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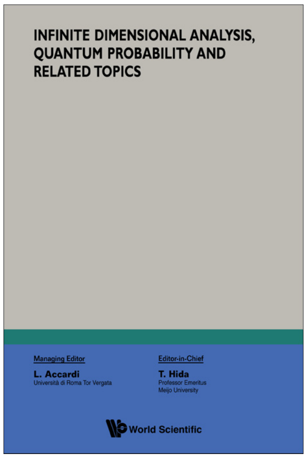

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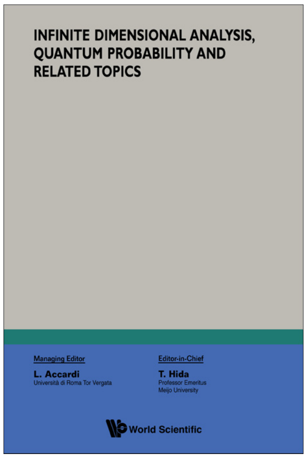

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A white noise approach to stochastic currents of Brownian motion

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In this paper, we study stochastic currents of Brownian motion $\xi(x)$, $x \in \mathbb{R}^d$, by using white noise analysis. For $x \in \mathbb{R}^d\backslash\{0\}$ and for $x = 0 \in \mathbb{R}$ we prove that the stochastic current $\xi(x)$ is a Hida distribution. Moreover for $x = 0 \in \mathbb{R}^d$ with $d > 1$ we show that the stochastic current is not a Hida distribution.

Keywords: Stochastic currents; extended Skorokhod integral; white noise analysis.

AMS Subject Classification 2020: 60H40, 60J65, 46F25

1. Introduction

The concept of current is fundamental in geometric measure theory. The simplest version of current is given by the functional

$$\varphi \mapsto \int_0^T (\varphi(\gamma(t)), \gamma'(t))_{\mathbb{R}^d}\, dt, \quad 0 < T < \infty,$$

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in a space of vector fields $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ and γ is a rectifiable curve in $\mathbb{R}^d$. Informally, this functional may be represented via its integral kernel

$$\zeta(x) = \int_0^T \delta(x - \gamma(t))\gamma'(t)\,dt,$$

where δ is the Dirac delta distribution on $\mathbb{R}^d$. The interested reader may find comprehensive account on the subject in Refs. 1 and 14.

The stochastic analog of the current $\zeta(x)$ rises if we replace the deterministic curve $\gamma(t)$, $t \in [0,T]$, by the trajectory of a stochastic process $X(t)$, $t \in [0,T]$, in $\mathbb{R}^d$. In this way, we obtain the following functional:

$$\xi(x) := \int_0^T \delta(x - X(t))\,dX(t). \tag{1.1}$$

The stochastic integral (1.1) has to be properly defined. Now we consider a d-dimensional Brownian motion $B(t)$, $t \in [0,T]$, and the main object of our study is

$$\xi(x) = \int_0^T \delta(x - B(t))\,dB(t). \tag{1.2}$$

In this work, the stochastic integral (1.2) is interpreted as an extension of the Skorokhod integral developed in Ref. 8. It coincides with the extension given by the adjoint of the Malliavin gradient. There have been some other approaches to study stochastic current, such as Malliavin calculus and stochastic integrals via regularization, see Refs. 2–4 and 6, among others.

An initial study of the stochastic current (1.2) using white noise theory was done in Ref. 7. The authors showed that $\xi(x)$ in (1.2) is well defined as a Hida distribution for all $x \in \mathbb{R}^d$ and all dimensions $d \in \mathbb{N}$. However the proof of Theorem 3.3 in Ref. 7 is not carefully written which lead the authors to an inaccurate conclusion. In fact, for $x = 0 \in \mathbb{R}^d$, $d > 1$, we show that $\xi(0)$ is not a Hida distribution. This is confirmed by first orders of the chaos expansion we obtained. Moreover, we got the impression that the authors were not checking integrability of the integrand in (1.1). Hence, they cannot apply Corollary 2.1 below. We in turn could check the assumptions of Corollary 2.1, below, for all nonzero $x \in \mathbb{R}^d$, $d \in \mathbb{N}$, and for $x = 0 \in \mathbb{R}$. The aim of this paper is to fill this gap and obtain kernels of first orders of the chaos expansion of $\xi(x)$.

The organization of the paper is as follows. Section 2 provides some background of white noise analysis. In Sec. 3, we prove the main results of this paper on the existence of the Brownian currents.

2. Gaussian White Noise Analysis

In this section, we summarize pertinent results from white noise analysis used throughout this work, and refer to Refs. 8, 10 and 13 and references therein for a detailed presentation.

2.1. *White noise space*

We start with the Gel'fand triple

$$S_d \subset L^2_d \subset S'_d,$$

where $S_d := S(\mathbb{R}, \mathbb{R}^d)$, $d \in \mathbb{N}$, is the space of vector valued Schwartz test functions, S'_d is its topological dual and the central Hilbert space $L^2_d := L^2(\mathbb{R}, \mathbb{R}^d)$ of square integrable vector valued measurable functions. For any $f \in L^2_d$ given by $f = (f_1, \dots, f_d)$ its norm is

$$|f|^2 = \sum_{i=1}^{d} \int_{\mathbb{R}} |f_i(x)|^2 \, dx.$$

Let $\mathcal{B}$ be the σ-algebra of cylinder sets on S'_d. Since S_d equipped with its standard topology is a nuclear space, by Minlos' theorem there is a unique probability measure μ_d on $(S'_d, \mathcal{B})$ with the characteristic function given by

$$C(\varphi) := \int_{S'_d} e^{i\langle w,\varphi\rangle} \, d\mu_d(w) = \exp\left(-\frac{1}{2}|\varphi|^2\right), \quad \varphi \in S_d.$$

Hence, we have constructed the white noise probability space $(S'_d, \mathcal{B}, \mu_d)$. In the complex Hilbert space $L^2(\mu_d) := L^2(S'_d, \mathcal{B}, \mu_d; \mathbb{C})$ a d-dimensional Brownian motion is given by

$$B(t, w) = (\langle w_1, \eta_t\rangle, \dots, \langle w_d, \eta_t\rangle), \quad w = (w_1, \dots, w_d) \in S'_d, \quad \eta_t := \mathbb{1}_{[0,t)}, \quad t \geq 0.$$

In other words, $(B(t))_{t\geq 0}$ consists of d independent copies of 1-dimensional Brownian motions. For all $F \in L^2(\mu_d)$ one has the Wiener–Itô–Segal chaos decomposition

$$F(w) = \sum_{n=0}^{\infty} \langle : w^{\otimes n} :, F_n\rangle, \quad F_n \in (L^2_{d,\mathbb{C}})^{\hat{\otimes} n},$$

where $: w^{\otimes n} : \in (S'_{d,\mathbb{C}})^{\hat{\otimes} n}$ denotes the nth order Wick power of $w \in S'_{d,\mathbb{C}}$ and $\langle \cdot, \cdot \rangle$ denotes the dual pairing on $(S'_{d,\mathbb{C}})^{\otimes n} \times (S_{d,\mathbb{C}})^{\otimes n}$ which is a bilinear extension of $(\cdot, \bar{\cdot})$, where $(\cdot, \cdot)$ is the inner product on $(L^2_{d,\mathbb{C}})^{\otimes n}$ in the sense of a Gel'fand triple. Here $V_{\mathbb{C}}$ denotes the complexification of the real vector space V and $\hat{\otimes} n$ denotes the nth power symmetric tensor product. Note that $\langle : \cdot^{\otimes n} :, \cdot\rangle$, $n \in \mathbb{N}_0$, in the second variable extends to $(L^2_{d,\mathbb{C}})^{\hat{\otimes} n}$ in the sense of an $L^2(\mu_d)$ limit.

2.2. *Hida distributions and characterization*

By the standard construction with the Hilbert space $L^2(\mu_d)$ as central space, we obtain the Gel'fand triple of Hida test functions and Hida distributions

$$(S_d) \subset L^2(\mu_d) \subset (S_d)'.$$

We denote the dual pairing between elements of $(S_d)'$ and (S_d) by $\langle\langle \cdot, \cdot \rangle\rangle$. For $F \in L^2(\mu_d)$ and $\varphi \in (S_d)$, with kernel functions F_n and φ_n, respectively, the dual pairing yields

$$\langle\!\langle F, \varphi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \langle F_n, \varphi_n \rangle.$$

This relation extends the chaos expansion to $\Phi \in (S_d)'$ with distribution valued kernels $\Phi_n \in (S'_{d,\mathbb{C}})^{\hat{\otimes} n}$ such that

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \langle \Phi_n, \varphi_n \rangle$$

for every generalized test function $\varphi \in (S_d)$ with kernels $\varphi_n \in (S_{d,\mathbb{C}})^{\hat{\otimes} n}$, $n \in \mathbb{N}_0$.

Instead of repeating the detailed construction of these spaces we present a characterization in terms of the S-transform.

Definition 2.1. Let $\varphi \in S_d$ be given. We define the Wick exponential by

$$e_{\mu_d}(\cdot, \varphi) := \frac{e^{\langle \cdot, \varphi \rangle}}{\mathbb{E}(e^{\langle \cdot, \varphi \rangle})} = C(\varphi) e^{\langle \cdot, \varphi \rangle} = \sum_{n=0}^{\infty} \frac{1}{n!} \langle : \cdot^{\otimes n} :, \varphi^{\otimes n} \rangle \in (S_d)$$

and the S-transform of $\Phi \in (S_d)'$ by

$$S\Phi(\varphi) := \langle\!\langle \Phi, e_{\mu_d}(\cdot, \varphi) \rangle\!\rangle.$$

Example 2.1. For $d \in \mathbb{N}$ the S-transform of d-dimensional white noise $(W(t))_{t \geq 0}$ is given by $SW(t)(\varphi) = \varphi(t)$, for all $\varphi \in S_d$, $t \geq 0$, see Ref. 8. Here $(W(t))_{t \geq 0}$ is the derivative of $(B(t))_{t \geq 0}$ as a Hida space valued process. That is, each of its components takes values in $(S_d)'$.

Definition 2.2. (U-functional) A function $F : S_d \to \mathbb{C}$ is called a U-functional if:

(1) For every $\varphi_1, \varphi_2 \in S_d$ the mapping $\mathbb{R} \ni \lambda \mapsto F(\lambda \varphi_1 + \varphi_2) \in \mathbb{C}$ has an entire extension to $z \in \mathbb{C}$.

(2) There are constants $0 < C_1, C_2 < \infty$ such that

$$|F(z\varphi)| \leq C_1 \exp(C_2 |z|^2 \|\varphi\|^2), \quad \forall\, z \in \mathbb{C}, \quad \varphi \in S_d$$

for some continuous norm $\| \cdot \|$ on S_d.

We are now ready to state the aforementioned characterization result.

Theorem 2.1. (cf. Refs. 10 and 16) *The S-transform defines a bijection between the space $(S_d)'$ and the space of U-functionals. In other words, $\Phi \in (S_d)'$ if and only if $S\Phi : S_d \to \mathbb{C}$ is a U-functional.*

Based on Theorem 2.1 a deeper analysis of the space $(S_d)'$ can be developed. The following corollary concerns the Bochner integration of functions with values in $(S_d)'$ (for more details and proofs see e.g., Refs. 8, 10 and 16 for the case $d = 1$).

Corollary 2.1. *Let* $(\Omega, \mathcal{F}, m)$ *be a measure space and* $\lambda \mapsto \Phi_\lambda$ *be a mapping from* Ω *to* $(S_d)'$. *We assume that the* S*-transform of* Φ_λ *fulfills the following two properties*:

(1) *The mapping* $\lambda \mapsto S\Phi_\lambda(\varphi)$ *is measurable for every* $\varphi \in S_d$.
(2) *The* U*-functional* $S\Phi_\lambda$ *satisfies*

$$|S\Phi_\lambda(z\varphi)| \leq C_1(\lambda)\exp(C_2(\lambda)|z|^2\|\varphi\|^2), \quad z \in \mathbb{C}, \quad \varphi \in S_d$$

for some continuous norm $\|\cdot\|$ *on* S_d *and for some* $C_1 \in L^1(\Omega, m)$, $C_2 \in L^\infty(\Omega, m)$.

Then

$$\int_\Omega \Phi_\lambda \, dm(\lambda) \in (S_d)'$$

and

$$S\left(\int_\Omega \Phi_\lambda \, dm(\lambda)\right)(\varphi) = \int_\Omega S\Phi_\lambda(\varphi) dm(\lambda), \quad \varphi \in S_d.$$

Moreover, *the integral exists as a Bochner integral in some Hilbert subspace of* $(S_d)'$.

Example 2.2. (Donsker's delta function) As a classical example of a Hida distribution we have the Donsker delta function. More precisely, the following Bochner integral is a well-defined element in $(S_d)'$,

$$\delta(x - B(t)) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(\lambda, x - B(t))_{\mathbb{R}^d}} \, d\lambda, \quad x \in \mathbb{R}^d.$$

The S-transform of $\delta(x - B(t))$ for any $z \in \mathbb{C}$ and $\varphi \in S_d$ is given by

$$S\delta(x - B(t))(z\varphi) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{1}{2t}\sum_{j=1}^{d}(x_j - \langle z\varphi_j, \eta_t\rangle)^2\right). \tag{2.1}$$

It is well known that the Wick product is a well-defined operation in Gaussian analysis, see for example Refs. 9, 11 and 12.

Definition 2.3. For any $\Phi, \Psi \in (S_d)'$ the Wick product $\Phi \diamondsuit \Psi$ is defined by

$$S(\Phi \diamondsuit \Psi) = S\Phi \cdot S\Psi. \tag{2.2}$$

Since the space of U-functionals is an algebra, by Theorem 2.1 there exists an element $\Phi \diamondsuit \Psi \in (S_d)'$ such that (2.2) holds.

3. Stochastic Currents of Brownian Motion

In this section, we investigate in the framework of white noise analysis the following functional:

$$\varphi \mapsto \int_0^T (\varphi(B(t)), dB(t))_{\mathbb{R}^d} \tag{3.1}$$

on a given space of vector fields $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$. The functional (3.1) can be represented via its integral kernel

$$\xi(x) := \int_0^T \delta(x - B(t))\, dB(t), \quad x \in \mathbb{R}^d.$$

We interpret the stochastic integral as an extended Skorokhod integral

$$\begin{aligned}\int_0^T \delta(x - B(t))\, dB(t) \\ &:= \left(\int_0^T \delta(x - B(t)) \diamond W_1(t)\, dt, \ldots, \int_0^T \delta(x - B(t)) \diamond W_d(t)\, dt\right) \\ &=: (\xi_1(x), \ldots, \xi_d(x)),\end{aligned}$$

where $W = (W_1, \ldots, W_d)$ is the white noise process as in Example 2.1. If the integrand is a square integrable function then this stochastic integral coincides with the Skorokhod integral. In this interpretation, we call $\xi(x)$ stochastic currents of Brownian motion.

Below we show that $\xi(x)$, $x \in \mathbb{R}^d \backslash \{0\}$ is a well-defined functional in $(S_d)'$. From now on, C is a real constant whose value is immaterial and may change from line to line.

Theorem 3.1. *For $x \in \mathbb{R}^d \backslash \{0\}$, $0 < T < \infty$, the Bochner integral*

$$\xi_i(x) = \int_0^T \delta(x - B(t)) \diamond W_i(t)\, dt \tag{3.2}$$

is a Hida distribution and its S-transform at $\varphi \in S_d$ is given by

$$S\left(\int_0^T \delta(x - B(t)) \diamond W_i(t)\, dt\right)(\varphi) = \frac{1}{(2\pi)^{d/2}} \int_0^T \frac{1}{t^{d/2}} e^{-\frac{|x - \langle \eta_t, \varphi\rangle|^2_{\mathbb{R}^d}}{2t}} \varphi_i(t)\, dt. \tag{3.3}$$

Proof. First we compute the S-transform of the integrand

$$(0, T] \ni t \mapsto \Phi_i(t) := \delta(x - B(t)) \diamond W_i(t).$$

Using Definition 2.3, Examples 2.1 and 2.2 for any $\varphi \in S_d$ we have

$$t \mapsto S\Phi_i(t)(\varphi) = S(\delta(x - B(t)))(\varphi)SW_i(t)(\varphi) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{1}{2t}|x - \langle \eta_t, \varphi\rangle|^2_{\mathbb{R}^d}\right)\varphi_i(t),$$

which is Borel measurable on $(0, T]$. Furthermore, for any $z \in \mathbb{C}$, $t \in (0, T]$ and all $\varphi \in S_d$ we obtain

$$\begin{aligned}
|S\Phi_i(t)(z\varphi)| &\leq \left|\frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{1}{2t}|x - \langle \eta_t, z\varphi\rangle|^2_{\mathbb{R}^d}\right) z\varphi_i(t)\right| \\
&\leq \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{1}{2t}|x|^2\right) \exp\left(\frac{1}{t}|x||\langle \eta_t, z\varphi\rangle|\right) \\
&\quad \cdot \exp\left(\frac{1}{2t}|z|^2|\langle \eta_t, \varphi\rangle|^2\right) |z\varphi_i(t)| \\
&\leq \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{1}{2t}|x|^2\right) \exp(|x||z||\varphi|_\infty) \exp\left(\frac{1}{2}|z|^2|\varphi|^2\right) \\
&\quad \cdot \exp(|z||\varphi|_\infty) \\
&\leq \frac{C}{(2\pi t)^{d/2}} \exp\left(-\frac{1}{2t}|x|^2\right) \exp\left(\frac{1}{2}|x|^2\right) \exp\left(\frac{1}{2}|z|^2|\varphi|^2\right) \\
&\quad \cdot \exp\left(\frac{1}{2}|z|^2|\varphi|^2_\infty\right) \\
&\leq \frac{C}{(2\pi t)^{d/2}} \exp\left(-\frac{1}{2t}|x|^2\right) \exp\left(\frac{1}{2}|x|^2\right) \exp\left(\frac{1}{2}|z|^2\|\varphi\|^2\right),
\end{aligned}$$

where $\|\cdot\|$ is the continuous norm on S_d defined by $\|\varphi\| := \sqrt{|\varphi|^2 + |\varphi|^2_\infty}$. The first factor $\frac{C}{(2\pi t)^{d/2}} \exp(-\frac{1}{2t}|x|^2)$ is integrable with respect to the Lebesgue measure dt on $[0, T]$. To be more precise, using the formula

$$\int_u^\infty y^{\nu-1} e^{-\mu y}\, dy = \mu^{-\nu}\Gamma(\nu, \mu u), \quad u > 0, \quad \mathrm{Re}(\mu) > 0,$$

where $\Gamma(\cdot, \cdot)$ is the complementary incomplete gamma function, one can show that

$$\int_0^T \frac{1}{t^{d/2}} \exp\left(-\frac{1}{2t}|x|^2\right) dt = 2^{d/2-1}|x|^{2-d}\Gamma\left(\frac{d}{2} - 1, \frac{|x|^2}{2T}\right).$$

As the second factor $\exp(\frac{1}{2}|x|^2)\exp(\frac{1}{2}|z|^2\|\varphi\|^2)$ is independent of $t \in (0, T]$, the result now follows from Corollary 2.1. □

Corollary 3.1. *For $x = 0$ and $d = 1$ the stochastic current $\xi(0)$ is a Hida distribution, that is, the Bochner integral*

$$\xi(0) = \int_0^T \delta(B(t)) \lozenge W(t)\, dt$$

is a Hida distribution. Moreover its S-transform at $\varphi \in S_1$ is given by

$$S\left(\int_0^T \delta(B(t)) \lozenge W(t)\, dt\right)(\varphi) = \frac{1}{\sqrt{2\pi}} \int_0^T \frac{1}{\sqrt{t}} e^{-\frac{\langle \eta_t, \varphi\rangle^2}{2t}} \varphi(t)\, dt.$$

Proof. By adapting the proof of Theorem 3.1 we obtain for any $z \in \mathbb{C}$, $t \in (0, T]$ and all $\varphi \in S_1$,

$$|S\Phi(t)(z\varphi)| \leq \frac{C}{(2\pi t)^{1/2}} \exp\left(\frac{1}{2}|z|^2 \|\varphi\|^2\right).$$

Since the function $(0, T] \ni t \mapsto t^{-1/2}$ is integrable with respect to the Lebesgue measure, Corollary 2.1 implies the statement of the corollary. □

Remark 3.1. We would like to comment on the chaos expansion of the stochastic current of Brownian motion. To this end, we identify the space L^2_d with the Hilbert space $L^2(m) := L^2(E, \mathcal{B}, m)$, where $E := \mathbb{R} \times \{1, \dots, d\}$, $\mathcal{B}$ is the product σ-algebra on E of the Borel σ-algebra on $\mathbb{R}$ and the power set of $\{1, \dots, d\}$ and $m = dx \otimes \Sigma$ is the product measure of the Lebesgue measure on $\mathbb{R}$ and the counting measure on $\{1, \dots, d\}$. That is, for all $f, g \in L^2(m)$ we have

$$(f, g)_{L^2(m)} = \int_E f(x, i) g(x, i)\, dm(x, i) = \sum_{i=1}^d \int_{\mathbb{R}} f(x, i) g(x, i)\, dx.$$

The nth order chaos of a Hida distribution can be computed by the nth order derivative of its S-transform at the origin. More precisely, for $\Psi \in (S_d)'$ and $\varphi \in (S_d)$ consider the function

$$\mathbb{R} \ni s \mapsto U(s) := S\Psi(s\varphi) \in \mathbb{C}.$$

Then the nth order chaos $\Psi^{(n)}$ of Ψ applied to $\varphi^{\otimes n} \in S_{d,\mathbb{C}}^{\hat{\otimes} n}$ is given by

$$\langle \Psi^{(n)}, \varphi^{\otimes n}\rangle = \frac{1}{n!} \frac{d^n}{ds^n} U(s)|_{s=0},$$

see Lemma 3.3.5 of Ref. 15. In our situation, we have

$$S\Phi_i(s\varphi) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{1}{2t}|x - \langle \eta_t, s\varphi\rangle|^2_{\mathbb{R}^d}\right) s\varphi_i(t), \quad i \in \{1, \dots, d\}.$$

Hence, for the stochastic currents of Brownian motion the first chaos are given by

$$\xi^{(0)}(x) = (0, \dots, 0),$$

$$\xi_i^{(1)}(x) = \left(\underbrace{0, \dots, 0}_{i-1}, \frac{1}{(2\pi)^{d/2}} \int_0^T \frac{1}{t^{d/2}} e^{-\frac{|x|^2}{2t}} \delta_t \, dt, 0, \dots, 0\right),$$

$$(\xi_i^{(2)}(x))_{j,k} = -\frac{1}{4(2\pi)^{d/2}} \int_0^T \frac{1}{t^{d/2+1}} e^{-\frac{|x|^2}{2t}} (\mathrm{Id}_{ki} x_j \eta_t \otimes \delta_t + \mathrm{Id}_{ji} x_k \delta_t \otimes \eta_t) dt,$$

where Id denotes the identity matrix on $\mathbb{R}^d$ and δ_t denotes the Dirac distribution at $t > 0$. Note that for $x = 0$ and $d > 1$ the first chaos $\xi_i^{(1)}(0)$ is divergent, hence in this case, $\xi(0)$ cannot be a Hida distribution. In all the other cases the integrals are well defined as Bochner integrals in a suitable Hilbert subspace of $(S_d')^{\hat{\otimes} n}$, $n = 0, 1, 2$. This follows from the estimates for integrability we derived in the proof of Theorem 3.1. Indeed, the estimates derived to apply Corollary 2.1 imply that the integrands Φ_i, $1 \le i \le d$, are Bochner integrable in some Hilbert subspace H_- of (S_d') equipped with a norm $\| \cdot \|_-$, see proof of Theorem 17 of Ref. 10. More precisely, there one shows that $\|\Phi_i\|_-$, $1 \le i \le d$, is integrable. That implies Bochner integrability of the kernels of nth order in the generalized chaos decomposition in a suitable Hilbert subspace of $S'(\mathbb{R}^n)$.

4. Conclusion and Outlook

In this paper, we give a mathematical rigorously meaning to the stochastic current $\xi(x)$, $x \in \mathbb{R}^d \backslash \{0\}$ and $\xi(0)$, $0 \in \mathbb{R}$, of Brownian motion in the framework of white noise analysis. On the other hand, for $x = 0 \in \mathbb{R}^d$, $d > 1$, we showed that $\xi(0)$ is not a Hida distribution. The first orders of the chaos expansion leave open whether the $\xi(x)$, $x \in \mathbb{R}^d \backslash \{0\}$, are regular generalized functions or even square integrable. That is, it is not obvious whether $\xi^{(n)}(x) \in (L^2_{d,\mathbb{C}})^{\hat{\otimes} n}$ or not for $x \in \mathbb{R}^d \backslash \{0\}$ and $n \in \mathbb{N}$.

There have been some other approaches to study stochastic current, such as Malliavin calculus and stochastic integrals via regularization, see Refs. 2–4 and 6, among others. In Ref. 3, ξ was constructed in a negative Sobolev space, i.e. in a generalized function space in the variable $x \in \mathbb{R}^d$. Then the constructed distribution was applied to a model of random vortex filaments in turbulent fluids.

Using the improved characterization of regular generalized functions from $\mathcal{G}'$, see Ref. 5, we plan to show more regularity of $\xi(x)$, $x \in \mathbb{R}^d \backslash \{0\}$. For general $d \in \mathbb{N}$ we are analysing, whether $\xi(x) \in \mathcal{G}'$. The improved characterization provided in Ref. 5 also enables to check, whether a given Hida distributions is even a square integrable function. We already obtained very promising estimates for the S-transform of ξ. These lead us to the following conjecture: For $d = 1$ and all $x \in \mathbb{R}$ is $\xi(x) \in L^2(\mu)$. As far as we know, this would be the first pointwise construction of stochastic current.

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