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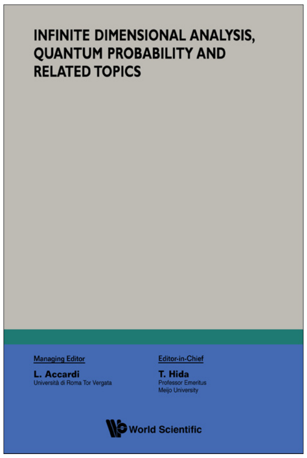

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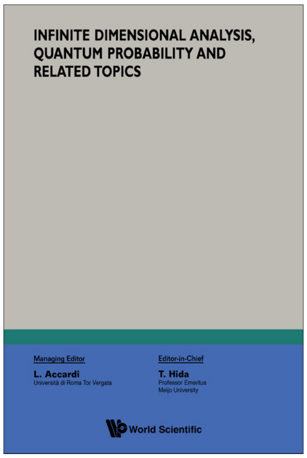

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Wolfgang Bock, Jose Luis da Silva, and Herry Pribawanto Suryawan

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Self-intersection local times for multifractional Brownian motion in higher dimensions: A white noise approach

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In this paper, we study the self-intersection local times of multifractional Brownian motion (mBm) in higher dimensions in the framework of white noise analysis. We show that when a suitable number of kernel functions of self-intersection local times of mBm are truncated then we obtain a Hida distribution. In addition, we present the expansion of the self-intersection local times in terms of Wick powers of white noises. Moreover, we obtain the convergence of the regularized truncated self-intersection local times in the sense of Hida distributions.

Keywords: Multifractional Brownian motion; self-intersection local times; white noise analysis.

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1. Introduction

Fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ is surely one of the most studied Gaussian processes in recent years. It was first introduced by Mandelbrot and van Ness as the centered Gaussian process B_H with covariance function

$$\mathbb{E}(B_H(t)B_H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad t,s > 0,$$

see Ref. 39. It would definitely be an impossible task to give all references about fBm, thus for an overview we refer to the papers[2,7,8,19,18,22,30,42] and monographs[10,41] and the references therein.

Properties like Hölder continuity of any order less than H, long-range dependence, self-similarity and stationarity of increments turn fBm an important process in models of many fields of applications. The use of fBm, however, restricts a model to a certain Hölder continuity H of the paths for all times of the process. For many applications time-dependent Hölder continuities would be desireable.

Lévy Véhel and Peltier[3] and Benassi *et al.*[6] independently introduced multifractional Brownian motion (mBm) B_h to overcome this restriction. The covariance of the centered Gaussian process B_h is given by

$$\mathbb{E}(B_h(t)B_h(s)) = \frac{C\left(\frac{1}{2}(h(t)+h(s))\right)^2}{C(h(t))C(h(s))}\left[\frac{1}{2}(t^{h(t)+h(s)} + s^{h(t)+h(s)} - |t-s|^{h(t)+h(s)})\right],$$

where $h : [0,T] \to (0,1)$ is a Hölder continuous function of exponent $\beta > 0$ and

$$C(x) := \left(\frac{2\pi}{\Gamma(2x+1)\sin(\pi x)}\right)^{1/2}$$

and Γ is the gamma function. Properties of this process such as Hölder continuity of the paths and Hausdorff dimension can be found in Refs. 6 and 16, as well as local times of mBm[4,14,15,12,40] and estimates for the local Hurst parameters.[9] In white noise analysis mBm was studied in Ref. 35 together with a corresponding stochastic calculus.

Self-intersection local time is an intensively studied object for about 80 years, see e.g., Ref. 37. Heuristically, the self-intersection local times serve to count the self-crossings of the trajectory of a stochastic process. In an informal way the self-intersection local time can be expressed by

$$I(T) := \int_0^T \int_0^T \delta(Y(t) - Y(s))\,\mathrm{d}s\,\mathrm{d}t,$$

where δ is the Dirac delta function and Y is a stochastic process. Indeed, the random variable $I(T)$ is intended to sum up the contributions from each pair of "times" $s,t \in [0,T]$ for which the process Y is at the same point. For Gaussian processes the self-intersection local times are defined as rigorous objects, see for the case of Brownian motion e.g., Refs. 1, 5, 17, 20, 25, 31, 36, 38, 47, 48, 49–51 and e.g., 4, 12, 18, 24, 28, 40 and 46 for fBm and mBm. One framework which serves to give a mathematical sound meaning to $I(T)$ above in the Gaussian setting is white noise analysis, see e.g., Refs. 26, 33 and 43, which is our choice in this work. The main results of this paper may be summarized as

(1) The truncated self-intersection local time of mBm is a Hida distribution, see Theorem 4.1.
(2) Derivation of the kernels functions of the truncated self-intersection local times of mBm, see Theorem 4.2.
(3) Proof of the convergence of the regularized truncated self-intersection local times of mBm to the truncated self-intersection local times of mBm, see Theorem 4.3.

The rest of the paper is organized as follows. In Sec. 2, we provide some background of white noise analysis. In Sec. 3, we recall the definition and some properties of mBm. Finally, Sec. 4 contains the main results and their proofs of this work.

2. Gaussian White Noise Analysis

In this section, we review some of the standard concepts and theorems of white noise analysis used throughout this work, and refer to Refs. 26, 27, 32 and 33 and references therein for a detailed presentation.

We start with the basic Gel'fand triple

$$S_d \subset L^2_d \subset S'_d,$$

where $S_d := S(\mathbb{R}, \mathbb{R}^d)$, $d \in \mathbb{N}$, is the space of vector valued Schwartz test functions, S'_d its topological dual and the central Hilbert space $L^2_d := L^2(\mathbb{R}, \mathbb{R}^d)$ of square integrable vector valued functions with norm

$$|f|_0^2 = \sum_{i=1}^{d} \int_{\mathbb{R}} f_i^2(x)\,\mathrm{d}x, \quad f \in L^2_d.$$

The space S_d is a nuclear and can be represented as a projective limit of a decreasing chain of Hilbert spaces $(H_p)_{p\in\mathbb{N}}$, see e.g., Refs. 23 and 45, that is

$$S_d = \bigcap_{p\in\mathbb{N}} H_p.$$

Hence, S_d is a countably Hilbert space in the sense of Gel'fand and Vilenkin.[23] We denote the corresponding norm on H_p by $|\cdot|_p$, with the convention $H_0 = L^2_d$. Let H_{-p} be the dual space of H_p and let $\langle\cdot,\cdot\rangle$ denote the dual pairing on $H_{-p} \times H_p$. H_p is continuously embedded into L^2_d. By identifying L^2_d with its dual via the Riesz isomorphism, we obtain the chain $H_p \subset L^2_d \subset H_{-p}$. Note that $S'_d = \bigcup_{p\in\mathbb{N}} H_{-p}$, that is S'_d is the inductive limit of the increasing chain of Hilbert spaces $(H_{-p})_{p\in\mathbb{N}}$, see e.g., Ref. 23. We denote the dual pairing of S'_d and S_d also by $\langle\cdot,\cdot\rangle$.

Let $\mathcal{B}$ be the σ-algebra generated by cylinder sets on S'_d. By Minlos' theorem there is a unique probability measure μ_d on $(S'_d, \mathcal{B})$ with the characteristic function given by

$$\int_{S'_d} e^{i\langle \boldsymbol{w}, \boldsymbol{\varphi}\rangle} \mathrm{d}\mu_d(\boldsymbol{w}) = \exp\left(-\frac{1}{2}|\boldsymbol{\varphi}|_0^2\right), \quad \boldsymbol{\varphi} \in S_d.$$

Hence, we have defined the white noise measure space $(S'_d, \mathcal{B}, \mu_d)$. The complex Hilbert space $L^2(\mu_d) := L^2(S'_d, \mathcal{B}, \mu_d)$ is canonically isomorphic to the Fock space of symmetric square integrable functions

$$L^2(\mu_d) \simeq \left(\bigoplus_{k=0}^{\infty} \mathrm{Sym} L^2(\mathbb{R}^k, k!\mathrm{d}^k x)\right)^{\otimes d}, \tag{2.1}$$

which implies the Wiener–Itô–Segal chaos decomposition for any element F in $L^2(\mu_d)$

$$F(\boldsymbol{w}) = \sum_{\boldsymbol{n}\in\mathbb{N}_0^d} \langle : \boldsymbol{w}^{\otimes \boldsymbol{n}} :, F_{\boldsymbol{n}}\rangle$$

with the kernel function $F_{\boldsymbol{n}}$ in the Fock space, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We introduce the following notation for simplicity:

$$\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d, \quad n = n_1 + \cdots + n_d, \quad \boldsymbol{n}! = n_1! \cdots n_d!$$

and for any $\boldsymbol{w} = (w_1, \ldots, w_d) \in S'_d$

$$: \boldsymbol{w}^{\otimes \boldsymbol{n}} : := : w_1^{\otimes n_1} : \otimes \cdots \otimes : w_d^{\otimes n_d} :,$$

where $: w^{\otimes n} :$ denotes the nth Wick tensor power of the element $w \in S'_1$, see e.g., Ref. 26. For any $F \in L^2(\mu_d)$, the isomorphism (2.1) yields

$$\|F\|^2_{L^2(\mu_d)} := \sum_{\boldsymbol{n}\in\mathbb{N}_0^d} \boldsymbol{n}! |F_{\boldsymbol{n}}|_0^2,$$

where the symbol $|\cdot|_0$ is also preserved for the norms on $L^2(\mathbb{R}, \mathbb{R}^d)_{\mathbb{C}}^{\hat{\otimes}\boldsymbol{n}}$, for simplicity. By the standard construction with the space of square-integrable functions with respect to μ_d as central space, we obtain the Gel'fand triple of Hida test functions and Hida distributions.

$$(S_d) \subset L^2(\mu_d) \subset (S_d)'.$$

In the following, we denote the dual pairing between elements of $(S_d)'$ and (S_d) by $\langle\langle \cdot, \cdot \rangle\rangle$. For $F \in L^2(\mu_d)$ and $\varphi \in (S_d)$, with kernel functions $F_{\boldsymbol{n}}$ and $\varphi_{\boldsymbol{n}}$, respectively, the dual pairing yields

$$\langle\langle F, \varphi\rangle\rangle = \sum_{\boldsymbol{n}\in\mathbb{N}_0^d} \boldsymbol{n}! \langle F_{\boldsymbol{n}}, \varphi_{\boldsymbol{n}}\rangle.$$

This relation extends the chaos expansion to $\Phi \in (S_d)'$ with distribution valued kernels $\Phi_{\boldsymbol{n}}$ such that

$$\langle\langle \Phi, \varphi \rangle\rangle = \sum_{\boldsymbol{n}\in\mathbb{N}_0^d} \boldsymbol{n}! \langle \Phi_{\boldsymbol{n}}, \varphi_{\boldsymbol{n}} \rangle,$$

for every generalized test function $\varphi \in (S_d)$ with kernels $\varphi_{\boldsymbol{n}}$.

Instead of reviewing the detailed construction of these spaces we give a characterization in terms of the S-transform.

Definition 2.1. Let $\boldsymbol{\varphi} \in S_d$ be given. We define the Wick exponential by

$$: \exp(\langle ., \boldsymbol{\varphi} \rangle) : := \sum_{\boldsymbol{n}\in\mathbb{N}_0^d} \frac{1}{\boldsymbol{n}!} \langle : .^{\otimes \boldsymbol{n}} :, \boldsymbol{\varphi}^{\otimes \boldsymbol{n}} \rangle \in (S_d)$$

and the S-transform of $\Phi \in (S_d)'$ by

$$S\Phi(\boldsymbol{\varphi}) := \langle\!\langle \Phi, : \exp(\langle ., \boldsymbol{\varphi} \rangle) : \rangle\!\rangle.$$

Definition 2.2. (U-functional) A function $F : S_d \to \mathbb{C}$ is called a U-functional whenever

(1) for every $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2 \in S_d$ the mapping $\mathbb{R} \ni \lambda \mapsto F(\lambda \boldsymbol{\varphi}_1 + \boldsymbol{\varphi}_2)$ has an entire extension to $z \in \mathbb{C}$,
(2) there are constants $K_1, K_2 > 0$ such that

$$|F(z\boldsymbol{\varphi})| \leq K_1 \exp(K_2 |z|^2 \|\boldsymbol{\varphi}\|^2), \quad \forall\, z \in \mathbb{C}, \boldsymbol{\varphi} \in S_d$$

for some continuous norm $\| \cdot \|$ on S_d.

We are now ready to state the aforementioned characterization result.

Theorem 2.1. (cf. Refs. 32 and 44) *The S-transform defines a bijection between the space $(S_d)'$ and the space of U-functionals. In other words, $\Phi \in (S_d)'$ if and only if $S\Phi : S_d \to \mathbb{C}$ is a U-functional.*

Based on Theorem 2.1 a deeper analysis of the space $(S_d)'$ can be done. The following corollaries deal with the convergence of sequences and the Bochner integration of families of generalized functions in $(S_d)'$ (for more details and proofs see e.g., Refs. 26, 32 and 44).

Corollary 2.1. *Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence in $(S_d)'$ such that*

(1) *for all $\boldsymbol{\varphi} \in S_d$, $(S\Phi_n(\boldsymbol{\varphi}))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$,*
(2) *there are $K_1, K_2 > 0$ such that for some continuous norm $\| \cdot \|$ on S_d one has*

$$|S\Phi_n(z\boldsymbol{\varphi})| \leq K_1 \exp(K_2 |z|^2 \|\boldsymbol{\varphi}\|^2), \quad \boldsymbol{\varphi} \in S_d,\ n \in \mathbb{N},\ \ z \in \mathbb{C}.$$

Then $(\Phi_n)_{n\in\mathbb{N}}$ converges strongly in $(S_d)'$ to a unique Hida distribution.

Corollary 2.2. *Let $(\Omega, \mathcal{F}, m)$ be a measure space and $\lambda \mapsto \Phi_\lambda$ be a mapping from Ω to $(S_d)'$. We assume that the S-transform of Φ_λ fulfills the following two properties:*

(1) *The mapping $\lambda \mapsto S\Phi_\lambda(\boldsymbol{\varphi})$ is measurable for every $\boldsymbol{\varphi} \in S_d$,*
(2) *The $S\Phi_\lambda$ obeys the estimate*

$$|S\Phi_\lambda(z\boldsymbol{\varphi})| \leq C_1(\lambda)\exp(C_2(\lambda)|z|^2\|\boldsymbol{\varphi}\|^2), \quad z \in \mathbb{C}, \boldsymbol{\varphi} \in S_d,$$

for some continuous norm $\|\cdot\|$ on S_d and for $C_1 \in L^1(\Omega, m)$, $C_2 \in L^\infty(\Omega, m)$.

Then,

$$\int_\Omega \Phi_\lambda \, \mathrm{d}m(\lambda) \in (S_d)'$$

and

$$S\left(\int_\Omega \Phi_\lambda \, \mathrm{d}m(\lambda)\right)(\boldsymbol{\varphi}) = \int_\Omega (S\Phi_\lambda)(\boldsymbol{\varphi}) \, \mathrm{d}m(\lambda), \quad \boldsymbol{\varphi} \in S_d.$$

At the end of this section, we introduce the notion of truncated kernels, defined via their Wiener–Itô–Segal chaos decomposition.

Definition 2.3. For $\Phi \in (S_d)'$ with kernels $\boldsymbol{\Phi_n}, \boldsymbol{n} \in \mathbb{N}_0^d$ and $k \in \mathbb{N}_0$ we define truncated Hida distribution by

$$\Phi^{(k)} = \sum_{\boldsymbol{n} \in \mathbb{N}_0^d : n \geq k} \langle :\cdot^{\otimes \boldsymbol{n}}:, \boldsymbol{\Phi_n}\rangle.$$

Obviously one has $\Phi^{(k)} \in (S_d)'$.

3. Multifractional Brownian Motion

mBm in dimension 1, was introduced by Peltier and Véhel[3] and Benassi *et al.*[6] as a generalization of fBm having a time-dependent Hurst parameter. We follow here the definition in Ref. 3.

Definition 3.1. Let $h : [0, T] \to [\alpha, \beta] \subset (0, 1)$ be a Hölder continuous function with exponent $\gamma > 0$, i.e. there exist $K > 0$, such that

$$|h(t) - h(s)| < K|t - s|^\gamma, \quad \text{for all } s, t \in [0, T].$$

A zero mean Gaussian process with covariance function

$$R_h(t, s) = \frac{C\left(\frac{1}{2}(h(t) + h(s))\right)^2}{C(h(t))C(h(s))}\left[\frac{1}{2}(t^{h(t)+h(s)} + s^{h(t)+h(s)} - |t - s|^{h(t)+h(s)})\right], \tag{3.1}$$

where the normalization constant is given by

$$C(x) := \left(\frac{2\pi}{\Gamma(2x + 1)\sin(\pi x)}\right)^{1/2}$$

and Γ is the gamma function is called mBm in dimension 1.

Remark 3.1. In fact the class of functions for h can be extended a little bit further than in Definition 3.1, see e.g., Ref. 35. It can be shown, see again e.g., Ref. 35 that R_h is indeed a covariance function.

Throughout this paper, we use the following notations. By $\hat{u}$ we denote the Fourier transform of u and let $L^1_{\mathrm{loc}}(\mathbb{R})$ be the set of measurable functions which are locally integrable in $\mathbb{R}$. Each $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ gives rise to an element in S_1', denoted by T_f, that is for any $\varphi \in S_1$, $\langle T_f, \varphi\rangle = \int_{\mathbb{R}} f(x)\varphi(x)\,\mathrm{d}x$.

In order to obtain a realization of mBm in the framework of white noise analysis we introduce the following operator, see Ref. 35. For any $H \in (0,1)$ we define the operator M_H, which is specified in the Fourier domain by

$$\widehat{(M_H u)}(y) = \frac{\sqrt{2\pi}}{C(H)}|y|^{1/2-H}\hat{u}(y), \quad y \in \mathbb{R}\backslash\{0\}.$$

The operator M_H is well defined in the space

$$L^2_H(\mathbb{R}) := \{u \in S_1' : \hat{u} = T_f;\ f \in L^1_{\mathrm{loc}}(\mathbb{R}) \text{ and } \|u\|_H < \infty\},$$

where the norm

$$\|u\|^2_H := \frac{1}{C(H)^2}\int_{\mathbb{R}} |x|^{1-2H}|\hat{u}(x)|^2\,\mathrm{d}x$$

is associated to the inner product on the Hilbert space $L^2_H(\mathbb{R})$ given by

$$(u,v)_H := \frac{1}{C(H)^2}\int_{\mathbb{R}} |x|^{1-2H}\hat{u}(x)\overline{\hat{v}}(x)\,\mathrm{d}x.$$

Another possible representation for the operator M_H is as follows Eq. (2.4) of Ref. 35:

$$(M_H\varphi)(x) = \gamma(H)\langle|\cdot|^{H-3/2}, \varphi(x+\cdot)\rangle =: \gamma(H)\langle\Theta_H, \varphi(x+\cdot)\rangle, \quad \varphi \in S_1(\mathbb{R}), \tag{3.2}$$

where

$$\gamma(H) := \frac{\sqrt{\Gamma(2H+1)\sin(\pi H)}}{2\Gamma(H-1/2)\cos(\pi(H-1/2)/2)}.$$

Note that Θ_H is a generalized function from S_1', that is for any $\psi \in S_1$ we have $\langle\Theta_H, \psi\rangle = \langle|y|^{H-3/2}, \psi(y)\rangle$. The operator M_H, $H \in (0,1)$ establishes an isometry between the Hilbert spaces $L^2_H(\mathbb{R})$ and $L^2(\mathbb{R})$ (see Proposition 2.10 of Ref. 35) which is bijective, see Theorem 2.14 of Ref. 35.

Next, we define the operator M_H for a measurable functional parameter $h : [0,T] \to (0,1)$. Let $\mathcal{E}(\mathbb{R})$ denote the space of simple functions of $\mathbb{R}$, that is the set of all finite linear combinations of functions $\mathbb{1}_{[a,b]}(\cdot)$ with $a, b \in \mathbb{R}$. Recall that R_h denotes the covariance function of a mBm with functional parameter h, see (3.1). We define in $\mathcal{E}(\mathbb{R})$ the bilinear form

$$(\mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]})_h := R_h(t,s)$$

and by linearity to general elements in $\mathcal{E}(\mathbb{R})$.

For all $h : [0,T] \longrightarrow [\alpha,\beta] \subset (0,1)$ as in Definition 3.1, the bilinear form $(\cdot,\cdot)_h$ is an inner product, see Proposition 3.1 of Ref. 35.

Example 3.1. The following are examples of functions h satisfying the assumptions in Definition 3.1:

(1) $h(t) = 0.5 + 0.4\sin(5\pi t)$ for any $t \in [0,T]$.
(2) $h(t) = 0.1 + (\frac{t}{2T})^{\beta}$, $\beta \in (0,1)$ and $t \in [0,T]$.

Note that the class can have time-varying Hölder continuity of the paths. Here, both functions take values in a compact subset of $(0,1)$. In fact every differentiable function is fulfilling the assumptions in Definition 3.1.

Definition 3.2. For h as Definition 3.1, we define the $L^2(\mu_1)$ random variable $B_h(t)$ by

$$B_h(t) := \langle \cdot, \eta_t\rangle := \langle \cdot, M_h 1\!\!1_{[0,t]}\rangle,$$

where the linear map M_h is defined by

$$M_h : \mathcal{E}(\mathbb{R}) \to L^2(\mathbb{R}), \quad 1\!\!1_{[0,t]} \mapsto \eta_t = M_h 1\!\!1_{[0,t]} := M_{h(t)} 1\!\!1_{[0,t]} := M_H 1\!\!1_{[0,t]}|_{H=h(t)}.$$

Remark 3.2. (1) One can show that the process $(\omega,t) \mapsto B_h(\omega,t)$ is a 1-dimensional mBm, see Ref. 35.

(2) There is a continuous modification of the process $B_h(t)$ by Kolmogorov's continuity theorem and we use the same notation for this continuous modification.
(3) The completion of $\mathcal{E}(\mathbb{R})$ with respect to $(\cdot,\cdot)_h$ is a Hilbert space denoted by $L^2_h(\mathbb{R})$. Moreover, the operator M_h is an isometry between $(\mathcal{E}(\mathbb{R}),(\cdot,\cdot)_h)$ and $(L^2(\mathbb{R}),(\cdot,\cdot))$, which can be extended to an isometry between $L^2_h(\mathbb{R})$ and $L^2(\mathbb{R})$.

The next proposition shows certain properties of 1-dimensional mBm.

Proposition 3.1. *The process $B_h(t)$, $t \geq 0$ has the following properties*:

(1) *It is a zero mean Gaussian process and the characteristic function of $B_h(t)$ is given by*

$$\begin{aligned}\mathbb{E}(e^{i\lambda B_h(t)}) &= \int_{S_1'(\mathbb{R})} e^{i\lambda\langle w, M_h 1\!\!1_{[0,t]}\rangle}\,\mathrm{d}\mu_1(w)\\ &= \exp\left(-\frac{\lambda^2}{2}|M_h 1\!\!1_{[0,t]}|^2_{L^2(\mathbb{R})}\right)\\ &= \exp\left(-\frac{\lambda^2}{2}t^{2h(t)}\right).\end{aligned}$$

(2) *Its increments are in general neither independent nor stationary. In particular, we have*

$$\mathbb{E}((B_h(t)-B_h(s))^2)=\int_{\mathbb{R}}(\eta_t(x)-\eta_s(x))^2\,\mathrm{d}x=s^{2h(s)}+t^{2h(t)}-R_h(t,s),$$

where the covariance $R_h(t,s)$ is defined in (3.1). If $s=0$ the above equality gives the variance of $B_h(t)$, that is $\mathbb{E}(B_h^2(t))=t^{2h(t)}$.

(3) *There exists $\varepsilon>0$ such that for all $m\in\mathbb{N}$ there exists $M_m>0$ such that*

$$\mathbb{E}((B_h(t)-B_h(s))^m)\geq M_m|t-s|^{m(h(t)\wedge h(s))},$$

for all $t,s>0$ such that $|t-s|<\varepsilon$. Here, for any $a,b\in\mathbb{R}$, $a\wedge b:=\min\{a,b\}$.

Proof. The two first properties are easy to check using the definition. The last property has been proved in Ref. 14 using the Hölder continuity of h. □

Now, we are ready to define the d-dimensional mBm.

Definition 3.3. (d-dimensional mBm) Let h be as in Definition 3.1. A d-dimensional mBm with functional parameter h is defined by

$$\boldsymbol{B}_h(t)=(B_{h,1}(t),\ldots,B_{h,d}(t)),\quad t\geq 0,$$

where $B_{h,i}(t)$, $i=1,\ldots,d$, are d independent 1-dimensional mBms.

Properties of $\boldsymbol{B}_h(t)$:

(1) The expectation is zero $\mathbb{E}(\boldsymbol{B}_h(t))=0$.

(2) The characteristic function of $\boldsymbol{B}_h(t)$ is given, for any $\boldsymbol{x}\in\mathbb{R}^d$, by

$$\int_{S'_d}e^{i(\boldsymbol{x},\boldsymbol{B}_h(\boldsymbol{w},t))_{\mathbb{R}^d}}\,d\mu_d(\boldsymbol{w})=\exp\left(-\frac{1}{2}t^{2h(t)}|\boldsymbol{x}|^2_{\mathbb{R}^d}\right).$$

(3) The covariance matrix of $\boldsymbol{B}_h(t)$ is

$$\mathrm{cov}(\boldsymbol{B}_h(t))=(\delta_{ij}t^{2h(t)})_{i,j=1}^d.$$

4. Self-Intersection Local Times of mBm

Heuristically, the self-intersection local times serve to count the self-crossings of the trajectory of a stochastic process. Informally the self-intersection local time is given by the time integral over a composition of Dirac delta function with the increment of the process. This Donsker's delta function is a well-studied object in white noise analysis, see e.g., Refs. 26, 33 and 43. The concept of local times for mBm was studied by many authors, see e.g., Refs. 4, 15 and 40 and references therein. In the framework of white noise analysis the local times of mBm was investigated in Ref. 12.

In this section, we show that the truncated self-intersection local time of mBm is a Hida distribution. Moreover, we determine its kernel functions, see Theorems 4.1

and 4.2. In addition, we prove that the truncated self-intersection local times of mBm may be approximated by a regularized truncated self-intersection local times in the sense of Hida distributions.

Proposition 4.1. *For $t, s > 0$ with $s < t$ the Bochner integral*

$$\delta(\boldsymbol{B}_h(t) - \boldsymbol{B}_h(s)) := \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{i(\boldsymbol{\lambda}, \boldsymbol{B}_h(t) - \boldsymbol{B}_h(s))_{\mathbb{R}^d}} \, \mathrm{d}\boldsymbol{\lambda} \tag{4.1}$$

is a Hida distribution and for any $\boldsymbol{\varphi} \in S_d$ its S-transform given by

$$S\delta(\boldsymbol{B}_h(t) - \boldsymbol{B}_h(s))(\boldsymbol{\varphi}) = \left(\frac{1}{\sqrt{2\pi}|\eta_{t,s}|_{L^2}}\right)^d \exp\left(-\frac{1}{2|\eta_{t,s}|_{L^2}^2} \sum_{j=1}^d (\varphi_j, \eta_{t,s})_{L^2}^2\right), \tag{4.2}$$

where $\eta_{t,s}(x) := \eta_t(x) - \eta_s(x)$, $|\eta_{t,s}|_{L^2}^2 := \int_{\mathbb{R}} |\eta_{t,s}(x)|^2 \, \mathrm{d}x$ and $(\varphi_j, \eta_{t,s})_{L^2}^2 := \int_{\mathbb{R}} \varphi_j(x)\eta_{t,s}(x) \, \mathrm{d}x$, $j = 1, \ldots, d$.

Proof. First, we compute the S-transform of the integrand in (4.1) for any $\boldsymbol{\varphi} \in S_d$:

$$Se^{i(\boldsymbol{\lambda}, \boldsymbol{B}_h(t) - \boldsymbol{B}_h(s))_{\mathbb{R}^d}}(\boldsymbol{\varphi}) = \exp\left(-\frac{1}{2}|\boldsymbol{\lambda}|_{\mathbb{R}^d}^2 |\eta_{t,s}|_{L^2}^2 + i \sum_{j=1}^d \lambda_j \int_{\mathbb{R}} \varphi_j(x)\eta_{t,s}(x) \, \mathrm{d}x\right). \tag{4.3}$$

It is clear that the S-transform is $\boldsymbol{\lambda}$-measurable for any $\boldsymbol{\varphi} \in S_d$. On the other hand, for any $z \in \mathbb{C}$ and all $\boldsymbol{\varphi} \in S_d$ we obtain

$$\begin{aligned}
|Se^{i(\boldsymbol{\lambda}, \boldsymbol{B}_h(t) - \boldsymbol{B}_h(s))_{\mathbb{R}^d}}(z\boldsymbol{\varphi})| &\leq \exp\left(-\frac{1}{2}|\boldsymbol{\lambda}|_{\mathbb{R}^d}^2 |\eta_{t,s}|_{L^2}^2\right) \exp(|z| \, |\boldsymbol{\lambda}|_{\mathbb{R}^d} |\eta_{t,s}|_{L^2} |\boldsymbol{\varphi}|_0) \\
&\leq \exp\left(-\frac{1}{2}|\boldsymbol{\lambda}|_{\mathbb{R}^d}^2 |\eta_{t,s}|_{L^2}^2\right) \exp\left(\frac{1}{4}|\boldsymbol{\lambda}|_{\mathbb{R}^d}^2 |\eta_{t,s}|_{L^2}^2 + |z|^2 |\boldsymbol{\varphi}|_0^2\right) \\
&\leq \exp\left(-\frac{1}{4}|\boldsymbol{\lambda}|_{\mathbb{R}^d}^2 |\eta_{t,s}|_{L^2}^2\right) \exp(|z|^2 |\boldsymbol{\varphi}|_0^2).
\end{aligned}$$

Thus, we have the following bound:

$$|Se^{i(\boldsymbol{\lambda}, \boldsymbol{B}_h(t) - \boldsymbol{B}_h(s))_{\mathbb{R}^d}}(z\boldsymbol{\varphi})| \leq \exp\left(-\frac{1}{4}|\boldsymbol{\lambda}|_{\mathbb{R}^d}^2 |\eta_{t,s}|_{L^2}^2\right) \exp(|z|^2 |\boldsymbol{\varphi}|_0^2),$$

where as a function of $\boldsymbol{\lambda}$, the first factor is integrable on $\mathbb{R}^d$ and the second factor is constant. The result (4.2) follows from (4.3) by integrating with respect to $\boldsymbol{\lambda}$ and Corollary 2.2. □

In the following theorem, we characterize the truncated self-intersection local times of mBm as a Hida distribution. Therefore, we use the notation $\delta^{(N)}$ for the truncated Donsker delta function $\delta^{(N)}(B(t)-B(s))$, $N \in \mathbb{N}_0$ which is the Hida distribution defined for any $\varphi \in S_d$ by its S-transform as

$$S\delta^{(N)}(B(t)-B(s))(\varphi) = \left(\frac{1}{\sqrt{2\pi}\,|\eta_{t,s}|_{L^2}}\right)^d \exp_N\left(-\frac{1}{2|\eta_{t,s}|_{L^2}^2}\sum_{j=1}^d (\varphi_j,\eta_{t,s})_{L^2}^2\right),$$

with $\exp_N(x) := \sum_{n=N}^{\infty} \frac{1}{n!}x^n$ for $x \in \mathbb{C}$.

In order to proceed with the first main result of this paper we need the following assumption.

Assumption 1. There exist $K > 0$ and $\kappa > 0$ such that operator M_h satisfy for all $t, s \in [0,T]$

$$\int_{\mathbb{R}} |(M_h \mathbb{1}_{[0,t]} - M_h \mathbb{1}_{[0,s]})(x)|\,\mathrm{d}x \le K|t-s|^{\kappa}.$$

Remark 4.1. (1) Note that for h constant or strictly monotone, fulfilling Definition 3.1, Assumption 1 is fulfilled.

(2) Assumption 1 can be viewed upon as a control of the increments of mBm.

From now on, C is a constant whose value is unimportant and may change from line to line.

Theorem 4.1. *Let h be as in Definition 3.1 and Assumption 1 hold and $d \in \mathbb{N}$, $N \in \mathbb{N}_0$ be such that $d\|h\|_\infty - 2N(\kappa - \|h\|_\infty) < 1$ ($\|h\|_\infty$ is the supremum norm of h on $[0,T]$), then the Bochner integral*

$$I_h^{(N)}([0,T]) := I_h^{(N)}(T) := \int_0^T \int_0^T \delta^{(N)}(\boldsymbol{B}_h(t) - \boldsymbol{B}_h(s))\,\mathrm{d}t\,\mathrm{d}s$$

is a Hida distribution.

Proof. The proof uses again Corollary 2.2 with respect to the Lebesgue measure in $[0,T]^2$. Let $[a,b] \subset [0,T]$ be such that $|b-a| < \varepsilon$. According to Proposition 3.1-3 and (4.2), we have

$$\begin{aligned} &S\delta^{(N)}(\boldsymbol{B}_h(t) - \boldsymbol{B}_h(s))(\boldsymbol{\varphi}) \\ &\quad = \left(\frac{1}{\sqrt{2\pi}\,|\eta_{t,s}|_{L^2}}\right)^d \exp_N\left(-\frac{1}{2|\eta_{t,s}|_{L^2}^2}\sum_{j=1}^d (\varphi_j,\eta_{t,s})_{L^2}^2\right), \end{aligned} \tag{4.4}$$

which is measurable in (t,s) for every $\boldsymbol{\varphi} \in S_d$. Using the supremum norm of $\boldsymbol{\varphi} \in S_d$ and Proposition 3.1–3 we obtain the following bound for any $z \in \mathbb{C}$:

$$\begin{aligned} &|S\delta^{(N)}(\boldsymbol{B}_h(t) - \boldsymbol{B}_h(s))(z\boldsymbol{\varphi})| \\ &\quad \le \left(\frac{C}{\sqrt{2\pi}\,|t-s|^{\|h\|_\infty}}\right)^d \exp_N\left(\frac{C}{|t-s|^{2\|h\|_\infty}}|z|^2\|\boldsymbol{\varphi}\|^2|t-s|^{2\kappa}\right) \\ &\quad \le \left(\frac{1}{\sqrt{2\pi}\,|t-s|^{\|h\|_\infty}}\right)^d \exp_N\left(C|t-s|^{2\kappa-2\|h\|_\infty}|z|^2\|\boldsymbol{\varphi}\|^2\right). \end{aligned} \tag{4.5}$$

The estimation

$$\exp_N(C|t-s|^{\kappa-2\|h\|_\infty}|z|^2\|\boldsymbol{\varphi}\|^2) \le |t-s|^{2N(\kappa-\|h\|_\infty)}\exp(C|z|^2\|\boldsymbol{\varphi}\|^2) \tag{4.6}$$

allows us to obtain the bound

$$\begin{aligned}&|S\delta^{(N)}(\boldsymbol{B}_h(t)-\boldsymbol{B}_h(s))(z\boldsymbol{\varphi})|\\ &\quad\le \left(\frac{1}{\sqrt{2\pi}}\right)^d |t-s|^{-d\|h\|_\infty+2N(\kappa-\|h\|_\infty)}\exp(C|z|^2\|\boldsymbol{\varphi}\|^2),\end{aligned}$$

which is integrable in $t,s\in[a,b]$ if and only if $d\|h\|_\infty - 2N(\kappa-\|h\|_\infty)<1$. The proof follows from Corollary 2.2. Finally, as $[0,T]$ is a finite interval, we may obtain the truncated self-intersection local time $L_h^{(N)}(T)$ on $[0,T]$ by a patch procedure. That is, if $\bigcup_{i=1}^n[a_i,b_i]$ is a partition of $[0,T]$ such that $|a_i-b_i|<\varepsilon$, then we define

$$I_h^{(N)}(T)=\sum_{i=1}^n I_h^{(N)}([a_i,b_i]).$$

□

Theorem 4.2. *Let h be as in Definition 3.1 and Assumption 1 hold and $d\in\mathbb{N}$, $N\in\mathbb{N}_0$ be such that $d\|h\|_\infty-2N(\kappa-\|h\|_\infty)<1$, then the kernel functions of $I_h^{(N)}(T)$ are given by*

$$\begin{aligned}F_{h,2\boldsymbol{n}}(u_1,\ldots,u_{2n}) &= \left(\frac{1}{\sqrt{2\pi}}\right)^d\int_0^T\int_0^T\left(-\frac{1}{2}\right)^n\frac{1}{|\eta_{t,s}|_{L^2}^{d+2n}}\\ &\quad\cdot\sum_{\substack{n_1,\ldots,n_d\in\mathbb{N}_0\\ n_1+\cdots+n_d=n}}\frac{1}{\boldsymbol{n}!}\prod_{j=1}^{2n}\eta_{t,s}(u_j)\,\mathrm{d}t\,\mathrm{d}s\end{aligned} \tag{4.7}$$

for each $\boldsymbol{n}\in\mathbb{N}^d$ such that $n\ge N$. All the other kernels $F_{h,\boldsymbol{n}}$ are zero.

Proof. The kernels of $I_h^{(N)}(T)$ are obtained by computing the S-transform. Therefore, we use Corollary 2.2 and integrate (4.4) over $[0,T]^2$. For any $\boldsymbol{\varphi}\in S_d$, we have

$$\begin{aligned}SI_h^{(N)}(T)(\boldsymbol{\varphi}) &= \left(\frac{1}{\sqrt{2\pi}}\right)^d\int_0^T\int_0^T\frac{1}{|\eta_{t,s}|_{L^2}^d}\sum_{n=N}^\infty\frac{(-1)^n}{2^n|\eta_{t,s}|_{L^2}^{2n}}\\ &\quad\cdot\sum_{\substack{n_1,\ldots,n_d\in\mathbb{N}_0\\ n_1+\cdots+n_d=n}}\frac{1}{n_1!\cdots n_d!}\prod_{j=1}^d(\varphi_j,\eta_{t,s})_{L^2}^{2n_j}\,\mathrm{d}t\,\mathrm{d}s\\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^d\int_0^T\int_0^T\sum_{n=N}^\infty\left(-\frac{1}{2}\right)^n\frac{1}{|\eta_{t,s}|_{L^2}^{d+2n}}\\ &\quad\cdot\sum_{\substack{n_1,\ldots,n_d\in\mathbb{N}_0\\ n_1+\cdots+n_d=n}}\frac{1}{\boldsymbol{n}!}\prod_{j=1}^d(\varphi_j,\eta_{t,s})_{L^2}^{2n_j}\,\mathrm{d}t\,\mathrm{d}s.\end{aligned}$$

Comparing it with the general form of the chaos expansion

$$I_h^{(N)}(T) = \sum_{\boldsymbol{n}\in\mathbb{N}_0^d} \langle : \boldsymbol{w}^{\otimes \boldsymbol{n}} :, F_{h,\boldsymbol{n}}\rangle$$

we obtain $F_{h,\boldsymbol{n}}$ as in (4.7). This completes the proof. □

Remark 4.2. The result of Theorem 4.1 states that for h constant

(1) and $d = 1$ all self-intersection local times are well-defined. In fact, for $h = H \in (0,1)$ we have $\kappa = 2$ and the integrability condition $d\|h\|_\infty - 2N(\kappa - \|h\|_\infty) < 1$ implies that $N > (1-H)(H-2)^{-1}/2$. As the right-hand side of this inequality is negative, then we can always choose $N = 0$,

(2) for $d = 2$ and $h = H = 1/2$, we have

$$\int_{\mathbb{R}} |(M_h \mathbb{1}_{[0,t]} - M_h \mathbb{1}_{[0,s]})(x)|\, \mathrm{d}x = \int_{\mathbb{R}} \mathbb{1}_{[s,t]}(x)\, \mathrm{d}x = |t-s|.$$

Hence in this case $\kappa = 1$. Thus

$$d\|h\|_\infty - 2N(\kappa - \|h\|_\infty) = 1 - N.$$

In this case, we have that $N > 1$ is necessary to fulfill the assumptions of Theorem 4.2. Hence, we have to subtract the expectation in order to obtain a well defined Hida distribution.

This is the well-known fact, that the self-intersection local time of a planar Brownian motion has to be centered, see e.g., Refs. 1, 24 and 47.

The considerations in Remark 4.2 motivate the study of a regularized version of $I_h(T)$, namely

$$I_{h,\gamma}(T) := \int_0^T \int_0^T \delta_\gamma(\boldsymbol{B}_h(t) - \boldsymbol{B}_h(s))\, \mathrm{d}t\, \mathrm{d}s, \quad \gamma > 0,$$

where

$$\delta_\gamma(\boldsymbol{B}_h(t) - \boldsymbol{B}_h(s)) := \left(\frac{1}{\sqrt{2\pi\gamma}}\right)^d \exp\left(-\frac{1}{2\gamma}|\boldsymbol{B}_h(t) - \boldsymbol{B}_h(s)|_{\mathbb{R}^d}^2\right).$$

Theorem 4.3. *Let $\gamma > 0$ be given and h be as in Definition 3.1 and Assumption 1 hold.*

(1) *The functional $I_{h,\gamma}(T)$ is a Hida distribution with kernel functions given by*

$$F_{h,\gamma,2\boldsymbol{n}}(u_1,\ldots,u_{2n}) = \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_0^T \int_0^T \left(-\frac{1}{2}\right)^n \frac{1}{(\gamma + |\eta_{t,s}|_{L^2}^2)^{n+d/2}} \cdot \sum_{\substack{n_1,\ldots,n_d\in\mathbb{N}_0 \\ n_1+\cdots+n_d=n}} \frac{1}{\boldsymbol{n}!} \prod_{j=1}^{2n} \eta_{t,s}(u_j)\, \mathrm{d}t\, \mathrm{d}s \tag{4.8}$$

for each $\boldsymbol{n} = (n_1,\ldots,n_d) \in \mathbb{N}_0^d$ and $F_{h,\gamma,\boldsymbol{n}} = 0$ if at least one of the n_j is an odd number.

(2) *For* $\gamma \to 0$ *the truncated functional* $I_{h,\gamma}^{(N)}(T)$ *converges strongly in* $(S_d)'$ *to the truncated self-intersection local times* $I_h^{(N)}(T)$.

Proof. (1) First, we compute the S-transform of $\delta_\gamma(\boldsymbol{B}_h(t)-\boldsymbol{B}_h(s))$. For any $\boldsymbol{\varphi} \in S_d$, we obtain

$$S\delta_\gamma(\boldsymbol{B}_h(t)-\boldsymbol{B}_h(s))(\boldsymbol{\varphi}) = \left(\frac{1}{\sqrt{2\pi(\gamma+|\eta_{t,s}|_{L^2}^2)}}\right)^d \exp\left(-\frac{1}{2(\gamma+|\eta_{t,s}|_{L^2}^2)}\right.$$
$$\left.\cdot\sum_{j=1}^d\left(\int_{\mathbb{R}}\varphi_j(x)\eta_{t,s}(x)\,\mathrm{d}x\right)^2\right),$$

which is measurable in t,s. Thus, for any $z\in\mathbb{C}$ and $\boldsymbol{\varphi}\in S_d$, by the Cauchy–Schwarz inequality we obtain the following bound:

$$|S\delta_\gamma(\boldsymbol{B}_h(t)-\boldsymbol{B}_h(s))(z\boldsymbol{\varphi})| \le \left(\frac{1}{\sqrt{2\pi(\gamma+|\eta_{t,s}|_{L^2}^2)}}\right)^d \exp\left(\frac{|\eta_{t,s}|_{L^2}^2}{2(\gamma+|\eta_{t,s}|_{L^2}^2)}|z|^2|\boldsymbol{\varphi}|_0^2\right)$$
$$\le \left(\frac{1}{\sqrt{2\pi\gamma}}\right)^d \exp(|z|^2|\boldsymbol{\varphi}|_0^2). \tag{4.9}$$

The first part of the assertion 1 follows from Corollary 2.2 since the integration is finite in a bounded set. Hence, $I_{h,\gamma}(T)\in(S_d)'$. To find the kernels of $I_{h,\gamma}(T)$, let $\boldsymbol{\varphi}\in S_d$ be given, then we have

$$SI_{h,\gamma}(T)(\boldsymbol{\varphi}) = \int_0^T\int_0^T S\delta_\gamma(\boldsymbol{B}_h(t)-\boldsymbol{B}_h(s))(\boldsymbol{\varphi})\,\mathrm{d}t\,\mathrm{d}s$$
$$= \left(\frac{1}{\sqrt{2\pi}}\right)^d\int_0^T\int_0^T\frac{1}{(\gamma+|\eta_{t,s}|_{L^2}^2)^{d/2}}\sum_{n=0}^\infty\frac{(-1)^n}{2^n(\gamma+|\eta_{t,s}|_{L^2}^2)^n}$$
$$\cdot\sum_{\substack{n_1,\dots,n_d\in\mathbb{N}_0\\ n_1+\cdots+n_d=n}}\frac{1}{n_1!\dots n_d!}\prod_{j=1}^d(\varphi_j,\eta_{t,s})_{L^2}^{2n_j}\,\mathrm{d}t\,\mathrm{d}s$$
$$= \left(\frac{1}{\sqrt{2\pi}}\right)^d\int_0^T\int_0^T\sum_{n=0}^\infty\left(-\frac{1}{2}\right)^n\frac{1}{(\gamma+|\eta_{t,s}|_{L^2}^2)^{n+d/2}}$$
$$\cdot\sum_{\substack{n_1,\dots,n_d\in\mathbb{N}_0\\ n_1+\cdots+n_d=n}}\frac{1}{\boldsymbol{n}!}\prod_{j=1}^d(\varphi_j,\eta_{t,s})_{L^2}^{2n_j}\,\mathrm{d}t\,\mathrm{d}s.$$

Comparing the latter expression with the kernels $F_{h,\gamma,\boldsymbol{n}}$ from the chaos expansion of $I_{h,\gamma}(T)$

$$I_{h,\gamma}(T) = \sum_{\boldsymbol{n}\in\mathbb{N}_0^d}\langle :\boldsymbol{w}^{\otimes\boldsymbol{n}}:, F_{h,\gamma,\boldsymbol{n}}\rangle,$$

we conclude that whenever one of the n_j in $\boldsymbol{n} = (n_1, \dots, n_d)$ is odd we have $F_{h,\gamma,\boldsymbol{n}} = 0$, otherwise they are given by the expression (4.8).

(2) To check the convergence we shall use Corollary 2.1 and the fact that

$$SI_{h,\gamma}^{(N)}(T)(\boldsymbol{\varphi}) = \int_0^T \int_0^T S\delta_\gamma^{(N)}(\boldsymbol{B}_h(t) - \boldsymbol{B}_h(s))(\boldsymbol{\varphi})\,\mathrm{d}t\,\mathrm{d}s.$$

Thus, for all $z \in \mathbb{C}$ and all $\boldsymbol{\varphi} \in S_d$ we estimate $SI_{h,\gamma}^{(N)}(T)(z\boldsymbol{\varphi})$ using the bound (4.9) by

$$\begin{aligned}|SI_{h,\gamma}^{(N)}(T)(z\boldsymbol{\varphi})| &\le \int_0^T \int_0^T \left|S\delta_\gamma^{(N)}(\boldsymbol{B}_h(t)) - \boldsymbol{B}_h(s)(z\boldsymbol{\varphi})\right| \mathrm{d}t\,\mathrm{d}s \\ &\le \left(\frac{1}{\sqrt{2\pi\gamma}}\right)^d \int_0^T \int_0^T \exp(|z|^2|\boldsymbol{\varphi}|_0^2)\mathrm{d}t\,\mathrm{d}s \\ &\le \left(\frac{1}{\sqrt{2\pi\gamma}}\right)^d T^2 \exp(|z|^2|\boldsymbol{\varphi}|_0^2),\end{aligned}$$

which shows the uniform boundedness condition. Moreover, using similar calculations as in Theorem 4.1, for any $t, s \in [a,b] \subset [0,T]$ such that $|a-b| < \varepsilon$, yields

$$\begin{aligned}|(SI_{h,\gamma}^{(N)}(T))(\boldsymbol{\varphi})| &\le \left(\frac{1}{\sqrt{2\pi(\gamma + |\eta_{t,s}|_{L^2}^2)}}\right)^d \exp_N\left(\frac{C}{2(\gamma + |\eta_{t,s}|_{L^2}^2)}|t-s|^{2\kappa}\|\boldsymbol{\varphi}\|^2\right) \\ &\le \left(\frac{C}{\sqrt{2\pi|t-s|^{\|h\|_\infty}}}\right)^d \exp_N\left(C|t-s|^{2(\kappa - \|h\|_\infty)}\|\boldsymbol{\varphi}\|^2\right) \\ &\le C(2\pi)^{-d/2}|t-s|^{2N(\kappa - \|h\|_\infty) - d\|h\|_\infty} \exp(C\|\boldsymbol{\varphi}\|^2),\end{aligned}$$

which is integrable in $t, s \in [a,b]$ if and only if $d\|h\|_\infty - 2N(\kappa - \|h\|_\infty) < 1$. Finally, an application of Lebesgue's dominated convergence theorem and Corollary 2.2 gives the statement 2 of the theorem. In order to obtain the result on the interval $[0,T]$ we use the same argument as in the proof of Theorem 4.1. □

5. Conclusion and Outlook

In this paper, we identified Donsker's delta function, truncated and regularized self-intersection local times of mBm satisfying Assumption 1 as Hida distributions. Moreover, we showed that the regularized truncated self-intersection local times converge strongly to truncated self-intersection local times in $(S_d)'$.

An application of the self-intersection local times is on the Edwards type model for locally non-homogeneous polymers. The time-dependent Hölder continuity models a different local persistence along the contour of a chain polymer. The self-intersection local times can be used to penalize self-crossings. Such a model was proposed by Edwards[21] for the Brownian case and mathematically treated by Varadhan.[47] Recently, the model was extended to the case of fBm, see e.g., Refs. 24 and 29.

A stochastic quantization of the corresponding Edwards measure via Dirichlet form techniques was used to determine a stochastic partial differential equation for the simulation of such polymer paths.[13,11] The extension of these techniques would surely be an important contribution to study the scaling behavior of heterogeneous polymers. This work can be seen as a starting point of this scientific journey.

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References

1. S. Albeverio, Y.-Z. Hu, M. Röckner and X. Y. Zhou, Stochastic quantization of the two-dimensional polymer measure, *Appl. Math. Optim.* **40** (1999) 341–354.
2. E. Alòs and D. Nualart, Stochastic integration with respect to the fractional Brownian motion, *Stoch. Stoch. Rep.* **75** (2003) 129–152.
3. A. Ayache and J. Lévy Véhel, Generalized multifractional Brownian motion: Definition and preliminary results, in *Fractals*: *Theory and Applications in Engineering* (Springer, London, 1999), pp. 17–32.
4. A. Ayache, N.-R. Shieh and Y. Xiao, Multiparameter multifractional Brownian motion: Local nondeterminism and joint continuity of the local times,' *Ann. Inst. Henri Poincaré Probab. Stat.* **47** (2011) 1029–1054.
5. R. F. Bass and D. Khoshnevisan, Intersection local times and Tanaka formulas, *Ann. Inst. H. Poincaré Probab. Statist.* **29** (1993) 419–451.
6. A. Benassi, S. Jaffard and D. Roux, Elliptic Gaussian random processes, *Rev. Mat. Iberoamericana* **13** (1997) 19–90.
7. C. Bender, An Itô formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter, *Stochastic Process. Appl.* **104** (2003) 81–106.
8. C. Bender, An S-transform approach to integration with respect to a fractional Brownian motion, *Bernoulli* **9** (2003) 955–983.
9. P. R. Bertrand, M. Fhima and A. Guillin, Local estimation of the Hurst index of multifractional Brownian motion by increment ratio statistic method, *ESAIM Probab. Stat.* **17** (2013) 307–327.
10. F. Biagini, Y. Hu, B. Øksendal and T. Zhang, *Stochastic Calculus for Fractional Brownian Motion and Applications*, Probability and its Applications (New York) (Springer-Verlag London, Ltd., London, 2008).
11. W. Bock, J. L. da Silva and T. Fattler, Analysis of stochastic quantization for the fractional Edwards measure, *Rep. Math. Phys.* **82** (2018) 187–202.
12. W. Bock, J. L. da Silva and H. P. Suryawan, Local times for multifractional Brownian motion in higher dimensions: A white noise approach, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **19** (2016) 1650026.
13. W. Bock, T. Fattler and L. Streit, Stochastic quantization for the fractional Edwards measure, *Acta Appl. Math.* **151** (2017) 81–88.
14. B. Boufoussi, M. Dozzi and R. Guerbaz, On the local time of multifractional Brownian motion, *Stochastics* **78** (2006) 33–49.

15. B. Boufoussi, M. Dozzi and R. Guerbaz, Sample path properties of the local time of multifractional Brownian motion, *Bernoulli* **13** (2007) 849–867.
16. B. Boufoussi, M. Dozzi and R. Guerbaz, Path properties of a class of locally asymptotically self similar processes, *Electron. J. Probab.* **13** (2008) 898–921.
17. M. de Faria, C. Drumond and L. Streit, The renormalization of self-intersection local times. I. The chaos expansion, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **3** (2000) 223–236.
18. C. Drumond, M. Oliveira and J. L. da Silva, Intersection local times of fractional Brownian motions with h(0,1) as generalized white noise functionals, *AIP Conf. Proc.* **1021** (2008) 34.
19. T. E. Duncan, Y. Hu and B. Pasik-Duncan, Stochastic calculus for fractional Brownian motion. I. Theory, *SIAM J. Control Optim.* **38** (2000) 582–612.
20. A. Dvoretzky, P. Erdös and S. Kakutani, Double points of paths of Brownian motion in n-space, *Acta Sci. Math.* (*Szeged*) **12** (1950) 75–81.
21. S. F. Edwards, The statistical mechanics of polymers with excluded volume, *Proc. Phys. Soc.* **85** (1965) 613–624.
22. R. J. Elliott and J. van der Hoek, A general fractional white noise theory and applications to finance, *Math. Finance* **13** (2003) 301–330.
23. I. M. Gel'fand, M. I. Graev and N. Y. Vilenkin, *Generalized Functions*, Vol. 5 (AMS Chelsea Publishing, Providence, RI, 2016). Integral geometry and representation theory, Translated from the 1962 Russian original by Eugene Saletan, Reprint of the 1966 English translation.
24. M. Grothaus, M. J. Oliveira, J. L. da Silva and L. Streit, Self-avoiding fractional Brownian motion — the Edwards model, *J. Stat. Phys.* **145** (2011) 1513–1523.
25. S. W. He, W. Q. Yang, R. Q. Yao and J. G. Wang, Local times of self-intersection for multidimensional Brownian motion, *Nagoya Math. J.* **138** (1995) 51–64.
26. T. Hida, H.-H. Kuo, J. Potthoff and L. Streit, *White Noise*: *An Infinite Dimensional Calculas*, Mathematics and its Applications, Vol 253 (Kluwer Academic Publishers Group, Dordrecht, 1993).
27. T. Hida and L. Streit, *Let Us Use White Noise* (World Scientific, Singapore, 2017).
28. Y. Hu and D. Nualart, Renormalized self-intersection local time for fractional Brownian motion, *Ann. Probab.* **33** (2005) 948–983.
29. Y. Hu and D. Nualart, Regularity of renormalized self-intersection local time for fractional Brownian motion, *Commun. Inf. Syst.* **7** (2007) 21–30.
30. Y. Hu and B. Øksendal, Fractional white noise calculus and applications to finance, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **6** (2003) 1–32.
31. P. Imkeller, V. Pérez-Abreu and J. Vives, Chaos expansions of double intersection local time of Brownian motion in $\mathbf{R}^d$ and renormalization, *Stochastic Process. Appl.* **56** (1995) 1–34.
32. Y. G. Kondratiev, P. Leukert, J. Potthoff, L. Streit and W. Westerkamp, Generalized functionals in Gaussian spaces: The characterization theorem revisited, *J. Funct. Anal.* **141** (1996) 301–318.
33. H.-H. Kuo, *White Noise Distribution Theory*, Probability and Stochastics Series, (CRC Press, Boca Raton, FL, 1996).
34. A. Lascheck, P. Leukert, L. Streit and W. Westerkamp, More about Donsker's delta function, *Soochow J. Math.* **20** (1994) 401–418.
35. J. Lebovits and J. Lévy Véhel, White noise-based stochastic calculus with respect to multifractional Brownian motion, *Stochastics* **86** (2014) 87–124.

36. J.-F. Le Gall, Sur le temps local d'intersection du mouvement brownien plan et la méthode de renormalisation de Varadhan, in *Séminaire de Probabilités, XIX*, 1983/84, Lecture Notes in Mathematics (Springer, Berlin, 1985), pp. 314–331.
37. P. Lévy, Le mouvement brownien plan, *Amer. J. Math.* **62** (1940) 487–550.
38. T. J. Lyons, The critical dimension at which quasi-every Brownian path is self-avoiding, *Adv. Appl. Probab.* **18** (1986) 87–99.
39. B. B. Mandelbrot and J. W. Van Ness, Fractional Brownian motions, fractional noises and applications, *SIAM Rev.* **10** (1968) 422–437.
40. M. M. Meerschaert and H.-P. Scheffler, Triangular array limits for continuous time random walks, *Stochastic Process. Appl.* **118** (2008) 1606–1633.
41. Y. S. Mishura, *Stochastic Calculus for Fractional Brownian Motion and Related Processes*, Lecture Notes in Mathematics, Vol. 1929 (Springer-Verlag, Berlin, 2008).
42. D. Nualart, Stochastic calculus with respect to fractional Brownian motion, *Ann. Fac. Sci. Toulouse Math.* (*6*) **15** (2006) 63–78.
43. N. Obata, *White Noise Calculus and Fock Space*, Lecture Notes in Mathematics, Vol. 1577 (Springer-Verlag, Berlin, 1994).
44. J. Potthoff and L. Streit, A characterization of Hida distributions, *J. Funct. Anal.* **101** (1991) 212–229.
45. M. Reed and B. Simon, *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness* (Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975).
46. J. Rosen, The intersection local time of fractional Brownian motion in the plane, *J. Multivariate Anal.* **23** (1987) 37–46.
47. S. R. S. Varadhan, Appendix to *Euclidean Quantum Field Theory*, by K. Symanszik, in *Local Quantum Theory*, ed. R. Jost (Academic Press, Inc., New York, 1969).
48. H. Watanabe, The local time of self-intersections of Brownian motions as generalized Brownian functionals, *Lett. Math. Phys.* **23** (1991) 1–9.
49. M. J. Westwater, On Edwards' model for long polymer chains, *Comm. Math. Phys.* **72** (1980) 131–174.
50. R. L. Wolpert, Wiener path intersections and local time, *J. Funct. Anal.* **30** (1978) 329–340.
51. M. Yor, Renormalisation et convergence en loi pour les temps locaux d'intersection du mouvement brownien dans $\mathbf{R}^3$, in *Séminaire de Probabilités*, *XIX*, 1983/84, Lecture Notes in Mathematics, Vol. 1123 (Springer, Berlin, 1985), pp. 350–365.