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# AN ANALYTICAL APPROACH FOR INVESTIGATING THE UNCERTAINTY IN DESIGN OF OBJECTIVE FUNCTIONS IN THE IHACRES RAINFALL RUN-OFF MODEL

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## ABSTRACT

This paper investigates the uncertainty in design of objective functions using analytical approach. Given input with uncertainty, we want to know the uncertainty in the output of the functions. We start from a simple function, and after finding the investigation results, we generalise the function to get wider-ranging results. These functions approximate the nonlinear module in the IHACRES model, which is a rainfall run-off model. Three cases are considered. In the first two simple cases, which the input has only one kind of uncertainty, we use the change-of-variable technique. In the third case, which the input has two different uncertainties but they are independent, we use the expectation formula since the cumulative distribution function is not defined.

**Keywords :** uncertainty analysis, random variable, IHACRES model, objective function

## ABSTRAK

Makalah ini menyelidiki ketidakpastian dalam desain fungsi tujuan menggunakan pendekatan analitis. Diberikan input yang mengandung ketidakpastian ke dalam fungsi tujuan, maka akan dicari ketidakpastian dari output yang dihasilkan fungsi tersebut. Pembahasan dalam makalah ini dimulai dari suatu fungsi sederhana, dan setelah didapatkan hasil penyelidikan, fungsi serta penyelidikannya diperumum. Fungsi-fungsi tersebut merupakan pendekatan dari modul nonlinear yang terdapat dalam model IHACRES, suatu model alir hujan. Ada tiga kasus yang dibahas. Dalam dua kasus pertama yang merupakan kasus-kasus sederhana, yang mempunyai input dengan satu jenis ketidakpastian, digunakan teknik pergantian peubah. Dalam kasus ketiga, yang mempunyai input dengan dua jenis ketidakpastian yang berbeda tetapi keduanya saling bebas, digunakan rumus nilai harapan karena fungsi distribusi kumulatifnya tidak terdefinisi untuk kasus ini.

**Kata kunci :** analisis ketidakpastian, peubah acak, model IHACRES, fungsi tujuan

## Introduction

In building a model, it is important to adequately account for variations in the errors in the observed and modelled quantities. Failure to do so means that the objective function is merely giving a measure of how well the modelled values represent the observed values, not how well the model is representing the system being modelled, unless the uncertainties in the observed and modelled values are sufficiently small (Croke, 2007).

The aim of this paper is to investigate the role of uncertainty in design of objective functions using analytical approach. The model that is being the focus of the project is the IHACRES, which is a rainfall run-off model. The investigation here is that how the input data, which of course has uncertainties, affects the uncertainties in the output of the model. In the investigation, we assume the input data as continuous random variable. Furthermore, we consider that there are two types of module in the IHACRES model, namely nonlinear and linear modules. The input data for the nonlinear module is rainfall and temperature, and the output is effective rainfall. The linear module has the effective rainfall, which is the output of the nonlinear module, as the input; and its output is stream flow. The structure of the IHACRES model is shown in Figure 1. More details about the IHACRES model can be found in Croke and Jakeman (2004; 2005), Croke *et al.* (2002; 2005), and Merritt *et al.* (2004).

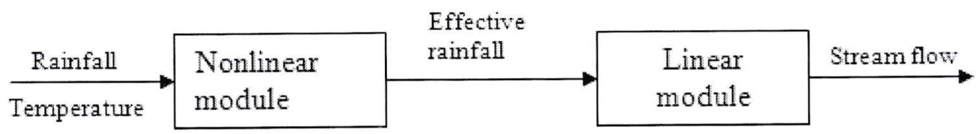


**Figure 1**. Generic structure of the IHACRES model.

The remaining of this paper is organised as follows. Section 2 represents some theoretical background, which consists of reviewing some equations involved in the IHACRES model and the theory of change-of-variable technique, which will be used in the next sections. Section 3 discusses how the input data affects the uncertainties of the output in the nonlinear module of the IHACRES model. We start from a simple case of function of random variable, and we generalise the case for approximating the equation that represents the nonlinear module of the IHACRES model. Section 4 concludes the investigation.

**Research Method**

In this section, we represent some theory that will be used in the next sections. Some equations involved in the IHACRES model and the idea in the change-of-variable are presented as follows. Our main reference for Subsection 2.1 is Ye *et al.* (1997), while references for Subsection 2.2 and Subsection 2.3 are Barr and Zehna (1983), Bertsekas and Tsitsiklis (2002), Ross (1972; 2002), and Thompson (2000).

## 1. Fundamental equations in IHACRES

Ye *et al.* (1997) has coded an adaptation of nonlinear module within IHACRES v2.0 with the effective rainfall is

$$u_k = \begin{cases} [c\cdot(\phi_k - l)]^n P_k & , \phi_k \geq l \\ 0 & , \phi_k < l \end{cases} \quad (1)$$

*where* $P_k$ is the observed rainfall; *c*, *l*, and *n* are parameters that represent mass balance, soil moisture index threshold, and nonlinear response terms respectively; and $\phi_k$ is a soil moisture index. The function $\phi_k$ is given by

$$\phi_k = \left(1 + \frac{1}{\tau_k}\right)\phi_{k-1} + P_k . \quad (2)$$

Here, the function $\tau_k$ is the drying rate, and is given by

$$\tau_k = \tau_w \exp(0.062 \cdot f \cdot (T_r - T_k)), \quad (3)$$

where $\tau_w$, $f$, and $T_r$ are parameters that represent reference drying rate, temperature modulation, and reference temperature respectively.

For simplicity, taking $f = 0$, $l = 0$, $n = 1$, we have

$$u_k = c[\gamma\, \phi_{k-1} P_k + P_k^2], \quad (4)$$

where $\phi_k = P_k + \gamma\, \phi_{k-1}$ and $\gamma = 1 + \dfrac{1}{\tau_w}$.

## 2. The change-of-variable technique

Suppose $X$ has density $f_X$ and $g$ is a monotone function with $g'(x)$ either positive for all $x$ or negative for all $x$. If $Y = g(X)$, we have, if $g$ is increasing,

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P[g(X) \leq y] \\ &= P[X \leq g^{-1}(y)] = F_X(g^{-1}(y)). \end{aligned} \quad (5)$$

If g is decreasing, we have

$$\begin{aligned} F_Y(y) &= P[X \geq g^{-1}(y)] \\ &= 1 - F_X(g^{-1}(y)). \end{aligned} \quad (6)$$

Thus in either case,

$$f_Y(y) = f_X(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right|, \quad (7)$$

Or

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}. \quad (8)$$

For example, if $Y = X^2$, then $R_Y \subset (0, \infty)$, and for any $y \in R_Y$,

$$(Y \leq y) = (-\sqrt{y} < X < \sqrt{y}), \quad (9)$$

so that $F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$. It then follows that

$$f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \cdot (-1) \cdot \frac{1}{2\sqrt{y}} = \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}} \quad (10)$$

for all $y \in R_Y$.

It is important to note that the formula is valid only for $y \in R_Y$, and care must be exercised in applying $f_X$ to both $\sqrt{y}$ and $-\sqrt{y}$.

## 3. Expectation and several identities

The expected value (or mean) of a random variable $X$, denoted by $E(X)$, is defined by

$$E(X) = \sum_x x\, p_X(x), \quad (11)$$

if $X$ is discrete, provided $\sum |x| p(x) < \infty$. If $X$ is continuous,

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx, \quad (12)$$

provided $\int_{-\infty}^{\infty} |x| f(x) < \infty$. The variance of $X$ is $\sigma_X^2 = E(X^2) - [E(X)]^2$, and the standard deviation of $X$ is the square root of its variance.

Several identities related to the expectation are as follows:

a. $E(aX + b) = aE(X) + b$

b. $E(A \cdot X) = E(A) \cdot E(X)$, if $A$ and $X$ are two independent random variable.

c. The variance of summation of two independent random variables is equal to the summation of each of its variance.

## Results and Discussions

Having the points above, next we will consider three problems starting from a simple one to the more general ones. Three problems that we present have the form $Y = X^2$, $Y = aX + X^2$, and $Y = AX + X^2$ where $a$ is constant, $A$ and $X$ are random variables. Note that these problems approximate the nonlinear module in the IHACRES model. Once again we take into account the explanation on uncertainty given in Barr and Zehna (1983), Bertsekas and Tsitsiklis (2002), Ross (1972; 2002), and Thompson (2000), as mentioned in Section 2.

### 1. Case 1: $Y = X^2$

From Section 2, we have $u_k = c[\gamma \phi_{k-1} P_k + P_k^2]$, where $\phi_k = P_k + \gamma\, \phi_{k-1}$, for some constant $\gamma$. Now, we assume that all $P_k$ form a continuous distribution. Let $x = P_k$ and $X$ is the related random variable, and let $\gamma \phi_{k-1} = 0$ for all $k$. Then writing $y = u_k / c$, we have new random variable $Y = X^2$.

Croke (2007) describes that the error propagation of an uncertainty through a function considers not only up to the first derivative but up to the second derivative. Given a function of random variable $Y = g(X)$, then the Taylor series expansion of $y = g(x)$ about the mean of $x$ is

$$y = g(\bar{x}) + \sum_{n=1}^{\infty} \frac{g^{(n)}(\bar{x})}{n!}(x - \bar{x})^n. \qquad (13)$$

Considering an ensemble of sufficiently large $N$ measurements of $x$ and considering the Taylor series expansion up to the second derivative, the mean of $y$ is given by

$$\bar{y} = g(\bar{x}) + \tfrac{1}{2} g''(\bar{x}) \sigma_x^2 \qquad (14)$$

and the variance is given by

$$\sigma_y^2 = [g'(\bar{x})]^2 \sigma_x^2 + g'(\bar{x}) \cdot g''(\bar{x}) \cdot {}^3M_x + \tfrac{1}{4}[g''(\bar{x})]^2 \cdot ({}^4M_x - \sigma_x^4), \qquad (15)$$

where ${}^nM_x = \sum_{i=1}^{N} \frac{(x_i - \bar{x})^n}{N-1}$.

Furthermore, Croke presents a Monte Carlo estimation of the mean, standard deviation and 95% confidence bounds for $Y = X^2$ when the uncertainty in $X$ is a uniform distribution with width 1.

Here, we want to investigate the related standard deviation of the same problem using the change-of-variable technique, given $Y = X^2$, where $X \sim U(\bar{x} - 0.5, \bar{x} + 0.5)$, $-1 < \bar{x} < 1$. It is clear that for the given random variable $X$, the density function is $f_X(x) = 1$, the mean is 0 and the standard deviation is $1/\sqrt{12}$. To simplify the next writing, let us denote

$$p = \min\{(\bar{x} - 0.5)^2, (\bar{x} + 0.5)^2\}, \qquad (16)$$

$$q = \max\{(\bar{x} - 0.5)^2, (\bar{x} + 0.5)^2\}. \qquad (17)$$

We want to investigate the standard deviation of $Y$. We do as follows. The range of $Y$ is $R_Y = (0, q)$. For any choice of $y > p$, either $f_X(-\sqrt{y})$ or $f_X(\sqrt{y})$ is equal to 0. Consequently,

$$f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{y}} & , 0 < y < p \\ \dfrac{1}{2\sqrt{y}} & , y > p. \end{cases} \qquad (18)$$

We separate the problem into two parts. The first is for $|\bar{x}| < 0.5$ and the second is for $|\bar{x}| \geq 0.5$. Let

$$s_1 = \tfrac{1}{5} p^{5/2} - \tfrac{2}{3} \bar{y} p^{3/2} + \bar{y}^2 p^{1/2}, \qquad (19)$$

$$s_2 = \tfrac{1}{5} q^{5/2} - \tfrac{2}{3} \bar{y} q^{3/2} + \bar{y}^2 q^{1/2}. \qquad (20)$$

Then for $|\bar{x}| < 0.5$, we get the mean of $Y$

$$\bar{y} = \int_0^q y\, f_Y(y) dy = \frac{1}{3}\left(p^{2/3} + q^{2/3}\right), \quad (21)$$

and the variance

$$\sigma_y^2 = \int_0^q (y - \bar{y})^2\, f_Y(y)\, dy = s_1 + s_2 . \quad (22)$$

For $|\bar{x}| \geq 0.5$, the mean of $Y$ is

$$\bar{y} = \int_p^q y\, f_Y(y) dy = \frac{1}{3}\left(q^{2/3} - p^{2/3}\right), \quad (23)$$

and the variance is

$$\sigma_y^2 = \int_p^q (y - \bar{y})^2\, f_Y(y)\, dy = s_2 - s_1 . \quad (24)$$

The graphs for $Y = X^2$ and its standard deviation are given in Figure 2,

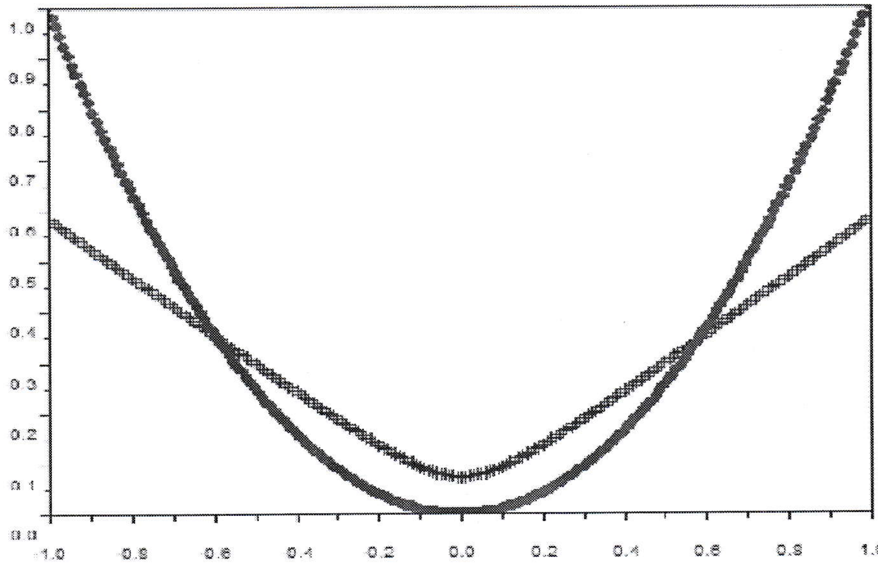


Figure 2: Graph for $Y = X^2$ (dot/solid line) and graph for its standard deviation (plus line).

where the dot/solid line represents $Y$, and the plus line represents the standard deviation of $Y$. Note that standard deviation is the square root of the variance. The standard deviation that we get here is an exact value. This result agrees well with the estimation using Monte Carlo simulated by Croke (2007) as shown in Figure 2.

2. **Case 2**: $Y = aX + X^2$

Now, we want to have a more general formula than Case 1, that is we want to investigate the standard deviation for an arbitrary value of $a$. Suppose we are given $X \sim U(\bar{x} - 0.5,\ \bar{x} + 0.5)$ where $-1 < \bar{x} < 1$.

Consider $Y = aX + X^2$ for an arbitrary real value of $a$. Let

$$p = \min\{a(\bar{x}-0.5)+(\bar{x}-0.5)^2,\ a(\bar{x}+0.5)+(\bar{x}+0.5)^2\}, \quad (25)$$

$$q = \max\{a(\bar{x}-0.5)+(\bar{x}-0.5)^2,\ a(\bar{x}+0.5)+(\bar{x}+0.5)^2\}. \quad (26)$$

The range of $Y$ is $R_Y = (y_{\min}, q)$, where $y_{\min}$ is the minimum value of $Y$ that is $y_{\min} = -a^2/4$. For any choice of $y > p$, either $f_X(-a/2-\sqrt{y+a^2/4})$ or $f_X(-a/2+\sqrt{y+a^2/4})$ is equal to 0.

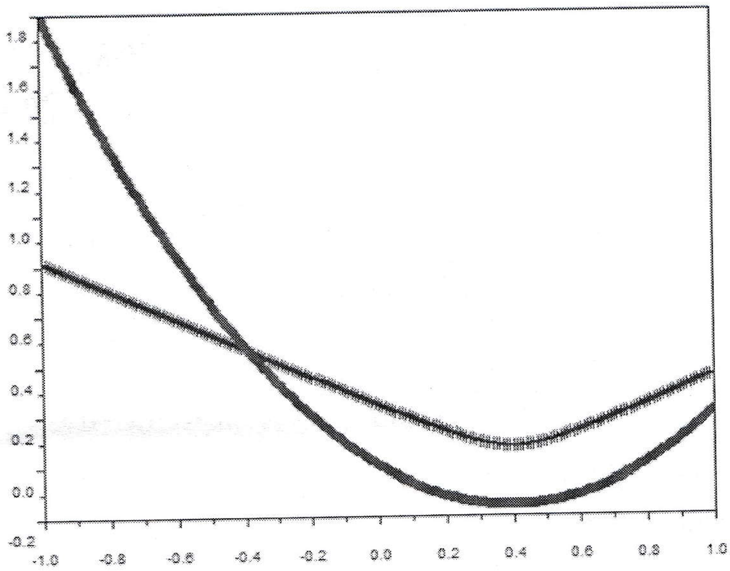


Figure 3: Graph for $Y = -0.8X + X^2$ (dot/solid line) and graph for its standard deviation (plus line).

Table 1: Some numbers involved in the computation of Case 2

| |
|---|
| $t_1 = \frac{1}{3}\sqrt{(p+a^2/4)^3} - a^2/4\sqrt{p+a^2/4}$ |
| $t_2 = \frac{1}{3}\sqrt{(q+a^2/4)^3} - a^2/4\sqrt{q+a^2/4}$ |
| $u_1 = \frac{1}{5}\sqrt{(p+a^2/4)^5}$ |
| $u_2 = \frac{1}{5}\sqrt{(q+a^2/4)^5}$ |
| $v_1 = (\frac{1}{6}a^2 + \frac{2}{3}\bar{y})\sqrt{(p+a^2/4)^3}$ |
| $v_2 = (\frac{1}{6}a^2 + \frac{2}{3}\bar{y})\sqrt{(q+a^2/4)^3}$ |
| $w_1 = (\frac{1}{16}a^4 + \frac{1}{2}\bar{y}a^2 + \bar{y}^2)\sqrt{p+a^2/4}$ |
| $w_2 = (\frac{1}{16}a^4 + \frac{1}{2}\bar{y}a^2 + \bar{y}^2)\sqrt{q+a^2/4}$ |

Consequently,

$$f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{y+a^2/4}}, & y_{\min} < y < p \\ \dfrac{1}{2\sqrt{y+a^2/4}}, & y > p. \end{cases} \quad (27)$$

Again, we separate the problem into two parts. The first is for $|\bar{x}+a/2| < 0.5$ and the second is for $|\bar{x}+a/2| \geq 0.5$. Let us define the numbers given in Table 1. Then for $|\bar{x}+a/2| < 0.5$, we get the mean of $Y$

$$\bar{y}=\int_{y_{\min}}^{q} y f_Y(y)dy = t_1 + t_2, \qquad (28)$$

and the variance

$$\sigma_y^2=\int_{y_{\min}}^{q}(y-\bar{y})^2 f_Y(y)dy$$
$$=(u_1-v_1+w_1)+(u_2-v_2+w_2). \qquad (29)$$

For $|\bar{x}+a/2|\geq 0.5$, the mean of $Y$ is

$$\bar{y}=\int_{p}^{q} y f_Y(y)dy = t_2 - t_1, \qquad (30)$$

and the variance is

$$\sigma_y^2=\int_{p}^{q}(y-\bar{y})^2 f_Y(y)dy$$
$$=(u_2-v_2+w_2)-(u_1-v_1+w_1). \qquad (31)$$

Taking the square root of the variance of $Y$ results the standard deviation. The formula for the mean and the standard deviation are valid for any $a$ constant.

Now, we want to simulate the results that we have got. For example, take $a=-0.8$ then the graphs for $Y$ and the related standard deviation are represented in Figure 3, where the dot/solid line represents $Y=-0.8X+X^2$, and the plus line represents its standard deviation.

3. **Case 3**: $Y=AX+X^2$

Now, we want to find the standard deviation of the output of the nonlinear module in the IHACRES model provided that the rainfall and the temperature are assumed to have continuous distribution.

Again, consider $u_k = c[\gamma\,\phi_{k-1}\,P_k + P_k^2]$ where $\phi_k = P_k + \gamma\,\phi_{k-1}$, for some constant $\gamma$. Now, we assume that all value of $\phi_{k-1}$ and $P_k$ form a continuous distribution. Let $x=P_k$ where $X$ is the related random variable, and let $a=\gamma\,\phi_{k-1}$ where $A$ is the related random variable. Then writing $y=u_k/c$, we have a new random variable $Y=AX+X^2$. Given $X\sim U(\bar{x}-0.5,\,\bar{x}+0.5)$ and $A\sim U(\bar{a}-0.5,\,\bar{a}+0.5)$, we cannot have the cumulative density function (cdf) for $Y$, since

$$F_Y(y)=P(Y\leq y)$$
$$=\int_{\bar{x}-0.5}^{\bar{x}+0.5} P(AX+X^2\leq y\,|\,X=x)\cdot f_X\,dx$$
$$=\int_{\bar{x}-0.5}^{\bar{x}+0.5} P(A\leq \frac{y}{x}-x)\,dx$$
$$=\int_{\bar{x}-0.5}^{\bar{x}+0.5} F_A(\frac{y}{x}-x)\,dx$$
$$=\int_{\bar{x}-0.5}^{\bar{x}+0.5} (\frac{y}{x}-x)\,dx$$
$$=(y\ln x-\tfrac{1}{2}x^2)\Big|_{\bar{x}-0.5}^{\bar{x}+0.5} \qquad (32)$$

is undefined for any interval which includes $x=0$. Therefore, we cannot use the change-of-variable technique.

Even though we do not have the cdf, we are still able to calculate the standard deviation as follows. Assuming $A$ and $X$ are two independent random variables and using the properties of expectation as a linear operator, we have

$$\bar{y} = E(A) \cdot E(X) + E(X^2), \quad (33)$$

$$\sigma_y^2 = E(Y^2) - \bar{y}^2, \quad (34)$$

where

$$E(Y^2) = E(A^2) \cdot E(X^2) + 2E(A) \cdot E(X^3) + E(X^4). \quad (35)$$

Each involved expectation formula in equation (35) is given in Table 2. They can be found by using the integration $E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$ since the random variables $A$ and $X$ are continuous.

Table 2: Several properties of expectation of uniform distribution with width 1

| |
|---|
| $E(A) = \bar{a}$ |
| $E(A^2) = \frac{1}{12} + \bar{a}^2$ |
| $E(X) = \bar{x}$ |
| $E(X^2) = \frac{1}{12} + \bar{x}^2$ |
| $E(X^3) = \frac{1}{4}(\bar{x} + 0.5)^4 - \frac{1}{4}(\bar{x} - 0.5)^4$ |
| $E(X^4) = \frac{1}{5}(\bar{x} + 0.5)^5 - \frac{1}{5}(\bar{x} - 0.5)^5$ |

Having those properties, we can simulate the objective function and its uncertainty. The graph for $Y = AX + X^2$, where $-1 < \bar{x}, \bar{a} < 1$ is given in Figure 4. The graph for the standard deviation of $Y = AX + X^2$ is given in Figure 5.

We see that the graph of the standard deviation is flatter than that of the objective function, which is the same phenomenon as in Case 1 and 2. This is correct obviously.

This analytical approach has been compared to a Monte Carlo estimation with the same distribution. We take 1000 random numbers uniformly distributed with width 1 for $A$ and $X$ with various means for each random variable. The results of the Monte Carlo estimations are very close to the analytical approach that we have done, and they differ in the order of $10^{-3}$.

**Conclusion**

We conclude this paper with the following remarks. The values of the function and its uncertainty translate, if the value of the input is changed. When the uncertainty of the soil moisture index is zero, the function and its uncertainty in the nonlinear module of the IHACRES model can be represented in a two-dimensional graph. Furthermore, when there are uncertainties in the soil moisture index and in the observed rainfall, the function and its uncertainty can be represented in a three-dimensional graph. Note that the change-of-variable can only be used when the cumulative distribution function is defined.

If the cumulative distribution function is not defined, we can still investigate the uncertainty of the value of the objective function using expectation formula and expectation identities. If the change-of-variable technique can be applied, it has more benefits. This is because when we know the cumulative distribution of the objective function, we have the opportunity to find more properties of the function. However, investigating the uncertainty using expectation identities is more efficient in terms of calculation.

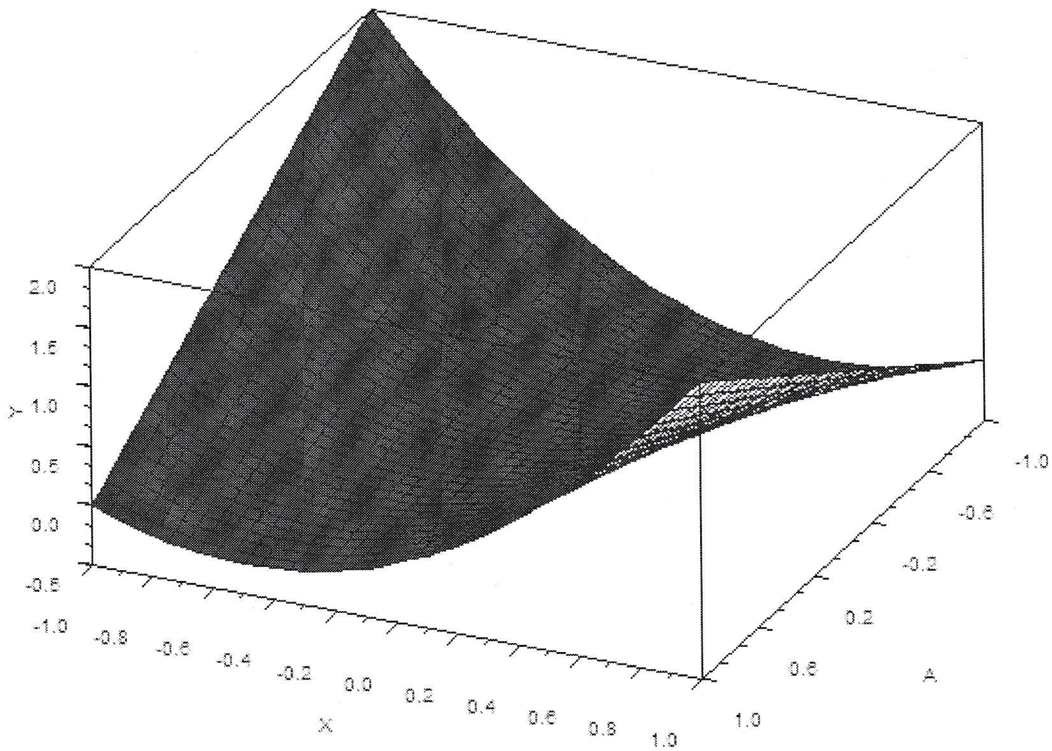


**Figure 4**. Graph for $Y = AX + X^2$.

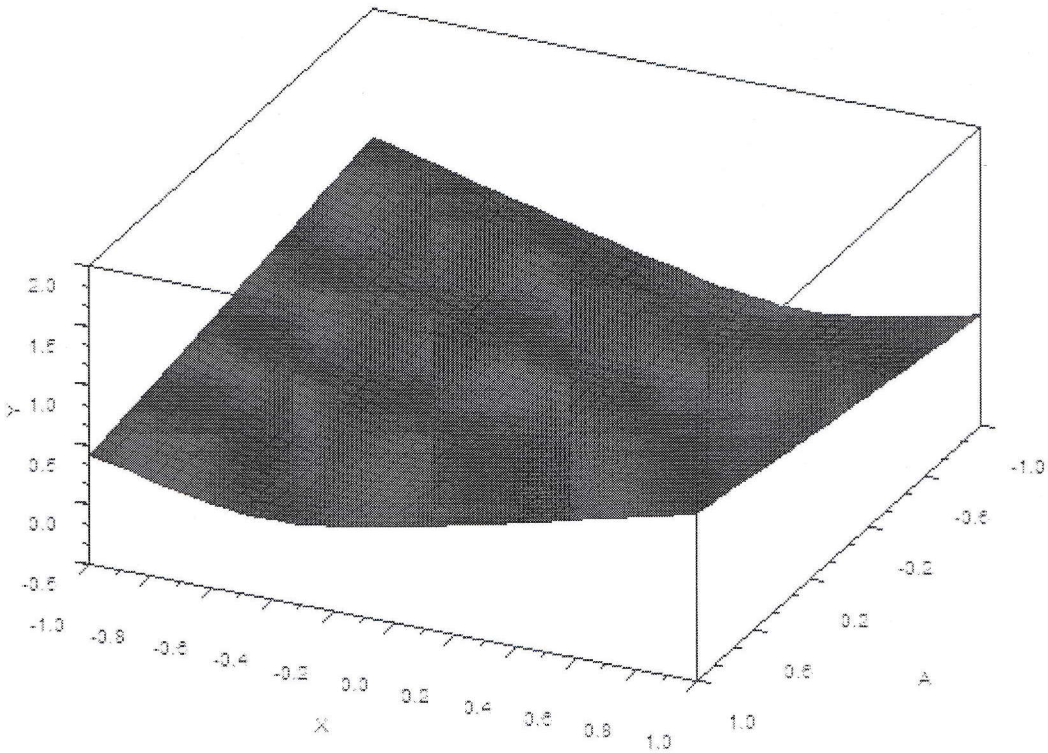


**Figure 5**. Graph for the standard deviation of $Y = AX + X^2$

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