INTERNATIONAL CONFERENCE ON RESEARCH, IMPLEMENTATION AND EDUCATION OF MATHEMATICS AND SCIENCES 2014


Global Trends and Issues on Mathematics and Sciences and the Education

## PROCEEDING

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## Table of Content

|  | page |
| :--- | :---: |
| Front Cover | i |
| Editorial Board and Reviewers | ii |
| Preface | iii |
| Forewords From The Head of Committee | iv |
| Forewords From The Dean of Faculty | vi |
|  |  |
| Table of Content | ix |

Plenary Session
$\begin{array}{ll}\text { Using Dynamic Visual Representations To Discover Possible } & \text { I-1 } \\ \text { Solutions In Solving Real-Life Open-Ended Problems }\end{array}$
Prof. Tran Vui

## Parallel Session

## MATHEMATICS

01 Probability Density Function of M/G/1 Queues under (0,k)
Control Policies: A Special Case
Isnandar Slamet, Ritu Gupta, Narasimaha R. Achuthan
$\boldsymbol{C}[\boldsymbol{a}, \boldsymbol{b}]$-Valued Measure And Some Of Its Properties
Firdaus Ubaidillah, Soeparna Darmawijaya, Ch. Rini Indrati
Applied Discriminant Analysis in Market Research
Hery Tri Sutanto
04 Characteristic Of Group Of Matrix 3x3 Modulo P, P A Prime Number

Ibnu Hadi, Yudi Mahatma
05 The Properties Of Group Of $\mathbf{3} \times \mathbf{3}$ Matrices Over Integers
M-31
Modulo Prime Number
Ibnu Hadi, Yudi Mahatma
06 Random Effect Model And Generalized Estimating Equations M-37 For Binary Panel Response

## Jaka Nugraha

07 The Properties of Ordered Bilinear Form Semigroup ..... M-45 in Term of Fuzzy Quasi-Ideals
Karyati, Dhoriva Urwatul Wutsqa
08 Symmetry Of Limit Cycles On A Liénard-Type Dynamical ..... M-55 System
Kus Prihantoso Krisnawan
09 Systems Of Interval Min-Plus Linear Equations And Its ..... M-61
Application On Shortest Path Problem With Interval Travel TimesM. Andy Rudhito and D. Arif Budi Prasetyo
10 Some Properties Of Primitive $\boldsymbol{\vartheta}$-Henstock Of Integrable ..... M-69 Function In Locally Compact Metric Space Of Vector Valued Function
Manuharawati
11 Learning Gauss-Jordan Elimination Using Ms Excel ..... M-77
Meifry Manuhutu
12 Linear Matrix Inequality Based Proportional Integral ..... M-83
Derivative Control For High Order Plant M. Khairudin
13 Bayesian With Full Conditional Posterior Distribution ..... M-93
Approach For Solution Of Complex Models
Pudji Ismartini
14 Optimal Control Analyze And Equilibrium Existence Of Seir ..... M-101 Epidemic Model With Bilinear Incidence And Time Delay In State And Control VariablesRubono Setiawan
15 Selection Of The Best Univariate Normality Test On The ..... M-111 Category Of Moments Using Monte Carlo Simulation
Sugiyanto and Etik Zukhronah
16 Additive Main Effect and Multiplicative Interaction on Fixed ..... M-119 Model of Two Factors Design
Suwardi Annas and Selfi Dian Purtanti
17 Gram-Schmidt Super Orthogonalization Process For Super ..... M-127 Linear Algebra

# SYSTEMS OF INTERVAL MIN-PLUS LINEAR EQUATIONS AND ITS APPLICATION ON SHORTEST PATH PROBLEM WITH INTERVAL TRAVEL TIMES 

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#### Abstract

The travel times in a network are seldom precisely known, and then could be represented into the interval of real number, that is called interval travel times. This paper discusses the solution of the iterative systems of interval min-plus linear equations its application on shortest path problem with interval travel times. The finding shows that the iterative systems of interval min-plus linear equations, with coefficient matrix is semi-definite, has a maximum interval solution. Moreover, if coefficient matrix is definite, then the interval solution is unique. The networks with interval travel time can be represented as a matrix over interval min-plus algebra. The networks dynamics can be represented as an iterative system of interval min-plus linear equations. From the solution of the system, can be deter-mined interval earliest starting times for each point can be traversed. Furthermore, we can determine the interval fastest time to traverse the network. Finally, we can determine the shortest path interval with interval travel times by determining the shortest path with crisp travel times.


Key words: Min-Plus Algebra, Linear System, Shortest Path, Interval.

## INTRODUCTION

Let $\mathbf{R}_{\varepsilon}:=\mathbf{R} \cup\{\varepsilon\}$ with $\mathbf{R}$ the set of all real numbers and $\varepsilon:=\infty$. In $\mathbf{R}_{\varepsilon}$ defined two operations : $\forall a, b \in \mathbf{R}_{\varepsilon}, a \oplus b:=\min (a, b)$ and $a \otimes b:=a+b$. We can show that $\left(\mathbf{R}_{\varepsilon}, \oplus, \otimes\right)$ is a commutative idempotent semiring with neutral element $\varepsilon=\infty$ and unity element $\boldsymbol{e}=0$. Moreover, $\left(\mathbf{R}_{\delta}, \oplus, \otimes\right)$ is a semifield, that is $\left(\mathbf{R}_{\delta}, \oplus, \otimes\right)$ is a commutative semiring, where for every $\mathbf{a} \in \mathbf{R}$ there exist -a such that $\boldsymbol{a} \otimes(-a)=0$. Thus, $\left(\mathbf{R}_{\odot} \oplus, \otimes\right)$ is amin-plus algebra, and is written as $\mathbf{R}_{\text {min }}$. One can define $x^{\otimes^{0}}:=0, x^{\otimes^{k}}:=x \otimes x^{\otimes^{k-1}}, \varepsilon^{\otimes^{0}}:=0$ and $\varepsilon^{\otimes^{k}}:=\varepsilon$, for $k=1,2$, ... ..The operations $\oplus$ and $\otimes$ in $\mathbf{R}_{\text {min }}$ can be extend to the matrices operations in $\mathbf{R}_{\text {min }}^{m \times n}$, with $\mathbf{R}_{\min }^{m \times n}:=\left\{A=\left(A_{i j}\right) \mid A_{i j} \in \mathbf{R}_{\text {min }}\right.$, for $i=1,2, \ldots, m$ and $\left.j=1,2, \ldots, n\right\}$, the set of all matrices over max-plus algebra. Specifically, for $A, B \in \mathbf{R}_{\min }^{n \times n}$ we define $(A \oplus B)_{i j}=A_{i j} \oplus B_{i j}$ and $\quad(A \otimes B)_{i j}=$ $\bigoplus_{k=1}^{n} A_{i k} \otimes B_{k j}$. We also define matrix $E \in \mathbf{R}_{\min }^{n \times n},(E)_{i j}:=\left\{\begin{array}{l}0, \text { if } i=j \\ \varepsilon, \text { if } i \neq j\end{array}\right.$ and $\varepsilon \in \mathbf{R}_{\min }^{m \times n},(\varepsilon)_{i j}:=\varepsilon$ for every $i$ and $j$. For anymatrices $A \in \mathbf{R}_{\min }^{n \times n}$, one can define $A^{\otimes^{0}}=E_{n}$ and $A^{\otimes^{k}}=A \otimes A^{\otimes k-1}$ for $k=$
$1,2, \ldots$. For any weighted, directed graph $\mathcal{G}=(\mathcal{V}, \mathcal{A})$ we can define a matrix $A \in \mathbf{R}_{\min }^{n \times n}, A_{i j}=$ $\left\{\begin{array}{ll}m(j, i) & \text { if }(j, i) \in \mathcal{A} \\ \varepsilon, & \text { if }(j, i) \notin \mathcal{A} .\end{array}\right.$, called the weight-matrix of graph $G$.

A matrix $A \in \mathbf{R}_{\min }^{n \times n}$ is said to be semi-definite if all of circuit in $g(A)$ have nonnegative weight, and it is said definite if all of circuit in $g(A)$ have positive weight. We can show that if any matrices $A$ is semi-definite, then $\forall p \geq n, A^{\otimes^{p}} \preceq_{\mathrm{m}} E \oplus A \oplus \ldots \oplus A^{\otimes^{n-1}}$. So, we can define $A^{*}:=$
 $n\}$. Notice that we can be seen $\mathbf{R}_{\min }^{n}$ as $\mathbf{R}_{\min }^{n \times 1}$. The elements of $\mathbf{R}_{\max }^{n}$ is called vector over $\mathbf{R}_{\min }$.In general, min-plus algebra is analogous to max-plus algebra. Further details about max-plus algebra, matrix and graph can be found in Baccelli et.al (2001) and Rudhito (2003).

The existence and uniqueness of the solution of the iterative system of min-plus linear equation and its application to determine the shortest path in the with crisp (real) travel times had been discussed in Rudhito (2013). The followings are some result in brief. Let $\boldsymbol{A} \in \mathbf{R}_{\min }^{n \times n}$ and $\boldsymbol{b} \in$ $\mathbf{R}_{\text {min }}^{n \times 1}$.If $A$ is semi-definite, then $\boldsymbol{x}^{*}=A^{*} \otimes \boldsymbol{b}$ is a solution of system $\boldsymbol{x}=\boldsymbol{A} \otimes \boldsymbol{X} \oplus \boldsymbol{b}$. Moreover, if $\boldsymbol{A}$ is definite, then the system has a unique solution.A oneway path networkSvith crisp activity times, is a directed, strongly connected, acyclic, crisp weighted graph $S=(\mathcal{V}, \mathcal{A})$, with $V=\{1,2$,, $\ldots, n\}$ suct that if $(i, j) \in \mathcal{A}$, then $i<j$. In this network, point represent crosspathway, arc expresses a pathway, while the weight of the arc representtravel time, so that the weights in the network is always positive.Let $X_{i}^{e}$ is earliest starting timesfor point $\boldsymbol{i}$ can be traversedand $\boldsymbol{x}^{\boldsymbol{e}}=[$ $\left.x_{1}^{e}, x_{2}^{e}, \ldots, x_{n}^{e}\right]^{\mathrm{T}}$. For the network with crisptravel times, with modes and Athe weight matrix of graph of the networks, then

$$
\boldsymbol{x}^{\boldsymbol{e}}=\left(E \oplus \boldsymbol{A} \oplus \ldots \oplus A^{\otimes n-1}\right) \otimes \boldsymbol{b}^{\boldsymbol{e}}=\boldsymbol{A}^{*} \otimes \boldsymbol{b}^{\boldsymbol{e}}
$$

with $\boldsymbol{b}^{\boldsymbol{e}}=[0, \varepsilon, \ldots, \varepsilon]^{\mathrm{T}}$. Furthermore, $x_{n}^{e}$ is the fastest times to traverse the network.Let $\boldsymbol{x}_{i}^{\prime}$ is be latest times leftpoint $i$ and $\boldsymbol{x}^{\prime}=\left[x_{1}^{\prime}, x_{2}^{\prime} \ldots ., x_{n}^{\prime}\right]$. For the network above, vector

$$
\boldsymbol{x}^{\prime}=-\left(\left(\boldsymbol{A}^{\mathrm{T}}\right)^{*} \otimes \boldsymbol{b}\right)
$$

with $\boldsymbol{b}=\left[\varepsilon, \varepsilon, \ldots,-x_{n}^{e}\right]^{\mathrm{T}}$.Define, a pathway $(i, j) \in \mathcal{A}$ in the one-way path network $S$ is called shortestpathwayif $x_{i}^{e}=x_{i}^{\prime}$ dan $x_{j}^{e}=x_{j}^{\prime}$. Define, A path $p \in$ Pin the one-way path network $S$ is calledshortest pathif all pathways belonging to p are shortest pathway. From this definition, we can show that apath $p \in$ Pis a shortest path if and only if $p$ has minimum weight, that is equal to $x_{n}^{e}$ . Also, a pathway is a shortest pathway if and only if it belonging to a shortest path.

## DISCUSSION

We discusses the solution of the iterative systems of interval min-plus linear equations its application on shortest path problem with interval travel times. The discussion begins by reviewing some basic concepts of interval min-plus algebra and matrices over interval min-plus algebra. Definition and concepts in the min-plus algebra analogous to the concepts in the maxplus algebra which can be seen in Rudhito (2011).

The (closed) interval x in $\mathbf{R}_{\text {min }}$ is a subset of $\mathbf{R}_{\text {min }}$ of the form

$$
\mathrm{x}=[\underline{\mathrm{x}}, \overline{\mathrm{x}}]=\left\{\boldsymbol{x} \in \mathbf{R}_{\mathrm{min}} \mid \underline{\mathrm{x}} \preceq_{\mathrm{m}} \boldsymbol{x} \preceq_{\mathrm{m}} \overline{\mathrm{x}}\right\} .
$$

The interval x in $\mathbf{R}_{\text {min }}$ is called min-plusinterval, which is in short is called interval.Define

$$
\mathbf{I}(\mathbf{R})_{\varepsilon}:=\left\{\mathrm{x}=[\underline{\mathrm{x}}, \overline{\mathrm{x}}] \mid \underline{\mathrm{x}}, \overline{\mathrm{x}} \in \mathbf{R}, \varepsilon \prec_{\mathrm{m}} \underline{\mathrm{x}} \preceq_{\mathrm{m}} \overline{\mathrm{x}}\right\} \cup\{\varepsilon\} \text {, where } \varepsilon:=[\varepsilon, \varepsilon] \text {. }
$$

In the $\mathbf{I}(\mathbf{R})_{\varepsilon}$, define operation $\bar{\oplus}$ and $\bar{\otimes}$ as

$$
\mathrm{x} \bar{\oplus} \mathrm{y}=[\underline{\mathrm{x}} \oplus \underline{y}, \overline{\mathrm{x}} \oplus \overline{\mathrm{y}}] \text { and } \mathrm{x} \bar{\otimes} \mathrm{y}=[\underline{\mathrm{x}} \otimes \underline{y}, \overline{\mathrm{x}} \otimes \overline{\mathrm{y}}], \forall \mathrm{x}, \mathrm{y} \in \mathbf{I}(\mathbf{R})_{\varepsilon} .
$$

Since $\left(\mathbf{R}_{s}, \oplus, \otimes\right)$ is an idempotent semiring and it has no zero divisors, with neutral element $\varepsilon$, we can show that $\mathbf{I}(\mathbf{R})_{\varepsilon}$ is closed with respect to the operation $\bar{\oplus}$ and $\bar{\otimes}$. Moreover, $\left(\mathbf{I}(\mathbf{R})_{\varepsilon}, \bar{\oplus}\right.$, $\bar{\otimes})$ is a comutative idempotent semiring with neutral element $\varepsilon=[\varepsilon, \varepsilon]$ and unity element $0=[0$, $0]$. This comutative idempotent semiring $\left(\mathbf{I}(\mathbf{R})_{\varepsilon}, \bar{\oplus}, \bar{\otimes}\right)$ is called interval min-plus algebra which is written as $\mathbf{I}(\mathbf{R})_{\text {min }}$.

Define $\mathbf{I}(\mathbf{R})_{\text {min }}^{m \times n}:=\left\{\mathrm{A}_{=}\left(\mathrm{A}_{i j}\right) \mid \mathrm{A}_{i j} \in \mathbf{I}(\mathbf{R})_{\text {min }}\right.$, for $i=1,2, \ldots, m$ and $\left.j=1,2, \ldots, n\right\}$. The element of $\mathbf{I}(\mathbf{R})_{\min }^{m \times n}$ are called matrices over interval min-plus algebra. Furthermore, this matrices are called interval matrices. The operations $\bar{\oplus}$ and $\bar{\otimes}$ in $\mathbf{I}(\mathbf{R})_{\min }$ can be extended to the matrices operations of in $\mathbf{I}(\mathbf{R})_{\text {max }}^{m \times n}$. Specifically, for $\mathrm{A}, \mathrm{B} \in \mathbf{l}(\mathbf{R})_{\text {min }}^{n \times n}$ and $\alpha \in \mathbf{l}(\mathbf{R})_{\text {min }}$ we define

$$
(\alpha \bar{\otimes} \mathrm{A})_{i j}=\alpha \bar{\otimes} \mathrm{A}_{i j},(\mathrm{~A} \oplus \mathrm{~B})_{i j}=\mathrm{A}_{i j} \bar{\oplus} \mathrm{~B}_{i j} \text { and }(\mathrm{A} \bar{\otimes} \mathrm{~B})_{i j}=\frac{n}{\bigoplus_{k=1}^{n}} \mathrm{~A}_{i k} \bar{\otimes} \mathrm{~B}_{k j} .
$$

MatricesA, $\mathrm{B} \in \mathbf{I}(\mathbf{R})_{\min }^{m \times n}$ are equalifA $\mathrm{A}_{i j}=\mathrm{B}_{i j}$, that is if $\mathrm{A}_{i j}=\mathrm{B}_{i j}$ and $\overline{\mathrm{A}_{i j}}=\overline{\mathrm{B}_{i j}}$ for everyiandj. We can show that $\left(\mathbf{I}(\mathbf{R})_{\min }^{n \times n}, \bar{\oplus}, \bar{\otimes}\right)$ is a idempotent semiring with neutral element is matrix $\varepsilon$, with $(\varepsilon)_{i j}=\varepsilon$ for every $i$ and $j$, and unity element is matrix $E$, with $(\mathrm{E})_{i j}:=\left\{\begin{array}{l}0, \text { if } i=j \\ \varepsilon, \text { if } i \neq j\end{array}\right.$. We can also show that $\mathbf{I}(\mathbf{R})_{\min }^{m \times n}$ is a semi-module over $\mathbf{I}(\mathbf{R})_{\text {min }}$.
For any matrix $\mathrm{A} \in \mathbf{l}(\mathbf{R})_{\text {min }}^{m \times n}$, define the matrices $\underline{\mathrm{A}}=\left(\underline{\mathrm{A}_{i j}}\right) \in \mathbf{R}_{\text {min }}^{m \times n}$ and $\overline{\mathrm{A}}=\left(\overline{\mathrm{A}_{i j}}\right) \in \mathbf{R}_{\text {min }}^{m \times n}$, which is called lower bound matrices and upper bound matrices ofA, respectively. Definematrices interval of A , that is

$$
[\underline{\mathrm{A}}, \overline{\mathrm{~A}}]=\left\{A \in \mathbf{R}_{\min }^{m \times n} \mid \underline{\mathrm{A}} \preceq_{\mathrm{m}} A \preceq_{\mathrm{m}} \overline{\mathrm{~A}}\right\} \text { and } \mathbf{I}\left(\mathbf{R}_{\min }^{m \times n}\right)^{*}=\left\{[\underline{\mathrm{A}}, \overline{\mathrm{~A}}] \mid \mathrm{A} \in \mathbf{I}(\mathbf{R})_{\min }^{n \times n}\right\} .
$$

Specifically, for $[\underline{\mathrm{A}}, \overline{\mathrm{A}}],[\underline{\mathrm{B}}, \overline{\mathrm{B}}] \in \mathbf{l}\left(\mathbf{R}_{\text {min }}^{m \times n}\right)^{*}$ and $\alpha \in \mathbf{l}(\mathbf{R})_{\text {min }}$ we define

$$
\begin{aligned}
\alpha \bar{\otimes}[\underline{\mathrm{A}}, \overline{\mathrm{~A}}]= & {[\underline{\alpha} \otimes \underline{\mathrm{A}}, \bar{\alpha} \otimes \overline{\mathrm{~A}}],[\underline{\mathrm{A}}, \overline{\mathrm{~A}}] \bar{\oplus}[\underline{\mathrm{B}}, \overline{\mathrm{~B}}]=[\underline{\mathrm{A}} \oplus \underline{\mathrm{~B}}, \overline{\mathrm{~A}} \oplus \overline{\mathrm{~B}}] } \\
& \text { and }[\underline{\mathrm{A}}, \overline{\mathrm{~A}}] \bar{\otimes}[\underline{\mathrm{B}}, \overline{\mathrm{~B}}]=[\underline{\mathrm{A}} \otimes \underline{\mathrm{~B}}, \overline{\mathrm{~A}} \otimes \overline{\mathrm{~B}}] .
\end{aligned}
$$

The matrices interval $[\underline{A}, \overline{\mathrm{~A}}]$ and $[\underline{\mathrm{B}}, \overline{\mathrm{B}}] \in \mathbf{I}\left(\mathbf{R}_{\text {min }}^{m \times n}\right)^{*}$ are equalif $\underline{\mathrm{A}}=\underline{\mathrm{B}}$ and $\overline{\mathrm{A}}=\overline{\mathrm{B}}$. We can show that $\left(\mathbf{I}\left(\mathbf{R}_{\min }^{n \times n}\right)^{*}, \bar{\oplus}, \bar{\otimes}\right)$ is an idempotent semiring with neutral element matrix interval $[\varepsilon$, $\varepsilon$ ] and the unity element is matrix interval [E, E]. We can also show that $\mathbf{I}\left(\mathbf{R}_{\text {min }}^{n \times n}\right)^{*}$ is a semimodule over $\mathbf{I}(\mathbf{R})_{\text {min }}$.

The semiring $\left(\mathbf{I}(\mathbf{R})_{\min }^{n \times n}, \bar{\oplus}, \bar{\otimes}\right)$ is isomorfic with semiring $\left(\mathbf{I}\left(\mathbf{R}_{\text {min }}^{n \times n}\right)^{*}, \bar{\oplus}, \bar{\otimes}\right)$. We can define a mapping $f$, where $f(\mathrm{~A})=[\underline{\mathrm{A}}, \overline{\mathrm{A}}], \forall \mathrm{A} \in \mathbf{I}(\mathbf{R})_{\min }^{n \times n}$. Also, the semimodule $\mathbf{I}(\mathbf{R})_{\min }^{n \times n}$ is isomorfic with semimodule $\mathbf{I}\left(\mathbf{R}_{\min }^{n \times n}\right)^{*}$. So, for every matrices interval $\mathrm{A} \in \mathbf{l}\left(\mathbf{R}_{\min }^{n \times n}\right)^{*}$ we can
determine matrices interval $[\underline{\mathrm{A}}, \overline{\mathrm{A}}] \in \mathbf{l}\left(\mathbf{R}_{\text {min }}^{n \times n}\right)^{*}$. Conversely, for every $[\underline{\mathrm{A}}, \overline{\mathrm{A}}] \in \mathbf{l}\left(\mathbf{R}_{\text {min }}^{n \times n}\right)^{*}$,then $\underline{\mathrm{A}}, \overline{\mathrm{A}} \in \mathbf{R}_{\min }^{n \times n}$, such that $\left[\underline{\mathrm{A}}_{i j}, \overline{\mathrm{~A}}_{i j}\right] \in \mathbf{l}(\mathbf{R})_{\text {min }}, \forall i$ and $j$. The matrix interval $[\underline{\mathrm{A}}, \overline{\mathrm{A}}]$ is called matrix interval associated with the intervalmatrix A and which is written $\mathrm{A} \approx[\mathrm{A}, \overline{\mathrm{A}}]$. So we have $\alpha \bar{\otimes} \mathrm{A} \approx[\underline{\alpha} \otimes \underline{\mathrm{A}}, \bar{\alpha} \otimes \overline{\mathrm{A}}], \mathrm{A} \oplus \mathrm{B} \approx[\underline{\mathrm{A}} \oplus \underline{\mathrm{B}}, \overline{\mathrm{A}} \oplus \overline{\mathrm{B}}]$ and $\mathrm{A} \bar{\otimes} \mathrm{B} \approx[\underline{\mathrm{A}} \otimes \underline{\mathrm{B}}, \overline{\mathrm{A}} \otimes \overline{\mathrm{B}}]$.

We define for any interval matrices $\mathrm{A} \in \mathbf{l}(\mathbf{R})_{\text {min }}^{n \times n}$, where $\mathrm{A} \approx[\underline{\mathrm{A}}, \overline{\mathrm{A}}]$, is said to be semidefinite (definite) if every matrices $A \in[\underline{A}, \overline{\mathrm{~A}}]$ is semi-definite (definite). We can show that interval matrices $\mathrm{A} \in \mathbf{I}(\mathbf{R})_{\text {max }}^{n \times n}$, where $\mathrm{A} \approx[\underline{\mathrm{A}}, \overline{\mathrm{A}}]$ is semi-definite (definite) if and only if $\overline{\mathrm{A}} \in \mathbf{R}$ ${ }_{\substack{n \times n \\ \text { max }}}$ semi-definite (definite).
$\operatorname{Define} \mathbf{I}(\mathbf{R})_{\text {min }}^{n}:=\left\{\mathbf{x}=\left[\mathrm{x}_{1}, \mathrm{x}_{2}, \ldots, \mathrm{x}_{n}\right]^{\mathrm{T}} \mid \mathrm{x}_{i} \in \mathbf{I}(\mathbf{R})_{\text {min }}, i=1,2, \ldots, n\right\}$. The set $\mathbf{l}(\mathbf{R})_{\text {min }}^{n}$ can be seen as set $\mathbf{I}(\mathbf{R})_{\min }^{n \times 1}$. The Elements of $\mathbf{I}(\mathbf{R})_{\text {min }}^{n}$ is called interval vector over $\mathbf{I}(\mathbf{R})_{\text {min }}$. The interval vector $\mathbf{x}$ associated with vector interval $[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$, that is $\mathbf{x} \approx[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$.

Definition1. Let $\in \mathbf{I}(\mathbf{R})_{\text {min }}^{n \times n}$ andb $\in \mathbf{I}(\mathbf{R})_{\text {min }}^{n} . A$ interval vector $\mathbf{x}^{*} \in \mathbf{I}(\mathbf{R})_{\text {min }}^{n}$ is calledinterval solutionof iterative system of interval min-plus linear equations $\mathbf{x}=\mathrm{A} \bar{\otimes} \mathbf{x} \quad \bar{\oplus}$ bifx* satisfy the system.

Theorem 1. Let $A \in \mathbf{I}(\mathbf{R})_{\text {max }}^{n \times n}$ and $\mathbf{b} \in \mathbf{I}(\mathbf{R})_{\text {min }}^{n \times 1}$. IfA is semi-definite, then intervalvector $\mathbf{x}^{*} \approx[$ $\left.\underline{A}^{*} \otimes \underline{\mathbf{b}}, \overline{\mathrm{~A}}^{*} \otimes \overline{\mathbf{b}}\right]$, is an interval solution of syster $\mathbf{x}=\mathrm{A} \bar{\otimes} \mathbf{x} \oplus \mathbf{b}$. Moreover, if A is definite, then interval solution is unique.
Proof.Proof is analogous to the case of max-plus algebra as seen in the Rudhito (2011)
Next will be discussed theearliest starting times intervalfor point $i$ can be traversed.The discussion is analogous to the case of (crisp) travel time (Rudhito, 2013), using the interval minplus algebra approach.
LetES $=\mathrm{x}_{i}^{e}$ is earliest starting times intervalfor point $i$ can be traversed, with $\mathrm{x}_{i}^{e}=\left[\underline{\mathrm{x}}_{i}^{e}, \overline{\mathrm{x}}_{i}^{e}\right]$.
$\mathrm{A}_{i j}=\left\{\begin{array}{cc}\text { intervaltraveltimefrompoint } j \text { to point } i \text { if }(j, i) \in \mathcal{A} \\ \varepsilon(=[+\infty,+\infty]) & \text { if }(j, i) \notin \mathcal{A}\end{array}\right.$.
We assume that $\mathrm{X}_{i}^{e}=0=[0,0]$ and with interval min-plus algebra notation we have

$$
\mathrm{x}_{i}^{e}= \begin{cases}0 & \text { if } i=1  \tag{1}\\ \overline{\bigoplus_{1 \leq j \leq n}}\left(\mathrm{~A}_{i j} \bar{\otimes} \mathrm{x}_{j}^{e}\right) & \text { if } i>1 .\end{cases}
$$

Let Ais the interval weight matrix of the interval-valued weighted graph of the networks, $\mathbf{x}^{e}=$ [ $\left.\mathrm{x}_{1}^{e}, \mathrm{x}_{2}^{e}, \ldots, \mathrm{x}_{n}^{e}\right]^{\mathrm{T}}$ dan $\mathbf{b}^{e}=[0, \varepsilon, \ldots, \varepsilon]^{\mathrm{T}}$, then equation (1) can be written in an iterative system of interval max-plus linear equations

$$
\begin{equation*}
\mathbf{x}^{e}=\mathrm{A} \bar{\otimes} \mathbf{x}^{e} \bar{\otimes} \mathbf{b}^{e} \tag{2}
\end{equation*}
$$

Since the project networks is acyclic directed graph, then there are no circuit, so according to the result in Rudhito(2011), Ais definite. And then according toTheorem 1,

$$
\begin{aligned}
\mathbf{x}^{e}=\mathrm{A}^{*} \bar{\otimes} \mathbf{b}^{e} \approx & {\left[\underline{\mathrm{~A}}^{*} \otimes \underline{\mathbf{b}}^{e}, \overline{\mathrm{~A}}^{*} \otimes \overline{\mathbf{b}}^{e}\right] } \\
& =\left[\left(\underline{\mathrm{E}} \oplus \underline{\mathrm{~A}} \oplus \ldots \underline{\mathrm{~A}}^{\otimes^{n-1}}\right) \otimes \underline{\mathbf{b}}^{e},\left(\overline{\mathrm{E}} \oplus \overline{\mathrm{~A}} \oplus \ldots \oplus \overline{\mathrm{~A}}^{\otimes^{n-1}}\right) \otimes \overline{\mathbf{b}}^{e}\right]
\end{aligned}
$$

is a unique solution of the system(2), that is the vector of earliest starting times intervalfor point $i$ can be traversed.

Notice that $\mathrm{x}_{n}^{e}$ is the fastest times interval to traverse the network. We summarize the description above in the Theorem 2.

Teorema 2. Given a one-way path networknetwork with interval travel times, with n node and Ais the weight matrix of the interval-valued weighted graph of networks. The interval vector of earliest starting times intervalfor point i can be traversed is given by

$$
\mathbf{x}^{e} \approx\left[\left(\underline{\mathrm{E}} \oplus \underline{\mathrm{~A}} \oplus \ldots \underline{\mathrm{~A}}^{\otimes^{n-1}}\right) \otimes \underline{\mathbf{b}}^{e},\left(\overline{\mathrm{E}} \oplus \overline{\mathrm{~A}} \oplus \ldots \oplus \overline{\mathrm{~A}}^{\otimes^{n-1}}\right) \otimes \overline{\mathbf{b}}^{e}\right]
$$

with $\mathbf{b}^{\mathbf{e}}=[0, \varepsilon, \ldots, \varepsilon]^{\mathrm{T}}$. Furthermore, $\mathrm{x}_{n}^{e}$ is thefastest timesinterval to traverse the network.
Bukti: (see description above) .
Example 1 Consider the project network in Figure 1.


Figure 1. A one-way path network network with interval travel times
We have

$$
\mathrm{A}=\left[\begin{array}{ccccccc}
\varepsilon, & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
{[1,3]} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
{[2,4]} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & {[2,3]} & {[3,5]} & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & {[2,3]} & {[0,0]} & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & {[2,3]} & {[4,7]} & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & {[7,9]} & {[5,8]} & {[6,8]} & \varepsilon
\end{array}\right] .
$$

Using MATLAB computer program, we have

$$
\begin{aligned}
& \underline{\mathrm{A}}^{*}=\left[\begin{array}{lllllll}
0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
1 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
2 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
3 & 2 & 3 & 0 & \varepsilon & \varepsilon & \varepsilon \\
3 & 2 & 2 & 0 & 0 & \varepsilon & \varepsilon \\
5 & 4 & 5 & 2 & 4 & 0 & \varepsilon \\
8 & 7 & 7 & 5 & 5 & 6 & 0
\end{array}\right], \overline{\mathrm{A}}^{*}=\left[\begin{array}{ccccccc}
0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
3 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
4 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
6 & 3 & 5 & 0 & \varepsilon & \varepsilon & \varepsilon \\
6 & 3 & 3 & 0 & 0 & \varepsilon & \varepsilon \\
9 & 6 & 8 & 3 & 7 & 0 & \varepsilon \\
14 & 11 & 11 & 8 & 8 & 8 & 0
\end{array}\right], \\
& \underline{\mathbf{x}}^{e}=[0,1,2,3,3,5,8]^{\mathrm{T}} \operatorname{dan} \overline{\mathbf{X}}^{e}=[0,3,4,6,6,9,14]^{\mathrm{T}} .
\end{aligned}
$$

So the vector of earliest starting times intervalfor point $i$ can be traversed is
$\mathbf{x}^{e}=[[0,0],[1,3],[2,4],[3,6],[3,6],[5,9],[8,14]]^{\mathrm{T}}$ and the fastest times interval to traverse the network $\mathrm{x}_{n}^{e}=[16,25]$.

Next given shortest path interval definition and theorem that gives way determination. Definitions and results is a modification of the definition of critical path-interval and theorem to determine the critical path method-interval, as discussed in Chanas and Zielinski (2001) and Rudhito (2011).We also give some examples for illustration.

Definition 2.A pathp $\in P$ is called aninterval-shortest pathinSif there exist aset of travel times $A_{i j}$ $\in\left[\underline{A}_{i j}, \bar{A}_{i j}\right],(i, j) \in \mathcal{A}$,such thatp is shortest path, after replacing the interval travel timest $A_{i j}$ with the travel timeA $A_{i j}$.

Definisi 3.A pathway $(k, l) \in$ Ais called aninterval-shortest pathway in Sif there exist aset of travel times $A_{i j} \in\left[\underline{A}_{i j}, \bar{A}_{i j}\right],(i, j) \in \mathcal{A}$,such that( $(k, I)$ is shortest pathway, after replacing the interval travel timesA ${ }_{i j}$ with the travel timeA $A_{i j}$.

The following theorem is given which relates the interval-shortest path and intervalshortest pathway.

Teorema 3.If path $p \in P$ is an interval-shortest path, then all pathways in the $p$ are intervalshortest pathway.

Proof : Let path $p \in$ is an interval shortest path, then according to Definition 2, there exist a set of times $A_{i j} \in\left[\underline{A}_{i j}, \bar{A}_{i j}\right],(i, j) \in \boldsymbol{A}$,such that p is shortest path, after replacing the interval travel times $\mathrm{A}_{i j}$ with the travel time $\boldsymbol{A}_{i j}$. Next, according to the definition of shortest path above, all pathways in $p$ are shortest pathways for a set of travel times $A_{i j} \in\left[\underline{A}_{i j}, \bar{A}_{i j}\right],(i, j) \in \boldsymbol{A}$.Thus according to Definiton 3, all pathways in $p$ are interval-shorstest pathways.

The following theorem is givena necessary and sufficient condition a path is an intervalshortestpath.

Teorema 4.A pathp $\in P$ is aninterval-shortest pathinSif and only if $p$ is a shortest path, with interval travel times $\mathrm{A}_{i j} \in\left[\underline{A}_{i j}, \bar{A}_{i j}\right],(i, j) \in \mathcal{A}$, have been replace with travel times $A_{i j}$ which is determined by the following formula

$$
A_{i j}=\left\{\begin{array}{l}
\bar{A}_{i j} j \operatorname{jika}(i, j) \notin p  \tag{3}\\
\underline{A}_{i j} \operatorname{jika}(i, j) \in p .
\end{array}\right.
$$

Bukti : $\Rightarrow$ : Letp is an interval-shortest path, then according to Definition 2, there exist a set of travel times $A_{i j}, A_{i j} \in\left[\underline{A}_{i j}, \bar{A}_{i j}\right],(i, j) \in \mathcal{A}$, such that $p$ is shortest pathway, after replacing the interval travel times $A_{i j}$ with travel times $\mathcal{A}_{i j},(i, j) \in \mathcal{A}$. If the travel times for all pathwayis located at $p$ isreduced from $A_{i j}$ to $\underline{A}_{i j}$ and for all pathway is not located $p$ is increased from $A_{i j}$ to $\bar{A}_{i j}$, then $p$ is a path with minimum weight in Sor new travel time formation. Thus path $p$ is a shortest path. $\Leftarrow$ : Since pathp a shortest path with a set of travel times $A_{i j} \in\left[\underline{\mathrm{~A}}_{i j}, \overline{\mathrm{~A}}_{i j}\right]$,which is determined by the formula(9), then according to Definition 2, pathp is an interval-shortest path.

Example 2.We consider the network in Example 1. We will determine all interval-shortest path in this network. For path $1 \rightarrow 3 \rightarrow 5 \rightarrow 7$, by applying formula (9), we have weight

$$
\left[\begin{array}{lllllll}
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & 3 & 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & 2 & 0 & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & 3 & 8 & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & 9 & 4 & 8 & \varepsilon
\end{array}\right] .
$$

Using MATLAB computer program, we have a shortest path $1 \rightarrow 3 \rightarrow 5 \rightarrow 7$ with minimum weight of path is 8 . Thus $1 \rightarrow 3 \rightarrow 5 \rightarrow 7$ is an interval-shortest path. The results of the calculations for all possible path in the network are given in Table 1 below.

Tabel 1Calculation results of all path

| No | Path $\boldsymbol{p}$ | Weight <br> Interval $\boldsymbol{p}$ | Shortest-path $\boldsymbol{p}^{*}$ (with <br> formula (9)) | Weight <br> of $p^{*}$ | Conclusion |
| :---: | :--- | :---: | :--- | :---: | :--- |
| 1 | $1 \rightarrow 3 \rightarrow 5 \rightarrow 7$ | $[8,14]$ | $1 \rightarrow 3 \rightarrow 5 \rightarrow 7$, | 8 | Interval-shortest |
| 2 | $1 \rightarrow 3 \rightarrow 5 \rightarrow 6 \rightarrow 7$ | $[15,23]$ | $1 \rightarrow 3 \rightarrow 5 \rightarrow 7$ | 11 | Not interval-shortest |
| 3 | $1 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 7$ | $[9,16]$ | $1 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 7$ <br> $1 \rightarrow 3 \rightarrow 5 \rightarrow 7$ | 9 | Interval-shortest |
| 4 | $1 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7$ | $[16,25]$ | $1 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 7$ <br> $1 \rightarrow 3 \rightarrow 5 \rightarrow 7$ | 12 | Not interval-shortest |
| 5 | $1 \rightarrow 3 \rightarrow 4 \rightarrow 6 \rightarrow 7$ | $[13,20]$ | $1 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 7$ <br> $1 \rightarrow 3 \rightarrow 5 \rightarrow 7$ | 12 | Not interval-shortest |
| 6 | $1 \rightarrow 3 \rightarrow 4 \rightarrow 7$ | $[12,18]$ | $1 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 7$ <br> $1 \rightarrow 3 \rightarrow 4 \rightarrow 7$ <br> $1 \rightarrow 3 \rightarrow 5 \rightarrow 7$ | 12 | Interval-shortest |
| 7 | $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 7$ | $[7,13]$ | $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 7$ | 7 | Interval-shortest |
| 8 | $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7$ | $[14,22]$ | $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 7$ | 10 | Not interval-shortest |
| 9 | $1 \rightarrow 2 \rightarrow 4 \rightarrow 6 \rightarrow 7$ | $[11,17]$ | $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 7$ | 10 | Not interval-shortest |
| 10 | $1 \rightarrow 2 \rightarrow 4 \rightarrow 7$ | $[10,15]$ | $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 7$ <br> $1 \rightarrow 2 \rightarrow 4 \rightarrow 7$ | 10 | Interval-shortest |

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