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Page count: 18
Word count: 7,395
Character count: 33,127
Submission date: 15-Apr-2023 09:07AM (UTC+0700)
Submission ID: 2064947618



Turnitin IDAQP2020

by Suryawan Herry

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Word count: 7395

Character count: 33127

Infinite Dimensional Analysis, Quantum Probability
and Related Topics
Vol. 23, No. 1 (2020) 2050007 (18 pages)
© World Scientific Publishing Company
DOI: 10.1142/S0219025720500071



Self-intersection local times for multifractional
Brownian motion in higher dimensions: A
white noise approach

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Received 9 September 2019

Revised 21 October 2019

Accepted 28 January 2020

Published 28 May 2020

Communicated by Yuri Kondratiev

In this paper, we study the self-intersection local times of multifractional Brownian motion (mBm) in higher dimensions in the framework of white noise analysis. We show that when a suitable number of kernel functions of self-intersection local times of mBm are truncated then we obtain a Hida distribution. In addition, we present the expansion of the self-intersection local times in terms of Wick powers of white noises. Moreover, we obtain the convergence of the regularized truncated self-intersection local times in the sense of Hida distributions.

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Keywords: Multifractional Brownian motion; self-intersection local times; white noise analysis.

Mathematics Subject Classification 2020: 60H40, 60G22, 60G18

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1. Introduction

Fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$ is surely one of the most studied Gaussian processes in recent years. It was first introduced by Mandelbrot and van Ness as the centered Gaussian process B_H with covariance function

$$\mathbb{E}(B_H(t)B_H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \quad t, s > 0,$$

see Ref. [39]. It would definitely be an impossible task to give all references about fBm, thus for an overview we refer to the papers [2, 7, 8, 19, 18, 22, 30, 42] and monographs [10, 41] and the references therein.

Properties like Hölder continuity of any order less than H , long-range dependence, self-similarity and stationarity of increments turn fBm an important process in models of many fields of applications. The use of fBm, however, restricts a model to a certain Hölder continuity H of the paths for all times of the process. For many applications time-dependent Hölder [62] continuities would be desirable.

Lévy Véhel and Peltier [3] and Benassi *et al.* [6] independently introduced multifractional Brownian motion (mBm) B_h to overcome this restriction [4]. The covariance of the centered Gaussian process B_h is given by

$$\mathbb{E}(B_h(t)B_h(s)) = \frac{C(\frac{1}{2}(h(t) + h(s)))^2}{C(h(t))C(h(s))} \left[\frac{1}{2}(t^{h(t)+h(s)} + s^{h(t)+h(s)} - |t - s|^{h(t)+h(s)}) \right],$$

where $h : [0, T] \rightarrow (0, 1)$ is a Hölder continuous function of exponent $\beta > 0$ and

$$C(x) := \left(\frac{2\pi}{\Gamma(2x+1) \sin(\pi x)} \right)^{1/2}$$

and Γ is the gamma function. Properties of this process such as Hölder continuity of the paths and Hausdorff dimension can be found in Refs. [6] and [16] as well as local times of mBm [4, 14, 15, 12, 40] and estimates for the local Hurst parameters [9]. In white noise analysis mBm was studied in Ref. [35] together with a corresponding stochastic calculus.

Self-intersection local time is an intensively studied object for about 80 years, see e.g., Ref. [37]. Heuristically, the self-intersection local times serve to count the self-crossings of the trajectory of a stochastic process. In an informal way the self-intersection local time can be expressed by

$$I(T) := \int_0^T \int_0^T \delta(Y(t) - Y(s)) \, ds \, dt,$$

where δ is the Dirac delta function and Y is a stochastic process. Indeed, the random variable $I(T)$ is intended to sum up the contributions from each pair of “times” $s, t \in [0, T]$ for which the process Y is at the same point. For Gaussian processes the self-intersection local times are defined as rigorous objects, see for the case of Brownian motion e.g., Refs. [1, 5, 17, 20, 25, 31, 36, 38, 47, 48, 49, 51] and e.g., [4, 12, 18, 24, 28, 40] and [46] for fBm and mBm. One framework which serves to

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give a mathematical sound meaning to $I(T)$ above in the Gaussian setting is white noise analysis, see e.g., Refs. [26, 33] and [43] which is our choice in this work. The main results of this paper may be summarized as

- (1) The truncated self-intersection local time of mBm is a Hida distribution, see Theorem 4.1
- (2) Derivation of the kernels functions of the truncated self-intersection local times of mBm, see Theorem 4.2
- (3) Proof of the convergence of the regularized truncated self-intersection local times of mBm to the truncated self-intersection local times of mBm, see Theorem 4.3

The rest of the paper is organized as follows. In Sec. 2 we provide some background of white noise analysis. In Sec. 3 we recall the definition and some properties of mBm. Finally, Sec. 4 contains the main results and their proofs of this work.

2. Gaussian White Noise Analysis

In this section, we review some of the standard concepts and theorems of white noise analysis used throughout this work, and refer to Refs. [26, 27, 32] and [33] and references therein for a detailed presentation.

We start with the basic Gel'fand triple

$$S_d \subset L_d^2 \subset S'_d,$$

where $S_d := S(\mathbb{R}, \mathbb{R}^d)$, $d \in \mathbb{N}$, is the space of vector valued Schwartz test functions, S'_d its topological dual and the central Hilbert space $L_d^2 := L^2(\mathbb{R}, \mathbb{R}^d)$ of square integrable vector valued functions with norm

$$|f|_0^2 = \sum_{i=1}^d \int_{\mathbb{R}} f_i^2(x) dx, \quad f \in L_d^2.$$

The space S_d is a nuclear and can be represented as projective limit of a decreasing chain of Hilbert spaces $(H_p)_{p \in \mathbb{N}}$, see e.g., Refs. [23] and [45] that is

$$S_d = \bigcap_{p \in \mathbb{N}} H_p.$$

Hence, S_d is a countably Hilbert space in the sense of Gel'fand and Vilenkin. [23] We denote the corresponding norm on H_p by $|\cdot|_p$, with the convention $H_0 = L_d^2$. Let H_{-p} be the dual space of H_p and let $\langle \cdot, \cdot \rangle$ denote the dual pairing on $H_{-p} \times H_p$. H_p is continuously embedded into L_d^2 . By identifying L_d^2 with its dual via the Riesz isomorphism, we obtain the chain $H_p \subset L_d^2 \subset H_{-p}$. Note that $S'_d = \bigcup_{p \in \mathbb{N}} H_{-p}$, that is S'_d is the inductive limit of the increasing chain of Hilbert spaces $(H_{-p})_{p \in \mathbb{N}}$, see e.g., Ref. [23]. We denote the dual pairing of S'_d and S_d also by $\langle \cdot, \cdot \rangle$.

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Let \mathcal{B} be the σ -algebra generated by cylinder sets on S'_d . By Minlos' theorem there is a unique probability measure μ_d on (S'_d, \mathcal{B}) with the characteristic function given by

$$\int_{S'_d} e^{i\langle \mathbf{w}, \boldsymbol{\varphi} \rangle} d\mu_d(\mathbf{w}) = \exp\left(-\frac{1}{2}|\boldsymbol{\varphi}|_0^2\right), \quad \boldsymbol{\varphi} \in S_d.$$

Hence, we have defined the white noise measure space $(S'_d, \mathcal{B}, \mu_d)$. The complex Hilbert space $L^2(\mu_d) := L^2(S'_d, \mathcal{B}, \mu_d)$ is canonically isomorphic to the Fock space of symmetric square integrable functions

$$L^2(\mu_d) \simeq \left(\bigoplus_{k=0}^{\infty} \text{Sym} L^2(\mathbb{R}^k, k! dx) \right)^{\otimes d}, \quad (2.1)$$

which implies the Wiener-Itô-Segal chaos decomposition for any element F in $L^2(\mu_d)$

$$F(\mathbf{w}) = \sum_{\mathbf{n} \in \mathbb{N}_0^d} \langle : \mathbf{w}^{\otimes \mathbf{n}} :, F_{\mathbf{n}} \rangle$$

with the kernel function $F_{\mathbf{n}}$ in the Fock space, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We introduce the following notation for simplicity:

$$\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}_0^d, \quad n = n_1 + \dots + n_d, \quad \mathbf{n}! = n_1! \cdots n_d!$$

and for any $\mathbf{w} = (w_1, \dots, w_d) \in S'_d$

$$: \mathbf{w}^{\otimes \mathbf{n}} : := : w_1^{\otimes n_1} : \otimes \cdots \otimes : w_d^{\otimes n_d} :,$$

where $: w^{\otimes n} :$ denotes the n th Wick tensor power of the element $w \in S'_1$, see e.g., Ref. [26]. For any $F \in L^2(\mu_d)$, the isomorphism (2.1) yields

$$\|F\|_{L^2(\mu_d)}^2 := \sum_{\mathbf{n} \in \mathbb{N}_0^d} \mathbf{n}! |F_{\mathbf{n}}|_0^2,$$

where the symbol $|\cdot|_0$ is also preserved for the norms on $L^2(\mathbb{R}, \mathbb{R}^d)_{\mathbb{C}}^{\otimes \mathbf{n}}$, for simplicity. By the standard construction with the space of square-integrable functions with respect to μ_d as central space, we obtain the Gel'fand triple of Hida test functions and Hida distributions.

$$(S_d) \subset L^2(\mu_d) \subset (S_d)'.$$

In the following, we denote the dual pairing between elements of $(S_d)'$ and (S_d) by $\langle\langle \cdot, \cdot \rangle\rangle$. For $F \in L^2(\mu_d)$ and $\boldsymbol{\varphi} \in (S_d)$, with kernel functions $F_{\mathbf{n}}$ and $\varphi_{\mathbf{n}}$, respectively, the dual pairing yields

$$\langle\langle F, \boldsymbol{\varphi} \rangle\rangle = \sum_{\mathbf{n} \in \mathbb{N}_0^d} \mathbf{n}! \langle F_{\mathbf{n}}, \varphi_{\mathbf{n}} \rangle.$$

This relation extends the chaos expansion to $\Phi \in (S_d)'$ with distribution valued kernels Φ_n such that

$$\langle \langle \Phi, \varphi \rangle \rangle = \sum_{n \in \mathbb{N}_0^d} n! \langle \Phi_n, \varphi_n \rangle,$$

for every generalized test function $\varphi \in (S_d)'$ and kernels φ_n .

Instead of reviewing the detailed construction of these spaces we give a characterization in terms of the S -transform.

Definition 2.1. Let $\varphi \in S_d$ be given. We define the Wick exponential by

$$:\exp(\langle \cdot, \varphi \rangle) := \sum_{n \in \mathbb{N}_0^d} \frac{1}{n!} \langle \cdot^{\otimes n}, \varphi^{\otimes n} \rangle \in (S_d)$$

and the S -transform of $\Phi \in (S_d)'$ by

$$S\Phi(\varphi) := \langle \langle \Phi, : \exp(\langle \cdot, \varphi \rangle) : \rangle \rangle.$$

Definition 2.2. (U -functional) A function $F : S_d \rightarrow \mathbb{C}$ is called a U -functional whenever

- (1) for every $\varphi_1, \varphi_2 \in S_d$ the mapping $\mathbb{R} \ni \lambda \mapsto F(\lambda\varphi_1 + \varphi_2)$ has an entire extension to $z \in \mathbb{C}$,
- (2) there are constants $K_1, K_2 > 0$ such that

$$|F(z\varphi)| \leq K_1 \exp(K_2 |z|^2 \|\varphi\|^2), \quad \forall z \in \mathbb{C}, \varphi \in S_d$$

for some continuous norm $\|\cdot\|$ on S_d .

We are now ready to state the aforementioned characterization result.

Theorem 2.1. (cf. Refs. [32] and [44]) *The S -transform defines a bijection between the space $(S_d)'$ and the space of U -functionals. In other words, $\Phi \in (S_d)'$ if and only if $S\Phi : S_d \rightarrow \mathbb{C}$ is a U -functional.*

Based on Theorem 2.1 a deeper analysis of the space $(S_d)'$ can be done. The following corollaries deal with the convergence of sequences and the Bochner integration of families of generalized functions in $(S_d)'$ (for more details and proofs see e.g., Refs. [26] [32] and [44]).

Corollary 2.1. *Let $(\Phi_n)_{n \in \mathbb{N}}$ be a sequence in $(S_d)'$ such that*

- (1) *for all $\varphi \in S_d$, $(S\Phi_n(\varphi))_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{C} ,*
- (2) *there are $K_1, K_2 > 0$ such that for some continuous norm $\|\cdot\|$ on S_d one has*

$$|S\Phi_n(z\varphi)| \leq K_1 \exp(K_2 |z|^2 \|\varphi\|^2), \quad \varphi \in S_d, n \in \mathbb{N}, z \in \mathbb{C}.$$

Then $(\Phi_n)_{n \in \mathbb{N}}$ converges strongly in $(S_d)'$ to a unique Hida distribution.

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Corollary 2.2. Let (Ω, \mathcal{F}, m) be a measure space and $\lambda \mapsto \Phi_\lambda$ be a mapping from Ω to $(S_d)'$. We assume that the S -transform of Φ_λ fulfills the following two properties:

- (1) The mapping $\lambda \mapsto S\Phi_\lambda(\varphi)$ is measurable for every $\varphi \in S_d$,
- (2) The $S\Phi_\lambda$ obeys the estimate

$$|S\Phi_\lambda(z\varphi)| \leq C_1(\lambda) \exp(C_2(\lambda)|z|^2\|\varphi\|^2), \quad z \in \mathbb{C}, \varphi \in S_d,$$

for some continuous norm $\|\cdot\|$ on S_d and for $C_1 \in L^1(\Omega, m)$, $C_2 \in L^\infty(\Omega, m)$.

Then,

$$\int_\Omega \Phi_\lambda \, dm(\lambda) \in (S_d)'$$

and

$$S\left(\int_\Omega \Phi_\lambda \, dm(\lambda)\right)(\varphi) = \int_\Omega (S\Phi_\lambda)(\varphi) \, dm(\lambda), \quad \varphi \in S_d.$$

At the end of this section, we introduce the notion of truncated kernels, defined via their Wiener–Itô–Segal chaos decomposition.

Definition 2.3. For $\Phi \in (S_d)'$ with kernels $\Phi_n, n \in \mathbb{N}_0^d$ and $k \in \mathbb{N}_0$ we define truncated Hida distribution by

$$\Phi^{(k)} = \sum_{n \in \mathbb{N}_0^d : n \geq k} \langle \cdot^{\otimes n} \cdot, \Phi_n \rangle.$$

Obviously one has $\Phi^{(k)} \in (S_d)'$.

3. Multifractional Brownian Motion

mBm in dimension 1, was introduced by Peltier and Véhel^[3] and Benassi *et al.*^[6] as a generalization of fBm having a time-dependent Hurst parameter. We follow here the definition in Ref. ^[3]

Definition 3.1. Let $h : [0, T] \rightarrow [\alpha, \beta] \subset (0, 1)$ be a Hölder continuous function with exponent $\gamma > 0$, i.e. there exists $\epsilon > 0$, such that

$$|h(t) - h(s)| < K|t - s|^\gamma, \quad \text{for all } s, t \in [0, T].$$

A zero mean Gaussian process with covariance function

$$R_h(t, s) = \frac{C\left(\frac{1}{2}(h(t) + h(s))\right)^2}{C(h(t))C(h(s))} \left[\frac{1}{2}(t^{h(t)+h(s)} + s^{h(t)+h(s)} - |t - s|^{h(t)+h(s)}) \right], \quad (3.1)$$

where the normalization constant is given by

$$C(x) := \left(\frac{2\pi}{\Gamma(2x+1)\sin(\pi x)} \right)^{1/2}$$

and Γ is the gamma function is called mBm in dimension 1.

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Remark 3.1. In fact the class of functions for h can be extended a little bit further than in Definition 3.1; see e.g., Ref. [35]. It can be shown, see again e.g., Ref. [35] that R_h is indeed a covariance function.

Throughout this paper, we use the following notations. By \hat{u} we denote the Fourier transform of u and let $L^1_{\text{loc}}(\mathbb{R})$ be the set of measurable functions which are locally integrable in \mathbb{R} . Each $f \in L^1_{\text{loc}}(\mathbb{R})$ gives rise to an element in S'_1 , denoted by T_f , that is for any $\varphi \in S_1$, $\langle T_f, \varphi \rangle = \int_{\mathbb{R}} f(x)\varphi(x) dx$.

In order to obtain a realization of mBm in the framework of white noise analysis we introduce the following operator, see Ref. [35]. For any $H \in (0, 1)$ we define the operator M_H , which is specified in the Fourier domain by

$$(\widehat{M_H u})(y) = \frac{\sqrt{2\pi}}{C(H)} |y|^{1/2-H} \hat{u}(y), \quad y \in \mathbb{R} \setminus \{0\}.$$

The operator M_H is well defined in the space

$$L^2_H(\mathbb{R}) := \{u \in S'_1 : \hat{u} = T_f; f \in L^1_{\text{loc}}(\mathbb{R}) \text{ and } \|u\|_H < \infty\},$$

where the norm

$$\|u\|_H^2 := \frac{1}{C(H)^2} \int_{\mathbb{R}} |x|^{1-2H} |\hat{u}(x)|^2 dx$$

is associated to the inner product on the Hilbert space $L^2_H(\mathbb{R})$ given by

$$(u, v)_H := \frac{1}{C(H)^2} \int_{\mathbb{R}} |x|^{1-2H} \hat{u}(x) \bar{\hat{v}}(x) dx.$$

Another possible representation for the operator M_H is as follows Eq. (2.4) of Ref. [35]

$$(M_H \varphi)(x) = \gamma(H) \langle |\cdot|^{H-3/2}, \varphi(x + \cdot) \rangle =: \gamma(H) \langle \Theta_H, \varphi(x + \cdot) \rangle, \quad \varphi \in S_1(\mathbb{R}), \quad (3.2)$$

where

$$\gamma(H) := \frac{\sqrt{\Gamma(2H+1)} \sin(\pi H)}{2\Gamma(H-1/2) \cos(\pi(H-1/2)/2)}.$$

Note that Θ_H is a generalized function from S'_1 , that is for any $\psi \in S_1$ we have $\langle \Theta_H, \psi \rangle = \langle |y|^{H-3/2}, \psi(y) \rangle$. The operator M_H , $H \in (0, 1)$ establishes an isometry between the Hilbert spaces $L^2_H(\mathbb{R})$ and $L^2(\mathbb{R})$ (see Proposition 2.10 of Ref. [35] which is bijective, see Theorem 2.14 of Ref. [35]).

Next, we define the operator M_h for a measurable functional parameter $h : [0, T] \rightarrow (0, 1)$. Let $\mathcal{E}(\mathbb{R})$ denote the space of simple functions of \mathbb{R} , that is the set of all finite linear combinations of functions $\mathbb{1}_{[a,b]}(\cdot)$ with $a, b \in \mathbb{R}$. Recall that R_h denotes the covariance function of a mBm with functional parameter h , see (3.1). We define in $\mathcal{E}(\mathbb{R})$ the bilinear form

$$(\mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]})_h := R_h(t, s)$$

and by linearity to general elements in $\mathcal{E}(\mathbb{R})$.

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For all $h : [0, T] \rightarrow [\alpha, \beta] \subset (0, 1)$ as in Definition 3.1 the bilinear form $(\cdot, \cdot)_h$ is an inner product, see Proposition 3.1 of Ref. [35]

Example 3.1. The following are examples of functions h satisfying the assumptions in Definition 3.1

- (1) $h(t) = 0.5 + 0.4 \sin(5\pi t)$ for any $t \in [0, T]$.
- (2) $h(t) = 0.1 + (\frac{t}{2T})^\beta$, $\beta \in (0, 1)$ and $t \in [0, T]$.

Note that the class can have time-varying Hölder continuity of the paths. Here, both functions take values in a compact subset of $(0, 1)$. In fact every differentiable function is fulfilling the assumptions in Definition 3.1

Definition 3.2. For h as Definition 3.1 we define the $L^2(\mu_1)$ random variable $B_h(t)$ by

$$B_h(t) := \langle \cdot, \eta_t \rangle := \langle \cdot, M_h \mathbb{1}_{[0,t]} \rangle,$$

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where the linear map M_h is defined by

$$M_h : \mathcal{E}(\mathbb{R}) \rightarrow L^2(\mathbb{R}), \quad \mathbb{1}_{[0,t]} \mapsto \eta_t = M_h \mathbb{1}_{[0,t]} := M_{h(t)} \mathbb{1}_{[0,t]} := M_H \mathbb{1}_{[0,t]}|_{H=h(t)}.$$

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Remark 3.2. (1) One can show that the process $(\omega, t) \mapsto B_h(\omega, t)$ is a 1-dimensional mBm, see Ref. [35]

- (2) There is a continuous modification of the process $B_h(t)$ by Kolmogorov's continuity theorem and we use the same notation for this continuous modification.
- (3) The complete $\mathcal{E}(\mathbb{R})$ with respect to $(\cdot, \cdot)_h$ is a Hilbert space denoted by $L_h^2(\mathbb{R})$. Moreover, the operator M_h is an isometry between $(\mathcal{E}(\mathbb{R}), (\cdot, \cdot)_h)$ and $(L^2(\mathbb{R}), (\cdot, \cdot))$, which can be extended to an isometry between $L_h^2(\mathbb{R})$ and $L^2(\mathbb{R})$.

The next proposition shows certain properties of 1-dimensional mBm.

Proposition 3.1. The process $B_h(t)$, $t \geq 0$ has the following properties:

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- (1) It is a zero mean Gaussian process and the characteristic function of $B_h(t)$ is given by

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$$\begin{aligned} \mathbb{E}(e^{i\lambda B_h(t)}) &= \int_{S'_1(\mathbb{R})} e^{i\lambda \langle w, M_h \mathbb{1}_{[0,t]} \rangle} d\mu_1(w) \\ &= \exp\left(-\frac{\lambda^2}{2} \|M_h \mathbb{1}_{[0,t]}\|_{L^2(\mathbb{R})}^2\right) \\ &= \exp\left(-\frac{\lambda^2}{2} t^{2h(t)}\right). \end{aligned}$$

- (2) Its increments are in general neither independent nor stationary. In particular, we have

$$\mathbb{E}((B_h(t) - B_h(s))^2) = \int_{\mathbb{R}} (\eta_t(x) - \eta_s(x))^2 dx = s^{2h(s)} + t^{2h(t)} - R_h(t, s),$$

where the covariance $R_h(t, s)$ is defined in (3.1). If $s = 0$ the above equality gives the variance of $B_h(t)$, that is $\mathbb{E}(B_h^2(t)) = t^{2h(t)}$.

- (3) There exists $\varepsilon > 0$ such that for all $m \in \mathbb{N}$ there exists $M_m > 0$ such that

$$\mathbb{E}((B_h(t) - B_h(s))^m) \geq M_m |t - s|^{m(h(t) \wedge h(s))},$$

for all $t, s > 0$ such that $|t - s| < \varepsilon$. Here, for any $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$.

Proof. The two first properties are easy to check using the definition. The last property has been proved in Ref. [14] using the Hölder continuity of h . \square

Now, we are ready to define the d -dimensional mBm.

Definition 3.3. (d -dimensional mBm) Let h be as in Definition 3.1. A d -dimensional mBm with functional parameter h is defined by

$$B_h(t) = (B_{h,1}(t), \dots, B_{h,d}(t)), \quad t \geq 0,$$

where $B_{h,i}(t)$, $i = 1, \dots, d$, are d independent 1-dimensional mBms.

Properties of $B_h(t)$:

- (1) The expectation is zero $\mathbb{E}(B_h(t)) = 0$.
 (2) The characteristic function of $B_h(t)$ is given, for any $x \in \mathbb{R}^d$, by

$$\int_{S'_d} e^{i(x, B_h(w, t))_{\mathbb{R}^d}} d\mu_d(w) = \exp\left(-\frac{1}{2} t^{2h(t)} |x|_{\mathbb{R}^d}^2\right).$$

- (3) The covariance matrix of $B_h(t)$ is

$$\text{cov}(B_h(t)) = (\delta_{ij} t^{2h(t)})_{i,j=1}^d.$$

4. Self-Intersection Local Times of mBm

Heuristically, the self-intersection local times serve to count the self-crossings of the trajectory of a stochastic process. Informally the self-intersection local time is given by the time integral over a composition of a delta function with the increment of the process. This Donsker's delta function is a well-studied object in white noise analysis, see e.g., Refs. [26, 33] and [43]. The concept of local times for mBm was studied by many authors, see e.g., Refs. [4, 15] and [40] and references therein. In the framework of white noise analysis the local times of mBm was investigated in Ref. [12].

In this section, we show that the truncated self-intersection local time of mBm is a Hida distribution. Moreover, we determine its kernel functions, see Theorems 4.1

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and [4.2](#) In addition, we prove that the truncated self-intersection local times of mBm may be approximated by a regularized truncated self-intersection local times in the sense of Hida distributions.

Proposition 4.1. For $t, s > 0$ with $s < t$ the Bochner integral

$$\delta(B_h(t) - B_h(s)) := \left(\frac{1}{2\pi} \right)^d \int_{\mathbb{R}^d} e^{i(\lambda, B_h(t) - B_h(s))_{\mathbb{R}^d}} d\lambda \quad (4.1)$$

is a Hida distribution and for any $\varphi \in S_d$ its S -transform given by

$$S\delta(B_h(t) - B_h(s))(\varphi) = \left(\frac{1}{\sqrt{2\pi}|\eta_{t,s}|_{L^2}} \right)^d \exp \left(-\frac{1}{2|\eta_{t,s}|_{L^2}^2} \sum_{j=1}^d (\varphi_j, \eta_{t,s})_{L^2}^2 \right), \quad (4.2)$$

where $\eta_{t,s}(x) := \eta_t(x) - \eta_s(x)$, $|\eta_{t,s}|_{L^2}^2 := \int_{\mathbb{R}} |\eta_{t,s}(x)|^2 dx$ and $(\varphi_j, \eta_{t,s})_{L^2}^2 := \int_{\mathbb{R}} \varphi_j(x) \eta_{t,s}(x) dx$, $j = 1, \dots, d$.

Proof. First, we compute the S -transform of the integrand in [\(4.1\)](#) for any $\varphi \in S_d$:

$$Se^{i(\lambda, B_h(t) - B_h(s))_{\mathbb{R}^d}}(\varphi) = \exp \left(-\frac{1}{2} |\lambda|_{\mathbb{R}^d}^2 |\eta_{t,s}|_{L^2}^2 + i \sum_{j=1}^d \lambda_j \int_{\mathbb{R}} \varphi_j(x) \eta_{t,s}(x) dx \right). \quad (4.3)$$

It is clear that the S -transform is λ -measurable for any $\varphi \in S_d$. On the other hand, for any $z \in \mathbb{C}$ and all $\varphi \in S_d$ we obtain

$$\begin{aligned} |Se^{i(\lambda, B_h(t) - B_h(s))_{\mathbb{R}^d}}(z\varphi)| &\leq \exp \left(-\frac{1}{2} |\lambda|_{\mathbb{R}^d}^2 |\eta_{t,s}|_{L^2}^2 \right) \exp(|z| |\lambda|_{\mathbb{R}^d} |\eta_{t,s}|_{L^2} |\varphi|_0) \\ &\leq \exp \left(-\frac{1}{2} |\lambda|_{\mathbb{R}^d}^2 |\eta_{t,s}|_{L^2}^2 \right) \exp \left(\frac{1}{4} |\lambda|_{\mathbb{R}^d}^2 |\eta_{t,s}|_{L^2}^2 + |z|^2 |\varphi|_0^2 \right) \\ &\leq \exp \left(-\frac{1}{4} |\lambda|_{\mathbb{R}^d}^2 |\eta_{t,s}|_{L^2}^2 \right) \exp(|z|^2 |\varphi|_0^2). \end{aligned}$$

Thus, we have the following bound:

$$|Se^{i(\lambda, B_h(t) - B_h(s))_{\mathbb{R}^d}}(z\varphi)| \leq \exp \left(-\frac{1}{4} |\lambda|_{\mathbb{R}^d}^2 |\eta_{t,s}|_{L^2}^2 \right) \exp(|z|^2 |\varphi|_0^2),$$

where as a function of λ , the first factor is integrable on \mathbb{R}^d and the second factor is constant. The result [\(4.2\)](#) follows from [\(4.3\)](#) by integrating with respect to λ and Corollary [2.2](#) \square

In the following theorem, we characterize the truncated self-intersection local times of mBm as a Hida distribution. Therefore, we use the notation $\delta^{(N)}$ for the

truncated Donsker delta function $\delta^{(N)}(B(t) - B(s))$, $N \in \mathbb{N}_0$ which is the Hida distribution defined for any $\varphi \in S_d$ by its S -transform as

$$S\delta^{(N)}(B(t) - B(s))(\varphi) = \left(\frac{1}{\sqrt{2\pi}|\eta_{t,s}|_{L^2}} \right)^d \exp_N \left(-\frac{1}{2|\eta_{t,s}|_{L^2}^2} \sum_{j=1}^d (\varphi_j, \eta_{t,s})_{L^2}^2 \right),$$

with $\exp_N(x) := \sum_{n=N}^{\infty} \frac{1}{n!} x^n$ for $x \in \mathbb{C}$.

In order to proceed with the first main result of this paper we need the following assumption.

Assumption 1. There exist $K > 0$ and $\kappa > 0$ such that operator M_h satisfy for all $t, s \in [0, T]$

$$\int_{\mathbb{R}} |(M_h \mathbb{1}_{[0,t]} - M_h \mathbb{1}_{[0,s]})(x)| dx \leq K|t - s|^\kappa.$$

Remark 4.1. (1) Note that for h constant or strictly monotone, fulfilling Definition 3.1 Assumption 1 is fulfilled.

(2) Assumption 1 can be viewed upon as a control of the increments of mBm.

From now on, C is a constant whose value is unimportant and may change from line to line.

Theorem 4.1. Let h be as in Definition 3.1 and Assumption 1 hold and $d \in \mathbb{N}$, $N \in \mathbb{N}_0$ be such that $d\|h\|_\infty - 2N(\kappa - \|h\|_\infty) < 1$ ($\|h\|_\infty$ is the supremum norm of h on $[0, T]$), then the Bochner integral

$$I_h^{(N)}([0, T]) := I_h^{(N)}(T) := \int_0^T \int_0^T \delta^{(N)}(B_h(t) - B_h(s)) dt ds$$

is a Hida distribution.

Proof. The proof uses again Corollary 2.2 with respect to the Lebesgue measure in $[0, T]^2$. Let $[a, b] \subset [0, T]$ be such that $|b - a| < \varepsilon$. According to Proposition 3.1-3 and (4.2), we have

$$\begin{aligned} S\delta^{(N)}(B_h(t) - B_h(s))(\varphi) \\ = \left(\frac{1}{\sqrt{2\pi}|\eta_{t,s}|_{L^2}} \right)^d \exp_N \left(-\frac{1}{2|\eta_{t,s}|_{L^2}^2} \sum_{j=1}^d (\varphi_j, \eta_{t,s})_{L^2}^2 \right), \end{aligned} \quad (4.4)$$

which is measurable in (t, s) for every $\varphi \in S_d$. Using the supremum norm of $\varphi \in S_d$ and Proposition 3.1-3 we obtain the following bound for any $z \in \mathbb{C}$:

$$\begin{aligned} |S\delta^{(N)}(B_h(t) - B_h(s))(z\varphi)| \\ \leq \left(\frac{C}{\sqrt{2\pi}|t - s|^{\|h\|_\infty}} \right)^d \exp_N \left(\frac{C}{|t - s|^{2\|h\|_\infty}} |z|^2 \|\varphi\|^2 |t - s|^{2\kappa} \right) \\ \leq \left(\frac{1}{\sqrt{2\pi}|t - s|^{\|h\|_\infty}} \right)^d \exp_N \left(C|t - s|^{2\kappa - 2\|h\|_\infty} |z|^2 \|\varphi\|^2 \right). \end{aligned} \quad (4.5)$$

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The estimation

$$\exp_N(C|t-s|^{\kappa-2\|h\|_\infty}|z|^2\|\varphi\|^2) \leq |t-s|^{2N(\kappa-\|h\|_\infty)} \exp(C|z|^2\|\varphi\|^2) \quad (4.6)$$

allows us to obtain the bound

$$\begin{aligned} & |S\delta^{(N)}(\mathbf{B}_h(t) - \mathbf{B}_h(s))(z\varphi)| \\ & \leq \left(\frac{1}{\sqrt{2\pi}}\right)^d |t-s|^{-d\|h\|_\infty+2N(\kappa-\|h\|_\infty)} \exp(C|z|^2\|\varphi\|^2), \end{aligned}$$

which is integrable in $t, s \in [a, b]$ if and only if $d\|h\|_\infty - 2N(\kappa - \|h\|_\infty) < 1$. The proof follows from Corollary 2.2. Finally, as $[0, T]$ is a finite interval, we may obtain the truncated self-intersection local time $L_h^{(N)}(T)$ on $[0, T]$ by a patch procedure. That is, if $\bigcup_{i=1}^n [a_i, b_i]$ is a partition of $[0, T]$ such that $|a_i - b_i| < \varepsilon$, then we define

$$I_h^{(N)}(T) = \sum_{i=1}^n I_h^{(N)}([a_i, b_i]). \quad \square$$

Theorem 4.2. *Let h be as in Definition 3.1 and Assumption 1 hold and $d \in \mathbb{N}$, $N \in \mathbb{N}_0$ be such that $d\|h\|_\infty - 2N(\kappa - \|h\|_\infty) < 1$, then the kernel functions of $I_h^{(N)}(T)$ are given by*

$$\begin{aligned} F_{h,2\mathbf{n}}(u_1, \dots, u_{2n}) &= \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_0^T \int_0^T \left(-\frac{1}{2}\right)^n \frac{1}{|\eta_{t,s}|_{L^2}^{d+2n}} \\ &\quad \cdot \sum_{\substack{n_1, \dots, n_d \in \mathbb{N}_0 \\ n_1 + \dots + n_d = n}} \frac{1}{\mathbf{n}!} \prod_{j=1}^{2n} \eta_{t,s}(u_j) dt ds \end{aligned} \quad (4.7)$$

for each $\mathbf{n} \in \mathbb{N}^d$ such that $n \geq N$. All the other kernels $F_{h,\mathbf{n}}$ are zero.

Proof. The kernels of $I_h^{(N)}(T)$ are obtained by computing the S -transform. Therefore, we use Corollary 2.2 and integrate (4.4) over $[0, T]^2$. For any $\varphi \in S_d$, we have

$$\begin{aligned} SI_h^{(N)}(T)(\varphi) &= \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_0^T \int_0^T \frac{1}{|\eta_{t,s}|_{L^2}^d} \sum_{n=N}^{\infty} \frac{(-1)^n}{2^n |\eta_{t,s}|_{L^2}^{2n}} \\ &\quad \cdot \sum_{\substack{n_1, \dots, n_d \in \mathbb{N}_0 \\ n_1 + \dots + n_d = n}} \frac{1}{n_1! \dots n_d!} \prod_{j=1}^d (\varphi_j, \eta_{t,s})_{L^2}^{2n_j} dt ds \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_0^T \int_0^T \sum_{n=N}^{\infty} \left(-\frac{1}{2}\right)^n \frac{1}{|\eta_{t,s}|_{L^2}^{d+2n}} \\ &\quad \cdot \sum_{\substack{n_1, \dots, n_d \in \mathbb{N}_0 \\ n_1 + \dots + n_d = n}} \frac{1}{\mathbf{n}!} \prod_{j=1}^d (\varphi_j, \eta_{t,s})_{L^2}^{2n_j} dt ds. \end{aligned}$$

Comparing it with the general form of the chaos expansion

$$I_h^{(N)}(T) = \sum_{\mathbf{n} \in \mathbb{N}_0^d} \langle \mathbf{w}^{\otimes \mathbf{n}}; F_{h,\mathbf{n}} \rangle$$

we obtain $F_{h,\mathbf{n}}$ as in (4.7). This completes the proof. \square

Remark 4.2. The result of Theorem 4.1 states that for h constant

- (1) and $d = 1$ all self-intersection local times are well-defined. In fact, for $h = H \in (0, 1)$ we have $\kappa = 2$ and the integrability condition $d\|h\|_\infty - 2N(\kappa - \|h\|_\infty) < 1$ implies that $N > (1 - H)(H - 2)^{-1}/2$. As the right-hand side of this inequality is negative, then we can always choose $N = 0$,
- (2) for $d = 2$ and $h = H = 1/2$, we have

$$\int_{\mathbb{R}} |(M_h \mathbb{1}_{[0,t]} - M_h \mathbb{1}_{[0,s]})(x)| dx = \int_{\mathbb{R}} \mathbb{1}_{[s,t]}(x) dx = |t - s|.$$

Hence in this case $\kappa = 1$. Thus

$$d\|h\|_\infty - 2N(\kappa - \|h\|_\infty) = 1 - N.$$

In this case, we have that $N > 0$ is necessary to fulfill the assumptions of Theorem 4.2. Hence, we have to subtract the expectation in order to obtain a well defined Hida distribution.

This is the well-known fact, that the self-intersection local time of a planar Brownian motion has to be centered, see e.g., Refs. [1, 24] and [47]

The considerations in Remark 4.2 motivate the study of a regularized version of $I_h(T)$, namely

$$I_{h,\gamma}(T) := \int_0^T \int_0^T \delta_\gamma(\mathbf{B}_h(t) - \mathbf{B}_h(s)) dt ds, \quad \gamma > 0,$$

where

$$\delta_\gamma(\mathbf{B}_h(t) - \mathbf{B}_h(s)) := \left(\frac{1}{\sqrt{2\pi\gamma}} \right)^d \exp \left(-\frac{1}{2\gamma} |\mathbf{B}_h(t) - \mathbf{B}_h(s)|_{\mathbb{R}^d}^2 \right).$$

Theorem 4.3. Let $\gamma > 0$ be given and h be as in Definition 3.1 and Assumption 1 hold.

- (1) The functional $I_{h,\gamma}(T)$ is a Hida distribution with kernel functions given by

$$F_{h,\gamma,2\mathbf{n}}(u_1, \dots, u_{2n}) = \left(\frac{1}{\sqrt{2\pi}} \right)^d \int_0^T \int_0^T \left(-\frac{1}{2} \right)^n \frac{1}{(\gamma + |\eta_{t,s}|_{L^2}^2)^{n+d/2}} \cdot \sum_{\substack{n_1, \dots, n_d \in \mathbb{N}_0 \\ n_1 + \dots + n_d = n}} \frac{1}{n!} \prod_{j=1}^{2n} \eta_{t,s}(u_j) dt ds \quad (4.8)$$

for each $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}_0^d$ and $F_{h,\gamma,\mathbf{n}} = 0$ if at least one of the n_j is an odd number.

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- (2) For $\gamma \rightarrow 0$ the truncated functional $I_{h,\gamma}^{(N)}(T)$ converges strongly in $(S_d)'$ to the truncated self-intersection local times $I_h^{(N)}(T)$.

Proof. (1) First, we compute the S -transform of $\delta_\gamma(\mathbf{B}_h(t) - \mathbf{B}_h(s))$. For any $\varphi \in S_d$, we obtain

$$S\delta_\gamma(\mathbf{B}_h(t) - \mathbf{B}_h(s))(\varphi) = \left(\frac{1}{\sqrt{2\pi(\gamma + |\eta_{t,s}|_{L^2}^2)}} \right)^d \exp \left(-\frac{1}{2(\gamma + |\eta_{t,s}|_{L^2}^2)} \cdot \sum_{j=1}^d \left(\int_{\mathbb{R}} \varphi_j(x) \eta_{t,s}(x) dx \right)^2 \right),$$

which is measurable in t, s . Thus, for any $z \in \mathbb{C}$ and $\varphi \in S_d$, by the Cauchy–Schwarz inequality we obtain the following bound:

$$|S\delta_\gamma(\mathbf{B}_h(t) - \mathbf{B}_h(s))(z\varphi)| \leq \left(\frac{1}{\sqrt{2\pi(\gamma + |\eta_{t,s}|_{L^2}^2)}} \right)^d \exp \left(\frac{|\eta_{t,s}|_{L^2}^2}{2(\gamma + |\eta_{t,s}|_{L^2}^2)} |z|^2 |\varphi|_0^2 \right) \\ \leq \left(\frac{1}{\sqrt{2\pi\gamma}} \right)^d \exp(|z|^2 |\varphi|_0^2). \quad (4.9)$$

The first part of the assertion 1 follows from Corollary 2.2 since the integration is finite in a bounded set. Hence, $I_{h,\gamma}(T) \in (S_d)'$. To find the kernels of $I_{h,\gamma}(T)$, let $\varphi \in S_d$ be given, then we have

$$SI_{h,\gamma}(T)(\varphi) = \int_0^T \int_0^T S\delta_\gamma(\mathbf{B}_h(t) - \mathbf{B}_h(s))(\varphi) dt ds \\ = \left(\frac{1}{\sqrt{2\pi}} \right)^d \int_0^T \int_0^T \frac{1}{(\gamma + |\eta_{t,s}|_{L^2}^2)^{d/2}} \sum_{n=0}^\infty \frac{(-1)^n}{2^n (\gamma + |\eta_{t,s}|_{L^2}^2)^n} \\ \cdot \sum_{\substack{n_1, \dots, n_d \in \mathbb{N}_0 \\ n_1 + \dots + n_d = n}} \frac{1}{n_1! \dots n_d!} \prod_{j=1}^d (\varphi_j, \eta_{t,s})_{L^2}^{2n_j} dt ds \\ = \left(\frac{1}{\sqrt{2\pi}} \right)^d \int_0^T \int_0^T \sum_{n=0}^\infty \left(-\frac{1}{2} \right)^n \frac{1}{(\gamma + |\eta_{t,s}|_{L^2}^2)^{n+d/2}} \\ \cdot \sum_{\substack{n_1, \dots, n_d \in \mathbb{N}_0 \\ n_1 + \dots + n_d = n}} \frac{1}{n!} \prod_{j=1}^d (\varphi_j, \eta_{t,s})_{L^2}^{2n_j} dt ds.$$

Comparing the latter expression with the kernels $F_{h,\gamma,\mathbf{n}}$ from the chaos expansion of $I_{h,\gamma}(T)$

$$I_{h,\gamma}(T) = \sum_{\mathbf{n} \in \mathbb{N}_0^d} \langle : w^{\otimes \mathbf{n}} :, F_{h,\gamma,\mathbf{n}} \rangle,$$

we conclude that whenever one of the n_j in $\mathbf{n} = (n_1, \dots, n_d)$ is odd we have $F_{h,\gamma,\mathbf{n}} = 0$, otherwise they are given by the expression (4.8).

(2) To check the convergence we shall use Corollary 2.1 and the fact that

$$SI_{h,\gamma}^{(N)}(T)(\varphi) = \int_0^T \int_0^T S\delta_\gamma^{(N)}(\mathbf{B}_h(t) - \mathbf{B}_h(s))(\varphi) dt ds.$$

Thus, for all $z \in \mathbb{C}$ and all $\varphi \in S_d$ we estimate $SI_{h,\gamma}^{(N)}(T)(z\varphi)$ using the bound (4.9) by

$$\begin{aligned} |SI_{h,\gamma}^{(N)}(T)(z\varphi)| &\leq \int_0^T \int_0^T |S\delta_\gamma^{(N)}(\mathbf{B}_h(t)) - \mathbf{B}_h(s)(z\varphi)| dt ds \\ &\leq \left(\frac{1}{\sqrt{2\pi\gamma}}\right)^d \int_0^T \int_0^T \exp(|z|^2|\varphi|_0^2) dt ds \\ &\leq \left(\frac{1}{\sqrt{2\pi\gamma}}\right)^d T^2 \exp(|z|^2|\varphi|_0^2), \end{aligned}$$

which shows the uniform boundedness condition. Moreover, using similar calculations as in Theorem 4.1 for any $t, s \in [a, b] \subset [0, T]$ such that $|a - b| < \varepsilon$, yields

$$\begin{aligned} |(SI_{h,\gamma}^{(N)}(T))(\varphi)| &\leq \left(\frac{1}{\sqrt{2\pi(\gamma + |\eta_{t,s}|_{L^2}^2)}}\right)^d \exp_N\left(\frac{C}{2(\gamma + |\eta_{t,s}|_{L^2}^2)}|t - s|^{2\kappa}\|\varphi\|^2\right) \\ &\leq \left(\frac{C}{\sqrt{2\pi|t - s|\|h\|_\infty}}\right)^d \exp_N(C|t - s|^{2(\kappa - \|h\|_\infty)}\|\varphi\|^2) \\ &\leq C(2\pi)^{-d/2}|t - s|^{2N(\kappa - \|h\|_\infty) - d\|h\|_\infty} \exp(C\|\varphi\|^2), \end{aligned}$$

which is integrable in $t, s \in [a, b]$ if and only if $d\|h\|_\infty - 2N(\kappa - \|h\|_\infty) < 1$. Finally, an application of Lebesgue's dominated convergence theorem and Corollary 2.2 gives the statement 2 of the theorem. In order to obtain the result on the interval $[0, T]$ we use the same argument as in the proof of Theorem 4.1 \square

5. Conclusion and Outlook

In this paper, we identified Donsker's delta function, truncated and regularized self-intersection local times of mBm satisfying Assumption 1 as Hida distributions. Moreover, we showed that the regularized truncated self-intersection local times converge strongly to truncated self-intersection local times in $(S_d)'$.

An application of the self-intersection local times is on the Edwards type model for locally non-homogeneous polymers. The time-dependent Hölder continuity models a different local persistence along the contour of a chain polymer. The self-intersection local times can be used to penalize self-crossings. Such a model was proposed by Edwards [21] for the Brownian case and mathematically treated by Varadhan [47]. Recently, the model was extended to the case of fBm, see e.g., Refs. [24] and [29].

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A stochastic quantization of the corresponding Edwards measure via Dirichlet form techniques was used to determine a stochastic partial differential equation for the simulation of such polymer paths.^{[13][11]} The extension of these techniques would surely be an important contribution to study the scaling behavior of heterogeneous polymers. This work can be seen as a starting point of this scientific journey.

Acknowledgments

We thank the anonymous reviewer for remarks which remarkably enhanced the quality of the paper. This work was partially supported by a grant from the *Niels Hendrik Abel Board*. Financial support of FCT – Fundação para a Ciência e a Tecnologia through the project UID/MAT/04674/2019 (CIMA Universidade da Madeira) is gratefully acknowledged.

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