UNIVERSITAS SANATA DHARMA YOGYAKARTA

Proceedings of

ICMAME

The 2015 International Conference on Mathematics, its Applications, and Mathematics Education

Sponsored by:

Australia Awards

Australia Awards Indonesia Alumni Grant Scheme 2015 under project title "International Conference on Science and Education: ANUGA Software for Flood Mitigation in Indonesia"

SANATA DHARMA UNIVERSITY

14 - 15 September 2015

PROCEEDINGS

THE 2015 INTERNATIONAL CONFERENCE ON MATHEMATICS, ITS APPLICATIONS, AND MATHEMATICS EDUCATION

Contributors:
R Setiawan, Eko Minarto, Pujiadi, Dwi Ratna S, Budi Setyono,Liyana Wulandhari , Buddin Al Hakim, Atmini Dhoruri, Dwi Lestari, Eminugroho Ratnasari, Firdaus, Liem Chin, Erwinna Chendra, Agus Sukmana, MV Any Herawati, Dafik, I.H. Agustin, K. Rizqy Aprilia, Boby Gunarso, G Pramesti, Febriani Astuti, Kartiko, Sri Sulistijowati Handajani, Iqbal Kharisudin, Dedi Rosadi, Abdurakhman, Suhartono, Fitriati,Pratikno, Jajang,Zulfa A,Regina Wahyudyah S.A, Andy Rudhito, Happy Christanti, Susi Susanti, Dita Apriliyanti, Dewi Handayani, Kriswandani, Mefa Indriati, Erdawati Nurdin, Nizaruddin, Budi Waluyo, Khafidurrohman Agustianto, Adhistya Erna Permanasari, Sri Suning Kusumawardani, Sudi Prayitno, Masrukan, Bambang Eko Susilo, Deddy Irawan,Suhito, M.A. Nuha, Rooselyna Ekawati, Stephanus Ivan Goenawan, and Ferry Rippun

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Cover Ilustration & Layout:
Made Setianto

First Edition, March 2016
Softcover; 216 pages.; 21 x 29,7 cm.

ISBN: 978-602-0830
EAN: 9-786020-830

PUBLISHER:

SANATA DHARMA UNIVERSITY PRESS
Lantai 1 Gedung Perpustakaan USD
Jl. Affandi (Gejayan) Mrican,
Yogyakarta 55281
Telp. (0274) 513301, 515253;
Ext.1527/1513; Fax (0274) 562383
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Sanata Dharma University Press Member of APPTI
(Association of University Publishers in Indonesia)

PREFACE

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This Proceedings of The 2015 International Conference on Mathematics, its Applications, and Mathematics Education (ICMAME 2015) is devoted to some of the accepted papers presented in ICMAME 2015. ICMAME 2015 was held by Sanata Dharma University in Yogyakarta, Indonesia, on 14-15 September 2015. This Conference was conducted to bring together mathematicians and other scientists working on new trends of mathematics, physics, its applications and also on mathematics education.

At least 100 submissions were received by the Committee for oral presentations. Peer-review was conducted after the Conference. Each full paper was reviewed by two or three referees. After review, 47 papers were accepted for publication. However, based on referees' recommendations, the Editors decided that 20 papers were selected for publication in a volume of Journal of Physics: Conference Series. The rest (27 papers) are included in this Proceedings Book.

More than 200 people participated in the Conference. They were from, based on alphabetical order, Australia, Brazil, Cambodia, Germany, Indonesia, Malaysia, The Philippines, Timor-Leste, and Vietnam. Among them, we had seven keynote/plenary speakers:

- Prof. Dr. Stephen Roberts (The Australian National University, Australia),
- Prof. Dr. Lutz Gross (The University of Queensland, Australia),
- Prof. em. Dr. Elmar Cohors-Fresenborg (The University of Osnabrueck, Germany),
- Dr. Eka Budiarto (Swiss German University, Indonesia),
- Dr. Yansen Marpaung (Sanata Dharma University, Indonesia),
- Dr. Herry Pribawanto Suryawan (Sanata Dharma University, Indonesia), and
- Dr. Hongki Julie (Sanata Dharma University, Indonesia).

We thank all of the speakers, participants and organising committee members for their contribution. In particular, we also thank our generous sponsor for the financial support to ICMAME 2015:

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TABLE OF CONTENTS

MATHEMATICS EDUCATION

Systems of Max-Plus Linear Equations with More Variables than Equations

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Abstract. This paper discusses the solution of systems of max-plus linear equations with more variables than equations through the reduced discrepancy matrix of the system. Let the entries of each column of the coefficient matrix are not all equal to infinite. We show that if there is a zero-row in reduced discrepancy matrix of the system, then the systems has no solution. Furthermore, if there is no zero-row in reduced discrepancy matrix of the system, then there are infinitely many solutions of the system.

Key words: Max-Plus algebra, system of linear equations, reduced discrepancy matrix.

1. Introduction

As in conventional algebra, we can also find system of linear equations in max-plus algebra. System of max-plus linear equations can also be represented by a matrix equation that is $A \otimes \boldsymbol{x} = \boldsymbol{b}$**.**

The solution of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ in max-plus algebra through the reduced "*discrepancy"*matrix has been discussed in [1] and [4]. However, they just concern about the existence and the uniqueness of the solution to $A \otimes \boldsymbol{x} = \boldsymbol{b}$ in general. They haven't concerned about the solution of the system of max-plus linear equations with more variables than equations in spesific yet. Therefore, in this article, we will discuss about the solution of the system of $A \otimes \boldsymbol{x} = \boldsymbol{b}$ with more variables than equations.

First, we will review some basic concepts of max-plus algebra, matrices over max-plus algebra and the solution of the system of $A \otimes \boldsymbol{x} = \boldsymbol{b}$. Futher details can be found in [2] and [4].

Let $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$ where $\mathbb{R}$ is a set of all real numbers and $\varepsilon := -\infty$. Defined two operations $\oplus$ dan $\otimes$ on $\mathbb{R}_\varepsilon$ such that

$$a \oplus b \ := max(a, b) \quad \text{dan} \quad a \otimes b \ := a + b \qquad , \forall a, b \in \mathbb{R}_\varepsilon.$$

$\mathbb{R}_{\max} = (\mathbb{R}_\varepsilon, \ \oplus, \ \otimes)$ is a commutative idempotent semiring. Furthermore, $\mathbb{R}_{\max}$ is a semifield. Then, $\mathbb{R}_{\max}$ is called as max-plus algebra. The relation " $\leq_{max}$ " on $\mathbb{R}_{max}$ defined by $a \leq_{max} b \Leftrightarrow a \oplus b = b$ is a partial order on $\mathbb{R}_{max}$.

The operations $\oplus$ dan $\otimes$ on $\mathbb{R}_{max}$ can be extended to set $\mathbb{R}_{\max}^{m\times n}$ where $\mathbb{R}_{\max}^{m\times n} = \{A = a_{ij} \mid a_{ij} \in \mathbb{R}_{max}, \text{for } i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n\}$. Let $A, B \in \mathbb{R}_{max}^{m\times p}$ and $C \in \mathbb{R}_{max}^{p\times n}$ then

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij} \text{ and } [A \otimes C]_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes c_{kj}.$$

The relation " $\leq_{max}$ " defined in $\mathbb{R}_{max}^{m\times n}$ where $A \leq_{max} B \Leftrightarrow A \oplus B = B$ is a partial order on $\mathbb{R}_{max}^{m\times n}$.

Defined $\mathbb{R}_{\max}^{n} = \left\{\boldsymbol{x} = [x_1, x_2, \dots, x_n]^T \mid x_j \in \mathbb{R}_{\max}, j = 1,2,\dots,n\right\}$. The element of $\mathbb{R}_{\max}^{n}$ is called vector over $\mathbb{R}_{max}$.

Definition 1.1. *Given $A \in \mathbb{R}_{max}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}_{max}^{m}$. Subsolution of the system of max-plus linear equations $A \otimes \boldsymbol{x} = \boldsymbol{b}$ is a vector $\boldsymbol{x}' \in \mathbb{R}_{max}^{n}$ that satisfies $A \otimes \boldsymbol{x}' \leq_{max} \boldsymbol{b}$.*

Definition 1.2. *A subsolution $\boldsymbol{x}^*$ of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ is called the greatest subsolution of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ if $\boldsymbol{x}' \leq_{max} \boldsymbol{x}^*$ for every subsolution $\boldsymbol{x}'$ of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$.*

Theorem 1.1. [4] *Given $A \in \mathbb{R}_{max}^{m\times n}$ with the entries of each column are not all equal ε and $\boldsymbol{b} \in \mathbb{R}^m$, then $-x^*_j = \max_i(-b_i + a_{ij})$ for every $i \in \{1,2,\dots,m\}$ and $j \in \{1,2,\dots,n\}$.*

Theorem 1.2. [3] *Given $A \in \mathbb{R}_{max}^{m\times n}$ with the entries of each column are not all equal ε and $\boldsymbol{b} \in \mathbb{R}^m$. $A \otimes \boldsymbol{x} = \boldsymbol{b}$ has a solution if and only if $\boldsymbol{x}^*$ is a solution.*

2. Main Result

Base on Theorem 1.2, we can conclude that the existence of the solution of the system of max-plus linear equation $A \otimes \boldsymbol{x} = \boldsymbol{b}$ is determined by the greatest subsolution. Let $A \in \mathbb{R}_{\max}^{m\times n}$ with the entries of each column are not all equal ε and $\boldsymbol{b} \in \mathbb{R}^m$. The case that we'll discuss is the solution of the system of max-plus linear equations $A \otimes \boldsymbol{x} = \boldsymbol{b}$ with more variables than more equations or $m < n$. The greatest subsolution is a candidate solution of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ that is vector $\boldsymbol{x}^*$ where

$$-\boldsymbol{x}^* = \begin{bmatrix} -x^*_1 \\ -x^*_2 \\ \vdots \\ -x^*_n \end{bmatrix} = \begin{bmatrix} \max\limits_i(-b_i + a_{i1}) \\ \max\limits_i(-b_i + a_{i2}) \\ \vdots \\ \max\limits_i(-b_i + a_{in}) \end{bmatrix} = \begin{bmatrix} \max\limits_i(a_{i1} - b_i) \\ \max\limits_i(a_{i2} - b_i) \\ \vdots \\ \max\limits_i(a_{in} - b_i) \end{bmatrix}$$
$$= \begin{bmatrix} \max\{a_{11} - b_1, a_{21} - b_2, \dots, a_{m1} - b_m\} \\ \max\{a_{12} - b_1, a_{22} - b_2, \dots, a_{m2} - b_m\} \\ \vdots \\ \max\{a_{1n} - b_1, a_{2n} - b_2, \dots, a_{mn} - b_m\} \end{bmatrix}$$

Then, we define discrepancy matrix denoted by $D_{A,b}$ as folows

$$D_{A,b} = \begin{bmatrix} a_{11} - b_1 & a_{12} - b_1 & \dots & a_{1n} - b_1 \\ a_{21} - b_2 & a_{22} - b_2 & \dots & a_{2n} - b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} - b_m & a_{m2} - b_m & \dots & a_{mn} - b_m \end{bmatrix}$$

Note that every $-x^*{}_j$ can be determined by taking the maximum of each column of $D_{A,b}$.

In order to predict the number of solutions of system $A \otimes \boldsymbol{x} = \boldsymbol{b}$, we define matrix $R_{A,b}$ that is reduced from $D_{A,b}$ as follows:

$$R_{A,b} = [r_{ij}] \text{ where } r_{ij} = \begin{cases} 1, & \text{if } d_{ij} = \text{maximum of column } j \\ 0, & \text{otherwise} \end{cases}$$

Next, we will give the examples of the solution of the max-plus linear equations $A \otimes \boldsymbol{x} = \boldsymbol{b}$ for $m < n$.

Example 2.1. Solve $A \otimes \boldsymbol{x} = \boldsymbol{b}$ if $A = \begin{bmatrix} 1 & 0 & 3 \\ \varepsilon & 4 & 2 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

A quick calculation gives $D_{A,b} = \begin{bmatrix} 0 & -1 & 2 \\ \varepsilon & -2 & -4 \end{bmatrix}$ and $R_{A,b} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Base on matrix $D_{A,b}$ we get $\boldsymbol{x}^* = [0,\ 1, -2]^T$. However, there is a row in $R_{A,b}$ that all entries are 0. It is the second row which means that there is no maximum in that row. It indicates that the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ has no solution. We can verify that through the calculation as follows:

$$A \otimes \boldsymbol{x}^* = \begin{bmatrix} 1 & 0 & 3 \\ \varepsilon & 4 & 2 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \max\{1,1,1\} \\ \max\{\varepsilon, 5, 0\} \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \boldsymbol{b}$$

So, $\boldsymbol{x}^*$ is just the greates subsolution of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ but not the solution.

Example 2.2. Solve $A \otimes \boldsymbol{x} = \boldsymbol{b}$ if $A = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 7 & 8 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, dan $\boldsymbol{b} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

A quick calculation gives $D_{A,b} = \begin{bmatrix} -2 & -2 & -3 \\ -2 & 1 & 2 \end{bmatrix}$ and $R_{A,b} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Base on matrix $D_{A,b}$ we get $\boldsymbol{x}^* = [2, -1, -2]^T$. Next, we will verify whether $\boldsymbol{x}^*$ is a solution or not.

$$A \otimes \boldsymbol{x}^* = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 7 & 8 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} \max\{4,1,-1\} \\ \max\{6,6,6\} \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \boldsymbol{b}$$

We can see that $\boldsymbol{x}^*$ is indeed the solution of $A \otimes \boldsymbol{x} = \boldsymbol{b}$**.** But, there is more than one 1 in the second row of $R_{A,b}$. In other words, there is more than one maximum in that row. It indicates that the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ has an infinite number of solutions. Base on Definition 1.2, we know that the elements of $\boldsymbol{x}^*$ are the upper bounds. So, the elements of vector $\boldsymbol{x}$ in this example must satisfy $x_1 \leq 2$, $x_2 \leq -1$ and $x_3 \leq -2$. On the first row of $R_{A,b}$, the maximum is in the first column then $x_1 = 2$. On the second row, the maximum is in the first, second and third column then there are three possible ways with either $x_1 = 2$, $x_2 = -1$ or $x_3 = -2$. If we change the value of x_1 then it will change the equation in the first row. So now as long as $x_2 \leq -1$ and $x_3 \leq -2$, the first and second equation will always be true. Therefore, every vector $\boldsymbol{x} = [2,\ a,\ b]^T$ where $a \leq -1$ and $b \leq -2$ is also a solution. So, the system of $A \otimes \boldsymbol{x} = \boldsymbol{b}$ in this example has an infinite number of solutions.

Matrix $D_{A,b}$ and $R_{A,b}$ play role in determining the characteristics of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$. Now, we will give the theorem about the exixtence of the solution of the system of max-plus linear equations $A \otimes \boldsymbol{x} = \boldsymbol{b}$.

Theorem 2.1. [1] *Given the system* $A \otimes \boldsymbol{x} = \boldsymbol{b}$ *where* $A \in \mathbb{R}_{max}^{m\times n}$ *with the entries of each column are not all equal* ε *and* $\boldsymbol{b} \in \mathbb{R}^m$.

1. *If there is a zero-row in matrix* $R_{A,b}$ *then the system has no solution.*
2. *If there is at least one 1 in each row of* $R_{A,b}$, *then* $\boldsymbol{x}^*$ *is the solution of the system* $A \otimes \boldsymbol{x} = \boldsymbol{b}$.

Proof.

1. Without lost of generality, suppose the zero-row of $R_{A,b}$ is the k^{th} and let $\boldsymbol{x}^*$ is the solution of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$, then $-x^*{}_j \geq \max_i(-b_i + a_{ij}) > -b_k + a_{kj}$. Thus, $-x^*{}_j > -b_k + a_{kj} \Leftrightarrow a_{kj} + x^*{}_j < b_k, \forall j$. Hence, $\boldsymbol{x}^*$ does not satisfy the k^{th} equation. It contradicts with $\boldsymbol{x}^*$ is the solution of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$. So, the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ has no solution.
2. We will proof the contrapositive. Suppose $\boldsymbol{x}^*$ is not the solution of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$. By Teorema 1.1, $-x^*{}_j \geq -b_k + a_{kj}\ , \forall k, j$. Thus, $\max_j(a_{kj} + x^*{}_j) \leq b_k$. If $\boldsymbol{x}^*$ is not the solution of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ then there is k such that $\max_j(a_{kj} + x^*{}_j) < b_k$. This is equivalent to $-x^*{}_j > -b_k + a_{kj}\ , \forall j$. Since $-x^*{}_j = \max(-b_l + a_{lj})$ for some l, then there is no element in the k^{th} row of $R_{A,b}$ thas is 1. ■

In order to determine the uniqueness of the solution of system of max-plus linear equations, we give the definition as follows

Definition 2.1. *The 1 in a row of* $R_{A,b}$ *is a* ***variable-fixing entry*** *if either*

1. *It is the only* ***1*** *in that row (a* ***lone-one)****, or*
2. *It is in the same column as a* ***lone-one.***

The remaining ***1s*** *are called* ***slack entries.***

The 1s that are circled in the above examples are the variable-fixing entries.

Theorem 2.2. [4] *Given the system* $A \otimes \boldsymbol{x} = \boldsymbol{b}$ *where* $A \in \mathbb{R}_{max}^{m\times n}$ *with the entries of each column are not all equal* ε *and* $\boldsymbol{b} \in \mathbb{R}^m$ *and the solution to the system exist.*

1. *If each row of* $R_{A,b}$ *has a* ***lone one****, then the solution of the systemis unique*
2. *If there are* ***slack entries*** *in* $R_{A,b}$, *then the system has infinite solutions.*

Corollary 2.1. *Given the system* $A \otimes \boldsymbol{x} = \boldsymbol{b}$ *where* $A \in \mathbb{R}_{max}^{m\times n}$ *with the entries of each column are not all equal* ε *and* $\boldsymbol{b} \in \mathbb{R}^m$ *and* $m < n$. *If there is no zero-row in* $R_{A,b}$ *then there are infinite solutions of the system.*

Proof. Matrix $R_{A,b}$ has no zero-row so there is at least one **1** in each row of $R_{A,b}$. Suppose that the solution of the system is unique then there is a ***lone one*** in each row of $R_{A,b}$. Meanwhile, $m < n$ which means that there are more variables than more equations in that system. Hence, there must be *slack entries* in $R_{A,b}$. This contradicts with the solution of the system is unique. So, there are infinite solutions of the system. ■

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