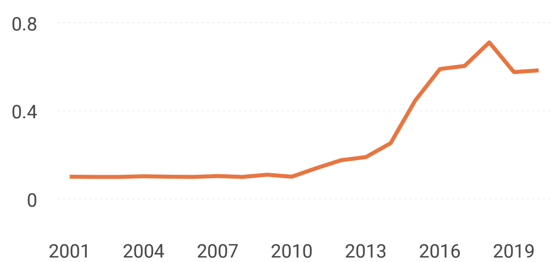
1
Multidiscipline Modeling in Materials and Structures
GBR
57%
similarity

2
Nonlinear Engineering
DEU
55%
similarity

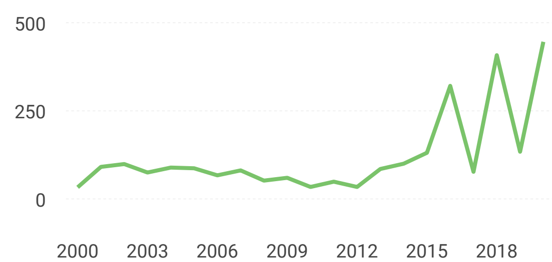
3
Propulsion and Power Research
NLD
51%
similarity

4
Ain Shams Engir Journal
EGY
48'
simila

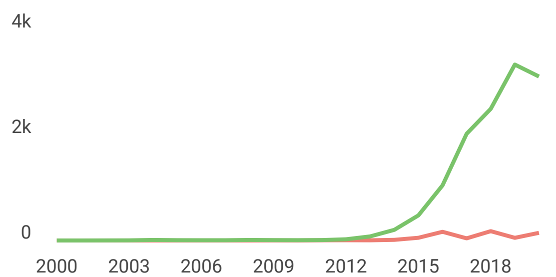
SJR



Total Documents



Total Cites Self-Cites



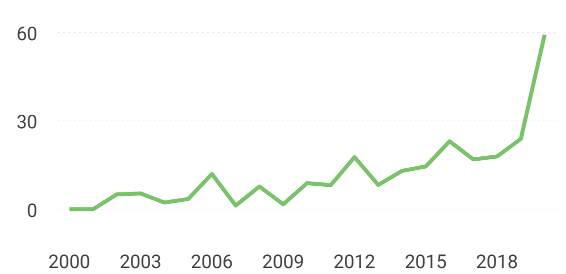
Citations per document

External Cites per Doc Cites per Doc

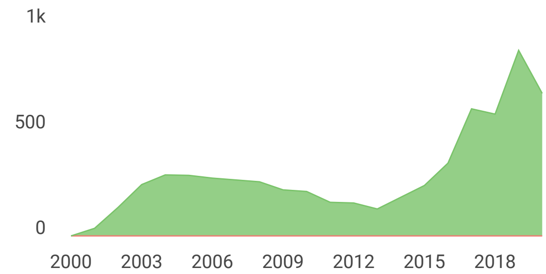
5

5

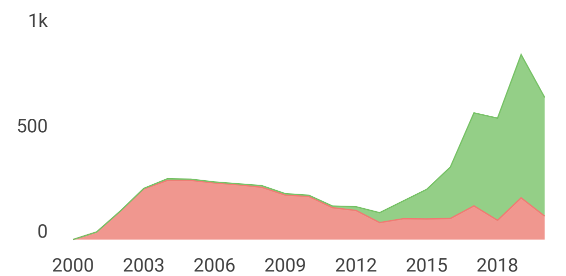
% International Collaboration



Citable documents Non-citable documents



Cited documents Uncited documents



← Show this widget in your own website

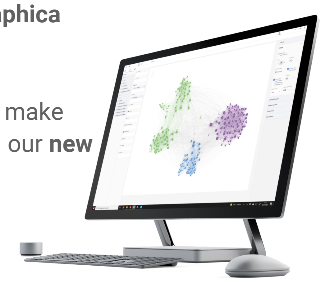
Just copy the code below and paste within your html code:

<a href="https://www.scimag

SCImago Graphica

Explore, visually communicate and make sense of data with our **new free tool**.

Get it



Metrics based on Scopus® data as of April 2021

Ibrahim 4 weeks ago

ScienceDirect

Alexandria Engineering Journal

Open access

5.6 CiteScore

3.732 Impact Factor

Submit your article | Guide for authors

Menu | Search in this journal

About the journal

Aims and scope | Editorial board | Abstracting and indexing

Alexandria Engineering Journal is an international journal devoted to publishing high quality papers in the field of engineering and applied science. *Alexandria Engineering Journal* is cited in the Engineering Information Services (EIS) and the Chemical Abstracts (CA). The papers published in *Alexandria Engineering Journal* are grouped into five sections, according to the following classification:

• Mechanical, Production, Marine and Textile Engineering

• Electrical Engineering, Computer Science and Nuclear Engineering

• Civil and Architecture Engineering

• Chemical Engineering and Applied Sciences

• Environmental Engineering

Alexandria Engineering Journal publishes original papers, critical reviews, technical papers, technical data, short notes, and letters to the editor. Papers covering experimental, theoretical, and computational aspects which contribute to the understanding of engineering and applied sciences or give an insight into engineering practices and processes are welcome. Authors from all over the world are invited to submit

FEEDBACK

manuscripts for possible publications in *Alexandria Engineering Journal*.

For queries related to the journal, please contact magdy@alexu.edu.eg

ISSN: 1110-0168

Alexandria Engineering Journal

Open access

Submit your article

Menu Search in this journal

About the journal

Aims and scope Editorial board Abstracting and indexing

FEEDBACK

Alexandria Engineering Journal

Open access

Submit your article

Menu

Search in this journal

FEEDBACK

Communications Engineering

FEEDBACK

FEEDBACK

FEEDBACK

Alexandria Engineering Journal
Open access

Submit your article

Menu

Search in this journal

FEEDBACK

FEEDBACK

FEEDBACK

organizations, along with the corresponding country or geographic region. Elsevier remains neutral with regard to any jurisdictional claims.

ISSN: 1110-0168

Alexandria Engineering Journal

Open access

5.6
CiteScore

3.732
Impact Factor

Volume 61, Issue 1

In progress (January 2022)

This issue is in progress but contains articles that are final and fully citable.

Download full issue

Previous vol/issue

Next vol/issue

FEEDBACK

FEEDBACK

FEEDBACK

FEEDBACK

FEEDBACK

FEEDBACK

FEEDBACK

FEEDBACK

FEEDBACK

FEEDBACK

Previous vol/issue Next vol/issue

ISSN: 1110-0168

FEEDBACK

Alexandria Engineering Journal (2022) **61**, 175–184

HOSTED BY

ELSEVIER

Alexandria University

Alexandria Engineering Journal

www.elsevier.com/locate/aej
www.sciencedirect.com

An order verification method for truncated asymptotic expansion solutions to initial value problems

Sudi Mungkasi

Department of Mathematics, Faculty of Science and Technology, Sanata Dharma University, Yogyakarta, Indonesia

Received 6 October 2020; revised 28 March 2021; accepted 24 April 2021
Available online 06 June 2021

KEYWORDS

Initial value problem;
Order verification method;
Truncated asymptotic expansion

Abstract The focus of this paper is to obtain explicit solutions to initial value problems, where numerical methods cannot provide one, and to verify the accuracy orders of the explicit solutions. One of available methods to obtain an explicit solution is the asymptotic (formal) expansion method. However, we must be sure with the accuracy order of the explicit solution. In this paper, an order verification method is proposed for truncated asymptotic formal expansion solutions to initial value problems. A least-squares fit of error data is used in the existing order verification method. The method that we propose does not involve any application of least-squares fit of error data, so is simpler, yet produces accurate expected accuracy orders of solutions of explicit truncated asymptotic formal expansions. With our proposed method, we are successful in verifying the accuracy orders of solutions of truncated asymptotic formal expansions to the linear and nonlinear initial value problems accurately.

1. Introduction

Initial value problems of differential equations occur in various fields [1,2]. Arqub et al. [3–7] have studied a number of problems relating to differential equations using kernel algorithms and fractional operators. Baleanu and Jajarmi with their coauthors [8–11] reported their approaches to solving harmonic oscillator and boundary value problems also relating to differential equations. Based on the wide range of applications of initial value problems of differential equations, accurate solutions to initial value problems are always desired. Generally, exact solutions to initial value problems are difficult to find. Therefore, approximate solutions are sought using either numerical or analytical method.

Numerical method provides approximate solutions in a discrete space. It can produce accurate solutions, but cannot express the solutions into explicit functions. When we need an explicit function as a solution, a numerical method (such as, a Runge–Kutta method) for solving the problem is not the option to use.

Approximate solutions in the forms of explicit functions can be obtained using analytical approaches in the continuous space, such as the perturbation method, which is also known as the asymptotic expansion method. This method has been

E-mail address: sudi@usd.ac.id

Peer review under responsibility of Faculty of Engineering, Alexandria University.

https://doi.org/10.1016/j.aej.2021.04.068
1110-0168

used for modelling as well as solving real problems. A number of applications of the asymptotic expansion method can be found in the literature, for example, those for fluid dynamics [12–17], chemistry-related problems [18], material problems [19,20], financial-economics models [21], and other problems [22].

Applying the asymptotic (formal) expansion method, we should be mindful of the accuracy orders of the truncated asymptotic expansion solutions. Abdel-Halim Hassan [23] solved linear and nonlinear initial value problems using asymptotic expansion method. They [23] also presented a method to check the accuracy orders of solutions obtained from truncated asymptotic (formal) expansions. The verification method of Abdel-Halim Hassan [23] was based on the work of Bosley [24] as well as Deeba and Xie [25].

Unfortunately, Abdel-Halim Hassan [23] failed to obtain the expected correct results of the truncated asymptotic expansion solutions. They [23] could not verify the correctness of their solutions of truncated asymptotic expansions to the linear initial value problem. In addition, their verification of the accuracy orders of solutions to the nonlinear initial value problem obtained using truncated asymptotic expansions was not simple, because their verification method used a least-squares fit of error data. Therefore, in this work, we reconsider the linear and nonlinear initial value problems solved by Abdel-Halim Hassan [23].

Our contributions are three folds. First, we provide the correct truncated asymptotic formal expansion solutions to both the linear and nonlinear initial value problems of Abdel-Halim Hassan [23]. Second, we propose a simple but accurate method to verify the orders of the truncated asymptotic formal expansion solutions. Third, using our proposed method, we are successful in verifying that the truncated asymptotic formal expansion solutions have the expected orders of accuracy.

The rest of this paper follows. We express the problem description in Section 2. The existing order verification method and our proposed one are provided in Section 3. Numerical results are reported and discussed in Section 4. Finally, Section 5 concludes the paper.

2. Problem description

In this section, we describe the problems that we consider and the research question that we want to answer. We consider two problems of Abdel-Halim Hassan [23]. The first is the linear initial value problem (LIVP)

$$v'' + (1 - \varepsilon t)v = 0 \tag{1}$$

with initial conditions

$$v(0) = 1, \quad v'(0) = 0. \tag{2}$$

The second is the nonlinear initial value problem (NLIVP)

$$v'' + v + \varepsilon v^3 = 0 \tag{3}$$

with initial conditions

$$v(0) = 1, \quad v'(0) = 0. \tag{4}$$

Here, t is the free variable and it usually denotes time; v is the dependent variable and it is dependent on t; and ε is a positive parameter.

We assume that solutions to the LIVP (1)–(2) and NLIVP (3)–(4) can be expressed in the power series in ε [2]

$$v(t) = \sum_{n=0}^{\infty} \varepsilon^n v_n(t) = v_0(t) + \varepsilon v_1(t) + \varepsilon^2 v_2(t) + \varepsilon^3 v_3(t) + \cdots. \tag{5}$$

This can be written as

$$v(t) = w_N(t) + O(\varepsilon^{N+1}), \tag{6}$$

where

$$\begin{aligned} w_N(t) &= \sum_{n=0}^{N} \varepsilon^n v_n(t) \\ &= v_0(t) + \varepsilon v_1(t) + \varepsilon^2 v_2(t) + \varepsilon^3 v_3(t) + \cdots + \varepsilon^N v_N(t), \end{aligned} \tag{7}$$

and $N = 0, 1, 2, \ldots$. Eq. (5) is the general form of asymptotic expansion solutions. The method to obtain an explicit form for Eq. (5) is called the asymptotic expansion method, also known as the asymptotic formal expansion method, or perturbation method. Eq. (7) is the truncated expansion solution involving the first $N + 1$ terms. As written in Eq. (6), the order of the error of the truncated expansion solution (7) is $O(\varepsilon^{N+1})$. In other words, $w_N(t)$ has the $(N + 1)$th order of accuracy, as per our assumption. However, we want to know how to verify that $w_N(t)$ has, indeed, the $(N + 1)$th order of accuracy.

Therefore, our research question is how to verify the orders of accuracy of solutions $w_N(t)$ of the LIVP (1)–(2) and NLIVP (3)–(4) obtained using truncated asymptotic formal expansions. We answer this research question by proposing a new method to verify that $w_N(t)$ has, indeed, the $(N + 1)$th order of accuracy.

We note several remarks as follows. The formulations of the LIVP (1)–(2) and NLIVP (3)–(4) may take different forms, and do not need to be unique in the sense of the forms exactly the same as Eqs. (1), (2), (3), (4), respectively. The formulation of the LIVP (1)–(2) and NLIVP (3)–(4) are stable under certain conditions. Based on the theory of stability by linearisation about the equilibrium point, the LIVP (1)–(2) is stable if $\varepsilon t < 1$, whereas the NLIVP (3)–(4) is stable if ε is sufficiently small ($\varepsilon \ll 1$). Readers interested in the stability theory of the LIVP (1)–(2) and NLIVP (3)–(4) are referred to the work of Coddington and Levinson [26], Haberman [27], as well as Verhulst [2]. In this paper we have used the Landau big "O" symbol for order of accuracy. This notation is understood as follows. Suppose that we have two functions $g(\varepsilon)$ and $h(\varepsilon)$. Then, we have the following definition: $g(\varepsilon) = O(h(\varepsilon))$ as $\varepsilon \to 0$, if there exists a constant c such that $g(\varepsilon) \leqslant ch(\varepsilon)$ as $\varepsilon \to 0$. We refer to the book of Verhulst [2] for this definition.

It is also worthwhile to note some properties of the asymptotic expansion method (perturbation method) for $\varepsilon \in (0, \varepsilon_0]$ with a positive ε_0 as follows. First, the order functions ε^n for all n are continuous, positive, and monotonically decreasing as $\varepsilon \to 0$. Second, the convergence of the method has been guaranteed on the domain scale 1 by Theorem 9.1 and Theorem 9.2 of Verhulst [2].

In the next section, we recall an existing order verification method and propose a new order verification method for truncated expansion solutions $w_N(t)$ of the LIVP (1)–(2) and NLIVP (3)–(4).

3. Order verification methods

We provide two order verification methods. The first is the existing method and the second is our proposed method, where the LIVP (1)–(2) and NLIVP (3)–(4) are our test case problems.

3.1. Existing method

In the existing order verification method, a technique about least-squares is used. This method was used by Abdel-Halim Hassan [23], Bosley [24], as well as Deeba and Xie [25]. We consider a fixed value of t, say $t = \tau > 0$, and a fixed value of N, where $N = 0, 1, 2, \ldots$. The steps of the existing order verification method are as follows:

1. First, the problem is solved using an available ordinary differential equation solver to get a reference solution $v(\tau)$; the reference solution $v(\tau)$ is either the exact solution or a very accurate approximate one at τ.
2. Second, the problem is solved using the asymptotic expansion method up to the term $\varepsilon^N v_N(\tau)$; the asymptotic expansion solution is $w_N(\tau)$.
3. Third, absolute errors $E_N(\tau, \varepsilon)$ of $w_N(\tau)$ are calculated for a number of different values of ε, where $0 < \varepsilon < 1$.
4. Fourth, in the log–log scale we seek the function of the least-squares fit of absolute errors $E_N(\tau, \varepsilon)$ with respect to the values of ε.
5. Fifth, the gradient of the curve of the least-squares fit function found in the fourth step is considered as the order of accuracy of w_N at $t = \tau$.

3.2. Proposed method

Here, we write the background of our proposed order verification method. This follows from the work of Mungkasi [28] for the van der Pol equation problem. We extend the work of Mungkasi [28] to the problems in this paper. Let us consider Eq. (6). The absolute error $E_N(t, \varepsilon)$ is

$$E_N(t, \varepsilon) = |v(t) - w_N(t)| = O(\varepsilon^{N+1}) = K\varepsilon^{N+1} \quad (8)$$

for a positive constant K. Now we consider two different values of ε, say ε_1 and ε_2, so the absolute errors are $E_N(t, \varepsilon_1)$ and $E_N(t, \varepsilon_2)$, respectively. Therefore, we have

$$E_N(t, \varepsilon_1) = K\varepsilon_1^{N+1} \quad (9)$$

and

$$E_N(t, \varepsilon_2) = K\varepsilon_2^{N+1}. \quad (10)$$

Taking the ratio between (9) and (10), we obtain

$$\frac{E_N(t, \varepsilon_1)}{E_N(t, \varepsilon_2)} = \frac{\varepsilon_1^{N+1}}{\varepsilon_2^{N+1}}. \quad (11)$$

Taking the logarithms of both sides and do some simplifications, we obtain

$$N + 1 = \frac{\log\left(\frac{E_N(t,\varepsilon_1)}{E_N(t,\varepsilon_2)}\right)}{\log\left(\frac{\varepsilon_1}{\varepsilon_2}\right)}, \quad (12)$$

which is the order of accuracy of $w_N(t)$. We should note that some analytical results confirm that this order of accuracy is guaranteed to be valid in the domain scale 1 (see the work of Verhulst [2], for example).

For our proposed method, we consider a fixed value of t, which we recommend $0 < t \leqslant 1$. We denote τ the fixed value of t of interest. We also fix the value of N, where $N = 0, 1, 2, \ldots$. Our proposal for order verification method takes the steps as follows. The first, second, and third steps are the same as those in the existing method, but with a note that we recommend to take $0 < t = \tau \leqslant 1$. Then, in the fourth step, we do not need to use any least-squares technique to verify the order of accuracy of w_N. Instead, the fourth step is implementing the formula (12) for various values of ε considered in the third step. This implementation is simpler to do and cheaper in terms of computation. Then, our fifth step is to check the values obtained in the fourth step if they equal or are close to $N + 1$. If as ε tends to 0, the values obtained in the fourth step approach to $N + 1$, then we conclude that the order of accuracy of w_N is indeed $N + 1$. A pseudocode of the proposed method (Proposed_Method) is written in Algorithm 1.

Algorithm 1. Proposed order verification method

```
procedure Proposed_Method(ε1, ε2, τ, N)        ▷Inputs are given
for k ← 1 : 2        ▷Loops through each ε
ε ← εk        ▷Epsilon value is fixed (ε = εk)
v(τ)←Reference solution with fixed ε at τ
wk(τ) ← Asymptotic expansion value with fixed ε at τ
Ek(τ) ← |v(τ) − wk(τ)|        ▷Absolute error is calculated
end for
Order ← log(E1/E2)/ log(ε1/ε2)        ▷Order is calculated
return Order        ▷Order is returned
end procedure
```

4. Results and discussion

In this section we solve the LIVP (1)–(2) and the NLIVP (3)–(4) using the asymptotic expansion method. Then, we confirm the order of accuracy of the asymptotic expansion solutions using our proposed order verification method. All simulations are carried out using the MATLAB software.

4.1. Results and discussion for the linear initial value problem

In this subsection we provide the correct derivations of asymptotic expansion solutions to the LIVP (1)–(2), where the previous work of Abdel-Halim Hassan [23] contained some obvious mistakes. Substituting the ε series solution (5) in Eqs. (1) and (2), we obtain infinitely many initial value problems

$$v_0'' + v_0 = 0, \quad v_0(0) = 1, \quad v_0'(0) = 0, \quad (13)$$

$$v_1'' + v_1 = tv_0, \quad v_1(0) = 0, \quad v_1'(0) = 0, \quad (14)$$

$$v_2'' + v_2 = tv_1, \quad v_2(0) = 0, \quad v_2'(0) = 0, \quad (15)$$

$$v_3'' + v_3 = tv_2, \quad v_3(0) = 0, \quad v_3'(0) = 0, \quad (16)$$

$$v_4'' + v_4 = tv_3, \quad v_4(0) = 0, \quad v_4'(0) = 0, \quad (17)$$

$$v_5'' + v_5 = tv_4, \quad v_5(0) = 0, \quad v_5'(0) = 0, \quad (18)$$

$$v_i'' + v_i = tv_{i-1}, \quad v_i(0) = 0, \quad v_i'(0) = 0, \tag{19}$$

where $i = 6, 7, 8, \ldots$. Suppose that we want to have asymptotic expansion solutions (5) up to the $\varepsilon^5 v_5$ term. In this case, we have to consider the first six initial value problems (13)–(18). Solving these problems (13)–(18) consecutively, we obtain

$$v_0(t) = \cos(t), \tag{20}$$

$$v_1(t) = -\frac{1}{4}\sin(t) + \frac{1}{4}t\cos(t) + \frac{1}{4}t^2\sin(t), \tag{21}$$

$$v_2(t) = \frac{7}{32}t^2\cos(t) - \frac{1}{32}t^4\cos(t) - \frac{7}{32}t\sin(t) + \frac{5}{48}t^3 \times \sin(t), \tag{22}$$

$$v_3(t) = \tfrac{35}{128}\sin(t) + \tfrac{35}{192}t^3\cos(t) + \tfrac{7}{96}t^4\sin(t) - \tfrac{35}{128}t\cos(t) - \tfrac{35}{128}t^2\sin(t) - \tfrac{7}{384}t^5\cos(t) - \tfrac{1}{384}t^6\sin(t), \tag{23}$$

$$v_4(t) = \tfrac{1085}{6144}t^4\cos(t) - \tfrac{1365}{2048}t^2\cos(t) - \tfrac{119}{9216}t^6\cos(t) + \tfrac{1}{6144}t^8\cos(t) + \tfrac{1365}{2048}t\sin(t) + \tfrac{175}{3072}t^5\sin(t) - \tfrac{1225}{3072}t^3\sin(t) - \tfrac{1}{512}t^7\sin(t), \tag{24}$$

$$v_5(t) = -\tfrac{15015}{8192}\sin(t) - \tfrac{109}{73728}t^8\sin(t) + \tfrac{3661}{73728}t^6\sin(t) + \tfrac{1}{122880}t^{10}\sin(t) - \tfrac{2275}{4096}t^4\sin(t) + \tfrac{15015}{8192}t^2\sin(t) + \tfrac{15015}{8192}t\cos(t) + \tfrac{11}{73728}t^9\cos(t) - \tfrac{5005}{4096}t^3\cos(t) + \tfrac{1547}{8192}t^5\cos(t) - \tfrac{23}{2304}t^7\cos(t). \tag{25}$$

We define $w_N(t)$ as

$$w_N(t) = \sum_{n=0}^{N} \varepsilon^n v_n(t), \tag{26}$$

an approximation of the exact solution $v(t)$ to the LIVP (1)–(2). This $w_N(t)$ is a truncated asymptotic expansion solution, where its error is of the order $O(\varepsilon^{N+1})$, as per our assumption. The first six asymptotic expansion solutions to the LIVP (1)–(2) are

$$w_0(t) = v_0(t), \tag{27}$$

$$w_1(t) = v_0(t) + \varepsilon v_1(t), \tag{28}$$

$$w_2(t) = v_0(t) + \varepsilon v_1(t) + \varepsilon^2 v_2(t), \tag{29}$$

$$w_3(t) = v_0(t) + \varepsilon v_1(t) + \varepsilon^2 v_2(t) + \varepsilon^3 v_3(t), \tag{30}$$

$$w_4(t) = v_0(t) + \varepsilon v_1(t) + \varepsilon^2 v_2(t) + \varepsilon^3 v_3(t) + \varepsilon^4 v_4(t), \tag{31}$$

$$w_5(t) = v_0(t) + \varepsilon v_1(t) + \varepsilon^2 v_2(t) + \varepsilon^3 v_3(t) + \varepsilon^4 v_4(t) + \varepsilon^5 v_5(t), \tag{32}$$

where $v_0(t), v_1(t), v_2(t), v_3(t), v_4(t), v_5(t)$ are given by Eqs. (20)–(25), respectively.

Our goal in this subsection is to verify that the accuracy orders of

$$w_0(t), w_1(t), w_2(t), w_3(t), w_4(t), w_5(t) \tag{33}$$

are indeed $1, 2, 3, 4, 5, 6$, respectively. In other words, their errors are of the orders

$$O(\varepsilon), O(\varepsilon^2), O(\varepsilon^3), O(\varepsilon^4), O(\varepsilon^5), O(\varepsilon^6) \tag{34}$$

respectively.

Using the same values of ε and $t = 5$ as those used by Abdel-Halim Hassan [23], we apply our proposed order verification method for functions (33). We want to verify that these solutions have, indeed, the first, second, third, fourth, fifth, and sixth orders of accuracy, respectively. The reference solution is obtained using the ode45 algorithm of the MATLAB software, where the relative error tolerance is taken the minimum possible, that is the relative error tolerance is 2.22045×10^{-14} and the absolute error tolerance is set to be 10^{-15}.

To know the behaviour of the asymptotic expansion solutions of the LIVP (1)–(2), we plot them together with the reference solution on the interval $0 \leqslant t \leqslant 10$. Fig. 1 shows the reference solution $v(t)$ and the asymptotic expansion solutions $w_0(t), w_1(t), w_2(t)$. Fig. 2 shows the reference solution $v(t)$ and the asymptotic expansion solutions $w_3(t), w_4(t), w_5(t)$. We observe that as N increases, the asymptotic expansion solution $w_N(t)$ gets closer to the reference solution $v(t)$ in a larger domain.

Our numerical results are recorded in Tables 1–3. Table 1 records the absolute errors and accuracy orders of $w_0(5)$ and $w_1(5)$; we observe in Table 1 that as ε tends to 0, the orders of accuracy of $w_0(5)$ and $w_1(5)$ converge to 1 and 2, respectively. Table 2 provides the absolute errors and accuracy orders of $w_2(5)$ and $w_3(5)$; we observe in Table 2 that as ε tends to 0, the orders of accuracy of $w_2(5)$ and $w_3(5)$ converge to 3 and 4, respectively. Table 3 presents the absolute errors and accuracy orders of $w_4(5)$ and $w_5(5)$; we observe in Table 3 that as ε tends to 0, the orders of accuracy of $w_4(5)$ and $w_5(5)$ converge to 5 and 6, respectively. These results indicate that we have been successful in verifying that $w_0(5), w_1(5), w_2(5), w_3(5), w_4(5), w_5(5)$ have, indeed, the first, second, third, fourth, fifth, and sixth orders of accuracy, respectively.

We note that for $w_5(5)$, when $\varepsilon = 0.015$, the error is 8.86938×10^{-11}; when $\varepsilon = 0.005$, the error is 1.15407×10^{-13}. The order of accuracy losses slightly from 6.007 to 6.048 when there is a change from $\varepsilon = 0.015$ to $\varepsilon = 0.005$. This is because the reference solution that we use is not the exact solution. In order to maintain the convergent behaviour of the order of accuracy with recpect to ε, we need a more accurate reference solution. However, the errors of $w_5(5)$ when $\varepsilon = 0.005$ is very close to the error tolerances that we set. Using the ode45 algorithm in the MATLAB software, we cannot take smaller relative error tolerance than 2.22045×10^{-14}. This makes that our reference solution cannot be more accurate than we have reported here. If the reference solution were the exact solution, we would be confident that the order of accuracy would be better when there was a change from $\varepsilon = 0.015$ to $\varepsilon = 0.005$.

Remark 1. We can compare our results in this subsection with those of Abdel-Halim Hassan [23] regarding the asymptotic solution and the order verification for the LIVP (1)–(2). Abdel-Halim Hassan [23] failed in verifying the accuracy orders of w_1, w_2, w_3, w_4, w_5, and they did not consider verifying

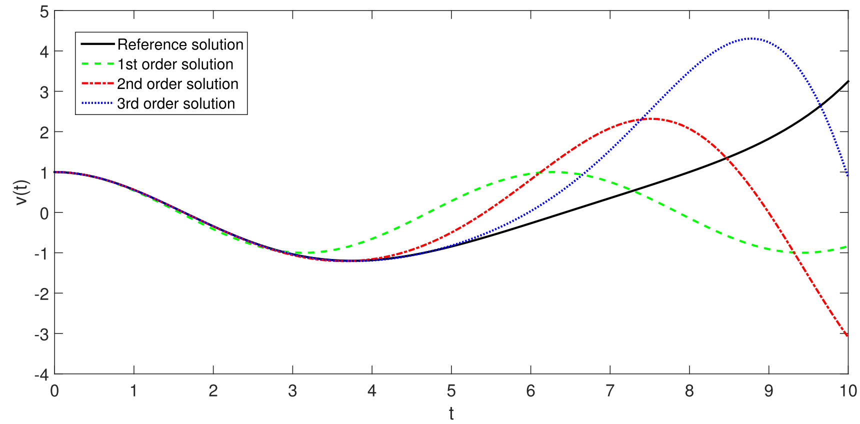


Fig. 1 Reference solution $v(t)$ and asymptotic expansion solutions $w_0(t), w_1(t), w_2(t)$ to the LIVP (1)–(2). Here $w_0(t), w_1(t), w_2(t)$ have the first, second, and third orders of accuracy, respectively.

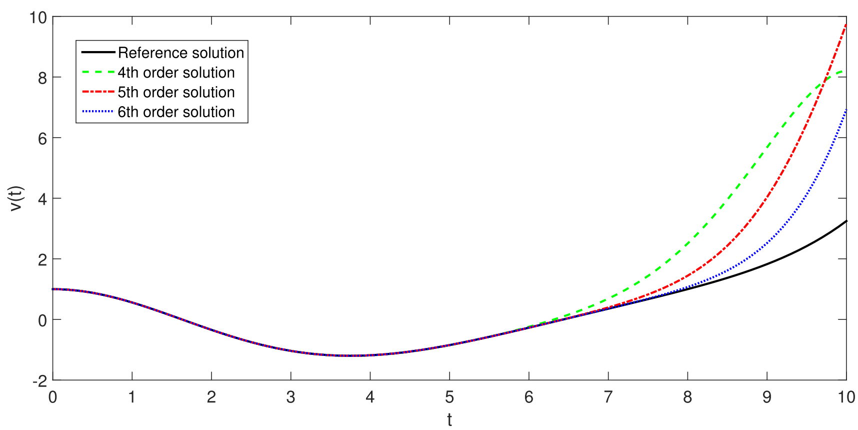


Fig. 2 Reference solution $v(t)$ and asymptotic expansion solutions $w_3(t), w_4(t), w_5(t)$ to the LIVP (1)–(2). Here $w_3(t), w_4(t), w_5(t)$ have the fourth, fifth, and sixth orders of accuracy, respectively.

Table 1 Verification of orders of accuracy for w_0 and w_1 of the LIVP. Error are calculated at $t = 5$.

ε	$E_0(5,\varepsilon)$	Order of $E_0(5,\varepsilon)$	$E_1(5,\varepsilon)$	Order of $E_1(5,\varepsilon)$
0.145	1.128785583	–	0.345935236	–
0.135	1.027719119	1.313	0.298858451	2.047
0.125	0.930188350	1.296	0.255317360	2.046
0.115	0.836180965	1.278	0.215299655	2.045
0.105	0.745680487	1.259	0.178788856	2.043
0.095	0.658666492	1.240	0.145764540	2.040
0.085	0.575114836	1.220	0.116202563	2.038
0.075	0.494997864	1.199	0.090075270	2.035
0.065	0.418284614	1.177	0.067351699	2.032
0.055	0.344941018	1.154	0.047997783	2.028
0.045	0.274930094	1.130	0.031976537	2.024
0.035	0.208212125	1.106	0.019248248	2.020
0.025	0.144744845	1.081	0.009770648	2.015
0.015	0.084483607	1.054	0.003499088	2.010
0.005	0.027381548	1.026	3.86708e−04	2.005

the order of accuracy of w_0. The source of failure was the incorrect Eqs. (3.4)-(3.8) in their paper. In Table 2 of the paper of Abdel-Halim Hassan [23], the gradients of the least-squares curves of the error data are about 1, which means that w_1, w_2, w_3, w_4, w_5 are all of the first order of accuracy. Obviously, the results of Abdel-Halim Hassan [23] are incorrect for the LIVP (1)–(2), whereas in this subsection we have provided the correct ones.

Table 2 Verification of orders of accuracy for w_2 and w_3 of the LIVP. Errors are calculated at $t = 5$.

ε	$E_2(5, \varepsilon)$	Order of $E_2(5, \varepsilon)$	$E_3(5, \varepsilon)$	Order of $E_3(5, \varepsilon)$
0.145	0.021600028	–	0.004229785	–
0.135	0.017716517	2.774	0.003129283	4.217
0.125	0.014283466	2.799	0.002264601	4.202
0.115	0.011288567	2.822	0.001597214	4.187
0.105	0.008715340	2.844	0.001092766	4.172
0.095	0.006543363	2.864	7.20841e−04	4.157
0.085	0.004748491	2.883	4.54751e−04	4.142
0.075	0.003303068	2.900	2.71314e−04	4.126
0.065	0.002176134	2.916	1.50656e−04	4.111
0.055	0.001333621	2.931	7.60091e−05	4.095
0.045	7.38545e−04	2.945	3.35214e−05	4.080
0.035	3.51191e−04	2.958	1.20719e−05	4.064
0.025	1.29292e−04	2.970	3.09224e−06	4.048
0.015	2.82007e−05	2.981	3.94330e−07	4.032
0.005	1.05428e−06	2.991	4.78994e−09	4.015

Table 3 Verification of orders of accuracy for w_4 and w_5 of the LIVP. Errors are calculated at $t = 5$.

ε	$E_4(5, \varepsilon)$	Order of $E_4(5, \varepsilon)$	$E_5(5, \varepsilon)$	Order of $E_5(5, \varepsilon)$
0.145	8.69381e−04	–	7.61207e−05	–
0.135	6.04322e−04	5.089	4.93845e−05	6.055
0.125	4.08680e−04	5.083	3.09987e−05	6.051
0.115	2.67645e−04	5.076	1.87227e−05	6.047
0.105	1.68755e−04	5.070	1.08048e−05	6.043
0.095	1.01665e−04	5.063	5.90377e−06	6.039
0.085	5.79296e−05	5.057	3.01722e−06	6.035
0.075	3.07868e−05	5.051	1.41831e−06	6.031
0.065	1.49582e−05	5.044	5.98681e−07	6.027
0.055	6.44746e−06	5.038	2.18879e−07	6.023
0.045	2.34910e−06	5.031	6.54065e−08	6.019
0.035	6.64427e−07	5.025	1.44236e−08	6.015
0.025	1.22766e−07	5.019	1.90821e−09	6.011
0.015	9.48662e−09	5.012	8.86938e−11	6.007
0.005	3.87900e−11	5.006	1.15407e−13	6.048

4.2. Results and discussion for the nonlinear initial value problem

In this subsection, we consider the NLIVP (3)–(4). Substitution the ε series solution (5) in Eqs. (3) and (4) leads to infinitely many problems (initial value problems), which the first six of the problems are

$$v_0'' + v_0 = 0, \quad v_0(0) = 1, \quad v_0'(0) = 0, \tag{35}$$

$$v_1'' + v_1 = -v_0^3, \quad v_1(0) = 0, \quad v_1'(0) = 0, \tag{36}$$

$$v_2'' + v_2 = -3v_0^2 v_1, \quad v_2(0) = 0, \quad v_2'(0) = 0, \tag{37}$$

$$v_3'' + v_3 = -3v_0 v_1^2 - 3v_0^2 v_2, \quad v_3(0) = 0, \quad v_3'(0) = 0, \tag{38}$$

$$v_4'' + v_4 = -6v_0 v_1 v_2 - 3v_0^2 v_3 - v_1^3, \quad v_4(0) = 0, \quad v_4'(0) = 0, \tag{39}$$

$$v_5'' + v_5 = -6v_0 v_1 v_3 - 3v_1^2 v_2 - 3v_0^2 v_4 - 3v_0 v_2^2, \quad v_5(0) = 0, \quad v_5'(0) = 0. \tag{40}$$

Problems expressed by Eqs. (35)–(40) have exact solutions and they are respectively given by

$$v_0(t) = \cos(t), \tag{41}$$

$$v_1(t) = -\frac{1}{8}\cos(t) + \frac{1}{8}\cos(t)^3 - \frac{3}{8}t\sin(t), \tag{42}$$

$$\begin{aligned} v_2(t) = \; & \tfrac{25}{256}\cos(t) - \tfrac{9}{64}t\cos(t)^2\sin(t) + \tfrac{33}{256}t\sin(t) \\ & - \tfrac{9}{128}t^2\cos(t) + \tfrac{1}{64}\cos(t)^5 - \tfrac{29}{256}\cos(t)^3, \end{aligned} \tag{43}$$

$$\begin{aligned} v_3(t) = \; & -\tfrac{161}{2048}\cos(t) + \tfrac{1}{512}\cos(t)^7 - \tfrac{55}{2048}\cos(t)^5 \\ & - \tfrac{15}{512}t\sin(t)\cos(t)^4 + \tfrac{53}{512}\cos(t)^3 - \tfrac{81}{1024}t^2\cos(t)^3 \\ & + \tfrac{81}{512}t\cos(t)^2\sin(t) + \tfrac{189}{2048}t^2\cos(t) - \tfrac{177}{2048}t\sin(t) \\ & + \tfrac{9}{1024}t^3\sin(t), \end{aligned} \tag{44}$$

$$\begin{aligned} v_4(t) = \; & \tfrac{17033}{262144}\cos(t) + \tfrac{1}{4096}\cos(t)^9 - \tfrac{81}{16384}\cos(t)^7 \\ & - \tfrac{21}{4096}t\sin(t)\cos(t)^6 + \tfrac{2297}{65536}\cos(t)^5 - \tfrac{225}{8192}t^2\cos(t)^5 \\ & + \tfrac{465}{8192}t\sin(t)\cos(t)^4 + \tfrac{4203}{32768}t^2\cos(t)^3 - \tfrac{24989}{262144}\cos(t)^3 \\ & - \tfrac{10431}{65536}t\cos(t)^2\sin(t) + \tfrac{243}{8192}t^3\sin(t)\cos(t)^2 \\ & + \tfrac{27}{32768}t^4\cos(t) \\ & - \tfrac{13077}{131072}t^2\cos(t) - \tfrac{441}{32768}t^3\sin(t) + \tfrac{17673}{262144}t\sin(t), \end{aligned} \tag{45}$$

$$
\begin{aligned}
v_5(t) = {} & \tfrac{213165}{2097152} t^2 \cos(t) - \tfrac{115217}{2097152} \cos(t) - \tfrac{459}{65536} t^4 \cos(t) \\
& + \tfrac{16695}{1048576} t^3 \sin(t) - \tfrac{81}{1310720} t^5 \sin(t) + \tfrac{46347}{524288} \cos(t)^3 \\
& - \tfrac{86211}{2097152} \cos(t)^5 + \tfrac{2211}{262144} \cos(t)^7 - \tfrac{107}{131072} \cos(t)^9 \\
& + \tfrac{1}{32768} \cos(t)^{11} - \tfrac{172719}{1048576} t^2 \cos(t)^3 + \tfrac{17037}{262144} t^2 \cos(t)^5 \\
& + \tfrac{2187}{262144} t^4 \cos(t)^3 - \tfrac{441}{65536} t^2 \cos(t)^7 - \tfrac{117009}{2097152} t \sin(t) \\
& - \tfrac{7155}{131072} t^3 \cos(t)^2 \sin(t) + \tfrac{1125}{65536} t^3 \cos(t)^4 \sin(t) \\
& + \tfrac{20385}{131072} t \cos(t)^2 \sin(t) - \tfrac{20925}{262144} t \cos(t)^4 \sin(t) \\
& + \tfrac{231}{16384} t \cos(t)^6 \sin(t) - \tfrac{27}{32768} t \cos(t)^8 \sin(t).
\end{aligned}
\tag{46}
$$

Based on $v_0(t), v_1(t), v_2(t), v_3(t), v_4(t), v_5(t)$ given by Eqs. (41)–(46), we obtain truncated asymptotic expansion solutions $w_N(t)$ of the form (26). In this subsection, we shall verify that the asymptotic expansion solutions $w_0(t), w_1(t), w_2(t), w_3(t), w_4(t), w_5(t)$ to the NLIVP (3)–(4) are, indeed, of the first, second, third, fourth, fifth, and sixth orders of accuracy, respectively.

As a note, we plot the asymptotic expansion solutions of the NLIVP (3)–(4) together with the reference solution on the interval $0 \leqslant t \leqslant 10$. Fig. 3 shows the reference solution $v(t)$ and the asymptotic expansion solutions $w_0(t), w_1(t), w_2(t)$. We observe that as N increases, the asymptotic expansion solution $w_N(t)$ gets closer to the reference solution $v(t)$ in a larger domain. Fig. 4 shows the reference solution $v(t)$ and the asymptotic expansion solutions $w_3(t), w_4(t), w_5(t)$. We observe that $v(t)$ and $w_3(t), w_4(t), w_5(t)$ almost coincide, which indicates that $w_3(t), w_4(t), w_5(t)$ are quite accurate in the considered domain.

Abdel-Halim Hassan [23] took $t = 3$ for verifying the orders of accuracy of solutions to the NLIVP (3)–(4) obtained using truncated asymptotic expansions, so we also take this value $t = 3$ into our proposed order verification method. Errors and observed orders of accuracy of solutions $w_0(3), w_1(3), w_2(3), w_3(3), w_4(3), w_5(3)$ to the NLIVP (3)–(4) are recorded in Tables 4–6. Table 4 records the absolute errors and accuracy orders of $w_0(3)$ and $w_1(3)$; we infer from Table 4 that as ε tends to 0, the orders of accuracy of $w_0(3)$ and $w_1(3)$ converge to 1 and 2, respectively. Table 5 provides the absolute errors and accuracy orders of $w_2(3)$ and $w_3(3)$; we infer from Table 5 that as ε tends to 0, the orders of accuracy of $w_2(3)$ and $w_3(3)$ converge to 3 and 4, respectively. Table 6 presents the absolute errors and accuracy orders of $w_4(3)$ and $w_5(3)$; we infer from Table 6 that as ε tends to 0, the orders of accuracy of $w_4(3)$ and $w_5(3)$ converge to 5 and 6, respectively. As additional notes, we observe that the behaviour of the order of accuracy of $w_0(t)$ at $t = 3$ is oscillating with respect to the considered values of ε; but as ε tends to 0, its order of accuracy

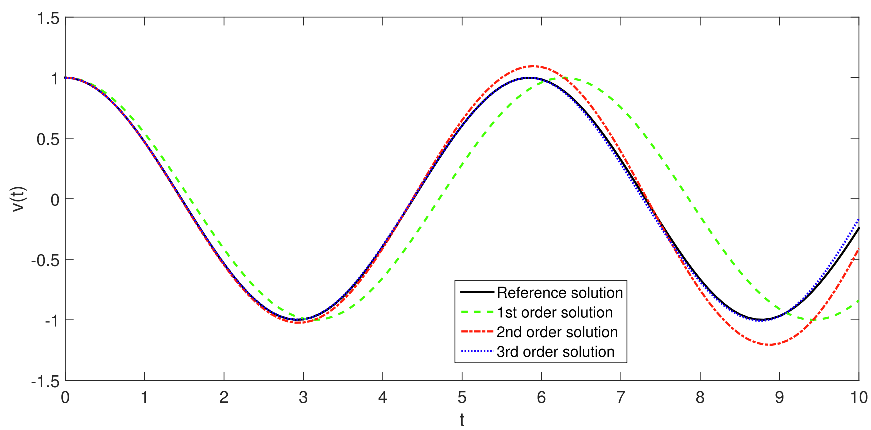


Fig. 3 Reference solution $v(t)$ and asymptotic expansion solutions $w_0(t), w_1(t), w_2(t)$ to the NLIVP (3)–(4). Here $w_0(t), w_1(t), w_2(t)$ have the first, second, and third orders of accuracy, respectively.

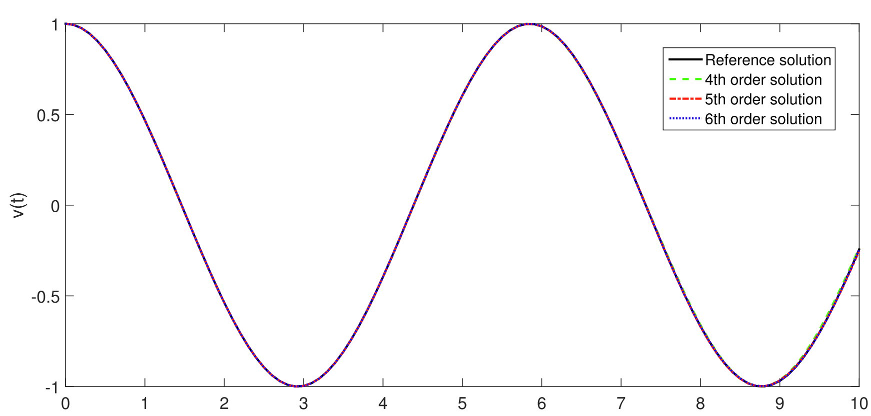


Fig. 4 Reference solution $v(t)$ and asymptotic expansion solutions $w_3(t), w_4(t), w_5(t)$ to the NLIVP (3)–(4). Here $w_3(t), w_4(t), w_5(t)$ have the fourth, fifth, and sixth orders of accuracy, respectively. All of these curves are very close each other and they almost coincide.

Table 4 Verification of orders of accuracy for w_0 and w_1 of the NLIVP. Errors are calculated at $t = 3$.

ε	$E_0(3, \varepsilon)$	Order of $E_0(3, \varepsilon)$	$E_1(3, \varepsilon)$	Order of $E_1(3, \varepsilon)$
0.207	0.006523211	–	0.025829969	–
0.195	0.007510366	−2.360	0.022967268	1.967
0.183	0.008335357	−1.641	0.020266730	1.970
0.171	0.008996899	−1.126	0.017729642	1.972
0.159	0.009493741	−0.739	0.015357252	1.974
0.147	0.009824676	−0.437	0.013150771	1.977
0.135	0.009988531	−0.194	0.011111369	1.979
0.123	0.009984172	0.005	0.009240181	1.981
0.111	0.009810501	0.171	0.007538306	1.983
0.099	0.009466455	0.312	0.006006805	1.985
0.087	0.008951004	0.433	0.004646709	1.987
0.075	0.008263151	0.539	0.003459016	1.989
0.063	0.007401929	0.631	0.002444690	1.991
0.051	0.006366400	0.713	0.001604672	1.992
0.039	0.005155652	0.786	9.39874e−04	1.994

Table 5 Verification of orders of accuracy for w_2 and w_3 of the NLIVP. Errors are calculated at $t = 3$.

ε	$E_2(3, \varepsilon)$	Order of $E_2(3, \varepsilon)$	$E_3(3, \varepsilon)$	Order of $E_3(3, \varepsilon)$
0.207	7.82711e−04	–	7.52010e−05	–
0.195	6.49320e−04	3.129	5.78597e−05	4.390
0.183	5.32637e−04	3.119	4.37874e−05	4.388
0.171	4.31374e−04	3.109	3.25241e−05	4.384
0.159	3.44285e−04	3.099	2.36484e−05	4.380
0.147	2.70158e−04	3.090	1.67778e−05	4.374
0.135	2.07823e−04	3.080	1.15677e−05	4.366
0.123	1.56146e−04	3.071	7.71092e−06	4.357
0.111	1.14027e−04	3.062	4.93614e−06	4.345
0.099	8.04050e−05	3.054	3.00737e−06	4.331
0.087	5.42489e−05	3.045	1.72228e−06	4.314
0.075	3.45622e−05	3.037	9.10652e−07	4.294
0.063	2.03780e−05	3.030	4.32605e−07	4.269
0.051	1.07577e−05	3.023	1.76600e−07	4.240
0.039	4.78884e−06	3.017	5.71591e−08	4.205

Table 6 Verification of orders of accuracy for w_4 and w_5 of the NLIVP. Errors are calculated at $t = 3$.

ε	$E_4(3, \varepsilon)$	Order of $E_4(3, \varepsilon)$	$E_5(3, \varepsilon)$	Order of $E_5(3, \varepsilon)$
0.207	3.86871e−05	–	1.04361e−05	–
0.195	2.91045e−05	4.766	7.33794e−06	5.898
0.183	2.14835e−05	4.780	5.04360e−06	5.903
0.171	1.55197e−05	4.795	3.37829e−06	5.909
0.159	1.09378e−05	4.809	2.19690e−06	5.914
0.147	7.49143e−06	4.823	1.38057e−06	5.920
0.135	4.96217e−06	4.837	8.33505e−07	5.926
0.123	3.15896e−06	4.851	4.79853e−07	5.931
0.111	1.91709e−06	4.865	2.60862e−07	5.937
0.099	1.09700e−06	4.879	1.32162e−07	5.943
0.087	5.82944e−07	4.893	6.12713e−08	5.949
0.075	2.81404e−07	4.907	2.53153e−08	5.955
0.063	1.19321e−07	4.921	8.95283e−09	5.962
0.051	4.20583e−08	4.935	2.53670e−09	5.968
0.039	1.11508e−08	4.949	5.10739e−10	5.975

Table 7 Verification of orders of accuracy for $w_0, w_1, w_2, w_3, w_4, w_5$ of the NLIVP at $t = 1$.

Value of ε	Order of $E_0(1,\varepsilon)$	Order of $E_1(1,\varepsilon)$	Order of $E_2(1,\varepsilon)$	Order of $E_3(1,\varepsilon)$	Order of $E_4(1,\varepsilon)$	Order of $E_5(1,\varepsilon)$
0.207	–	–	–	–	–	–
0.195	0.963	1.950	2.952	3.952	4.952	5.952
0.183	0.965	1.953	2.954	3.955	4.955	5.955
0.171	0.967	1.956	2.957	3.958	4.958	5.958
0.159	0.969	1.959	2.960	3.961	4.960	5.960
0.147	0.971	1.962	2.963	3.963	4.963	5.963
0.135	0.973	1.965	2.966	3.966	4.966	5.966
0.123	0.976	1.968	2.968	3.969	4.969	5.969
0.111	0.978	1.971	2.971	3.972	4.972	5.972
0.099	0.980	1.974	2.974	3.975	4.974	5.975
0.087	0.982	1.976	2.977	3.977	4.977	5.977
0.075	0.984	1.979	2.980	3.980	4.980	5.980
0.063	0.987	1.982	2.983	3.983	4.983	5.983
0.051	0.989	1.985	2.986	3.986	4.986	5.986
0.039	0.991	1.989	2.989	3.989	4.989	5.990

approaches to 1, as shown in Table 4. Furthermore, as ε tends to 0, the orders of accuracy of $w_1(3), w_2(3), w_3(3), w_4(3), w_5(3)$ clearly approach to $2, 3, 4, 5, 6$, respectively.

The expected orders of accuracy reveal clearer when we take the value of t in the domain scale 1, say $t = 1$, as shown in Table 7. We observe from Table 7 that the orders of accuracy of $w_0(1), w_1(1), w_2(1), w_3(1), w_4(1), w_5(1)$ approach to $1, 2, 3, 4, 5, 6$, respectively for the NLIVP (3)–(4). These results of order verifications are clearer than those of Abdel-Halim Hassan [23]. As a remark, we have successfully implemented our proposed order verification mehod to verify the orders of accuracy of solutions to the NLIVP (3)–(4) obtained using truncated asymptotic expansions.

With the success of our proposed method, it is worthwhile to indicate a follow-up research. A possible future research direction is to extend the proposed method in this paper for the fractional order problems, such as those studied by Jajarmi and Baleanu [29] as well as Mohammadi et al. [30].

5. Conclusion

We have obtained explicit solutions to initial value problems, where numerical methods cannot provide one, and verified the accuracy orders of the explicit solutions. A method has been proposed to verify the accuracy orders of explicit truncated asymptotic expansion solutions to initial value problems. We have solved linear and nonlinear initial value problems using the asymptotic expansion method. In addition, we have the correct asymptotic (formal) expansion solutions to the initial value problems. The correctness of our solutions is confirmed by our proposed order verification method. Furthermore, our proposed method is successful in verifying the expected correct orders of truncated asymptotic expansion solutions for both linear and nonlinear initial value problems. A possible future research direction is to explore and implement the proposed method for differential equations of fractional orders.

Declaration of Competing Interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The author is thankful to Sanata Dharma University and the Directorate of Research and Community Services (DRPM) of the Ministry of Research and Technology (Kemenristek) / National Research and Innovation Agency (BRIN) of the Republic of Indonesia. Part of the work conducted in 2020 was financially supported by Sanata Dharma University under contract number 035/Penel./LPPM-USD/V/2020. The rest of the work conducted in 2021 was financially supported by DRPM Kemenristek / BRIN under a World Class Research scheme with the announcement number B/112/E3/RA.00/2021. Comments and suggestions, which have improved the quality of this paper, of two anonymous reviewers are gratefully acknowledged sincerely.

References

[1] S. Meftah, A new approach to approximate solutions for nonlinear differential equation, Int. J. Math. Math. Sci. 2018 (2018) 5129502, https://doi.org/10.1155/2018/5129502.

[2] F. Verhulst, Nonlinear Differential Equations and Dynamical Systems, Springer, Berlin, 1996, https://doi.org/10.1007/978-3-642-61453-8.

[3] O.A. Arqub, M. Al-Smadi, Atangana-Baleanu fractional approach to the solutions of Bagley-Torvik and Painlevé equations in Hilbert space, Chaos, Solitons & Fractals 117 (2018) 161–167, https://doi.org/10.1016/j.chaos.2018.10.013.

[4] O.A. Arqub, B. Maayah, Numerical solutions of integrodifferential equations of Fredholm operator type in the sense of the Atangana-Baleanu fractional operator, Chaos, Solitons & Fractals 117 (2018) 117–124, https://doi.org/10.1016/j.chaos.2018.10.007.

[5] O.A. Arqub, B. Maayah, Fitted fractional reproducing kernel algorithm for the numerical solutions of ABC – Fractional Volterra integro-differential equations, Chaos, Solitons & Fractals 126 (2019) 394–402, https://doi.org/10.1016/j.chaos.2019.07.023.

[6] O.A. Arqub, B. Maayah, Modulation of reproducing kernel Hilbert space method for numerical solutions of Riccati and Bernoulli equations in the Atangana-Baleanu fractional sense, Chaos, Solitons & Fractals 125 (2019) 163–170, https://doi.org/10.1016/j.chaos.2019.05.025.

[7] O.A. Arqub, N. Shawagfeh, Application of reproducing kernel algorithm for solving Dirichlet time-fractional diffusion-Gordon types equations in porous media, J. Porous Media 22 (4) (2019) 411–434, https://doi.org/10.1615/JPorMedia.2019028970.

[8] D. Baleanu, B. Ghanbari, J.H. Asad, A. Jajarmi, H.M. Pirouz, Planar system-masses in an equilateral triangle: Numerical study within fractional calculus, CMES-Comput. Model. Eng. Sci. 124 (3) (2020) 953–968, https://doi.org/10.32604/cmes.2020.010236.

[9] S.S. Sajjadi, D. Baleanu, A. Jajarmi, H.M. Pirouz, A new adaptive synchronization and hyperchaos control of a biological snap oscillator, Chaos, Solitons & Fractals 138 (2020) 109919, https://doi.org/10.1016/j.chaos.2020.109919.

[10] D. Baleanu, A. Jajarmi, S.S. Sajjadi, J.H. Asad, The fractional features of a harmonic oscillator with position-dependent mass, Commun. Theor. Phys. 72 (5) (2020) 055002, https://doi.org/10.1088/1572-9494/ab7700.

[11] A. Jajarmi, D. Baleanu, A new iterative method for the numerical solution of high-order non-linear fractional boundary value problems, Front. Phys. 8 (2020) 220, https://doi.org/10.3389/fphy.2020.00220.

[12] H. Guo, C. Gui, P. Lin, M. Zhao, Multiple solutions and their asymptotics for laminar flows through a porous channel with different permeabilities, IMA J. Appl. Math. 85 (2) (2020) 280–308, https://doi.org/10.1093/imamat/hxaa006.

[13] T.S. Chan, F. Yang, A. Carlson, Directional spreading of a viscous droplet on a conical fibre, J. Fluid Mech. 894 (2020) A26, https://doi.org/10.1017/jfm.2020.240.

[14] K.D. Housiadas, G.C. Georgiou, New analytical solutions for weakly compressible Newtonian Poiseuille flows with pressure-dependent viscosity, Int. J. Eng. Sci. 107 (2016) 13–27, https://doi.org/10.1016/j.ijengsci.2016.07.001.

[15] H. Salari, H. Hassanzadeh, J. Abedi, Approximate analytical solutions for steady-state nonisothermal convection-diffusion-reaction in a slab, Appl. Math. Comput. 264 (2015) 141–159, https://doi.org/10.1016/j.amc.2015.04.080.

[16] M.V. Kunte, V.R. Sonti, Asymptotic wavenumber expansions of a strongly orthotropic fluid-filled circular cylindrical shell, Wave Motion 50 (3) (2013) 402–414, https://doi.org/10.1016/j.wavemoti.2012.10.003.

[17] S. Lynch, C. Knessl, Drift-diffusion past an ellipse: Singular perturbation analysis of the illuminated region, Stud. Appl. Math. 125 (2) (2010) 113–151, https://doi.org/10.1111/j.1467-9590.2009.00477.x.

[18] J. Ma, J.P. Keener, The computation of biomarkers in pharmacokinetics with the aid of singular perturbation methods, J. Math. Biol. 77 (5) (2018) 1407–1430, https://doi.org/10.1007/s00285-018-1257-3.

[19] V.E. Ragozina, Y.E. Ivanova, Spherically symmetric shock waves in materials with a nonlinear stress-strain dependence, Mater. Sci. Forum 945 (2019) 807–812, https://doi.org/10.4028/www.scientific.net/MSF.945.807.

[20] L.V. Stepanova, S.A. Igonin, Perturbation method for solving the nonlinear eigenvalue problem arising from fatigue crack growth problem in a damaged medium, Appl. Math. Model. 38 (14) (2014) 3436–3455, https://doi.org/10.1016/j.apm.2013.11.057.

[21] B. Stehlíková, Perturbation analysis of a nonlinear equation arising in the Schaefer-Schwartz model of interest rates, Mathematica Slovaca 68 (3) (2018) 617–624, https://doi.org/10.1515/ms-2017-0129.

[22] M. Li, X. Pu, S. Wang, Quasineutral limit for the compressible quantum Navier–Stokes–Maxwell equations, Commun. Math. Sci. 16 (2) (2018) 363–391, https://doi.org/10.4310/CMS.2018.v16.n2.a3.

[23] I.H. Abdel-Halim Hassan, The asymptotic expansion and numerical verification method for linear and nonlinear initial value problem, Appl. Math. Comput. 180 (1) (2006) 29–37, https://doi.org/10.1016/j.amc.2005.11.146.

[24] D.L. Bosley, A technique for the numerical verification of asymptotic expansions, SIAM Rev. 38 (1) (1996) 128–135, https://doi.org/10.1137/1038006.

[25] E. Deeba, S. Xie, The asymptotic expansion and numerical verification of van der Pol's equation, J. Comput. Anal. Appl. 3 (2) (2001) 165–171, https://doi.org/10.1023/A:1010189225921.

[26] E.A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

[27] R. Haberman, Mathematical Models: Mechanical Vibrations, Population Dynamics, and Traffic Flow, Prentice-Hall, Englewood Cliffs, 1977.

[28] S. Mungkasi, Numerical verification of the orders of accuracy of truncated asymptotic expansion solutions to the van der Pol equation, J. Math. Chem. 59 (1) (2021) 216–223, https://doi.org/10.1007/s10910-020-01191-6.

[29] A. Jajarmi, D. Baleanu, On the fractional optimal control problems with a general derivative operator, Asian J. Control 23 (2) (2021) 1062–1071, https://doi.org/10.1002/asjc.2282.

[30] F. Mohammadi, L. Moradi, D. Baleanu, A. Jajarmi, A hybrid functions numerical scheme for fractional optimal control problems: Application to nonanalytic dynamic systems, J. Vib. Control 24 (21) (2018) 5030–5043, https://doi.org/10.1177/1077546317741769.