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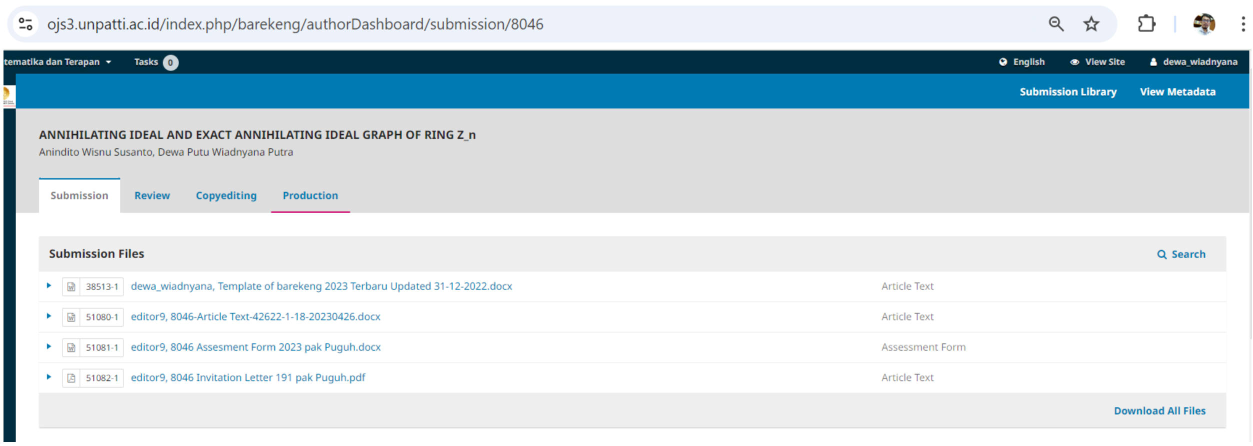

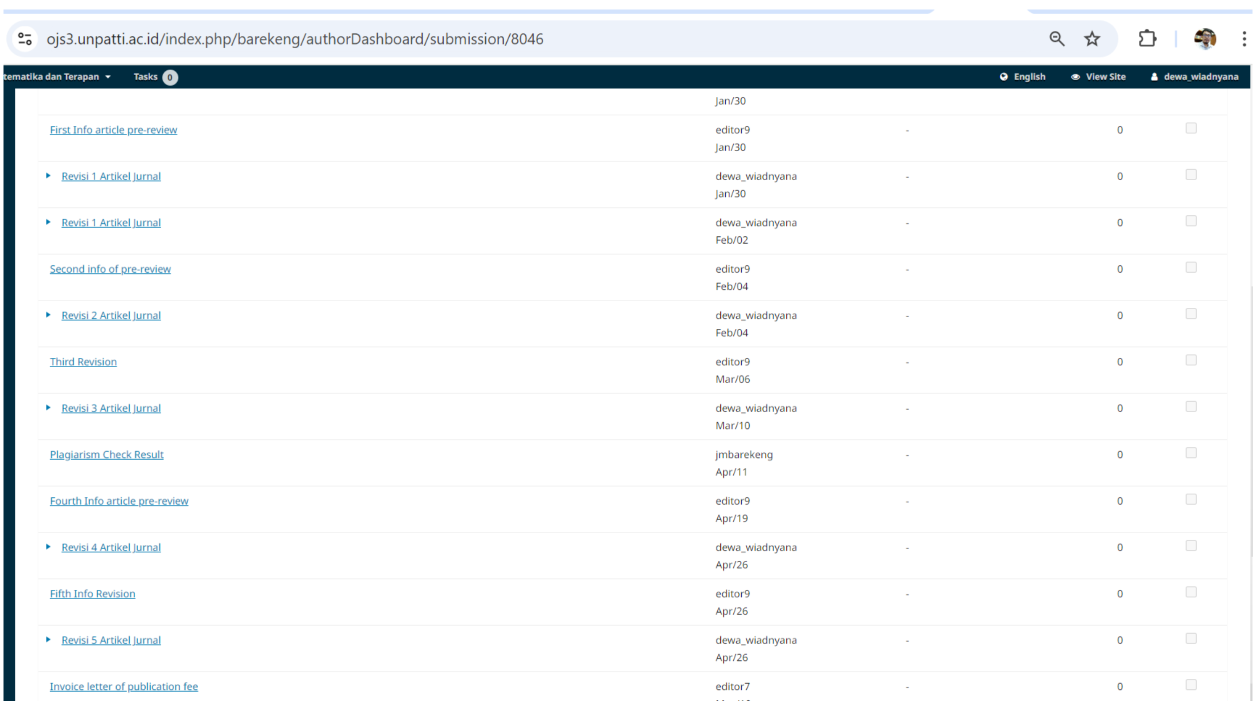

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ANNIHILATING IDEAL AND EXACT ANNIHILATING IDEAL GRAPH OF RING $\mathbb{Z}_n$

Anindito Wisnu Susanto [1], **Dewa Putu Wiadnyana Putra** [2*]

[1,2] *Department of Mathematics Education, Faculty of Teacher Training and Education, Sanata Dharma University*
Jl. Affandi, Mrican, Caturtunggal, Depok, Sleman, Yogyakarta, 55281, Indonesia

Corresponding author's e-mail: [2*] ***dewa@usd.ac.id***

ABSTRACT

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The existence of annihilator in the ring motivates the emergence of studies on Annihilating Ideal and Exact Annihilating Ideal Graphs. The purpose of this research is to describe the characteristics of an (exact) annihilating ideal of ring $\mathbb{Z}_n$. The method used in this research is literature study. The results of this study discuss finiteness, adjacency, connectedness, vertices, and types of $\mathbb{AG}(\mathbb{Z}_n)$ and $\mathbb{EAG}(\mathbb{Z}_n)$. Furthermore, the number of vertices of an Annihilating Ideal Graph is determined by the factorization of n. The adjacency of two vertices is determined by the divisibleness of n. The results also show that $\mathbb{EAG}(\mathbb{Z}_n)$ is a subgraph of $\mathbb{AG}(\mathbb{Z}_n)$. $\mathbb{EAG}(\mathbb{Z}_n)$ can be represented as a union of several complete graphs.

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1. INTRODUCTION

The use of graphs in representing algebraic structures has been carried out since at least 1878 in [1]. This representation starts from representing a group structure into a graph. The vertices of a graph are all elements of a group and changes to an element due to operations on the group are represented by directed edges. Furthermore, [2], [3] began to associate graphs with a broader structure, namely rings. Investigation of the ring structure is carried out through the colored representation of the graph. The representation of an algebraic structure on a graph opens up opportunities for visual investigation of the properties of a particular structure. An essential part in the process of representing a particular algebraic structure to a graph is how to define the connection between the vertices of the graph. Different ways of defining adjacent vertices can lead to different variations of properties as well.

One of the interesting things in the ring, which is about zero divisor. A non-zero element a is said to be a zero divisor if it can be found a non-zero element b such that $ab = 0$. From this structure, [4] proposed the origin of the zero divisor graph. The vertices of the graph are all zero divisors. Two vertices are adjacent if and only if the product of the two elements is zero. Many interesting properties result from this concept, one of which is about the combinatorics of a finite ring [5]–[7].

The concept is similar to zero divisor in the ring is the Annihilator. Badawi has started a study on annihilator graphs [8]. In its development, annihilating graphs are generalized into annihilating Ideal graphs. In [9], it is stated that Annihilator is an ideal I, namely $Ann(I) = \{r \in R | rl = 0 \, \forall \, l \in L\}$. If $Ann(I)$ is not a trivial set, then I is called an ideal annilator. In 2011, [10] started to represent a structure consisting of annihilator ideals into a graph. The graph that is formed is named Annihilating Ideal Graph. In line with the development of zero divisor graphs, [11] is continuing the study of Exact Annihilating Ideal graphs. The development of ideal annihilating and exact annihilating properties of graphs is studied separately. The general relationship between these two graphs began to be investigated by [12].

An integer modulo n, $\mathbb{Z}_n$ is a ring that has very interesting properties. This $\mathbb{Z}_n$ structure is widely used in graphs, for example in coloring Antimagic graphs [13] and Domination ratio [14]. The factorization theorem on integers is a motivation for developing graph studies involving a ring of integers modulo n. One of the graph studies carried out was a study on non-coprime for $\mathbb{Z}_n$ [15]. Based on this, the study of annihilating ideal graph for $\mathbb{Z}_n$ rings is interesting to do.

2. RESEARCH METHODS

This research is a literature research that examines the properties of annihilating ideal and exact annihilating ideal graphs on integer rings modulo n, $\mathbb{Z}_n$. The definition of (Exact) Annihilating Ideal based on [10], [11] is as follows.

Definition 1. Annihilating Ideal [10]
An Ideal I of commutative ring R with identity is a Annihilating Ideal if there exist non zero ideal J of R such that $IJ = 0$. The set of all Annihilating Ideal of ring R is denoted by $\mathbb{A}(R)$.

Definition 2. Exact Annihilating Ideal [11]
An ideal I of commutative ring R with identity is Exact Annihilating Ideal if there exist non zero ideal J of R such that $Ann(I) = J$ and $Ann(J) = I$. The set of all Exact Annihilating Ideal of ring R denoted by $\mathbb{EA}(R)$.

Based on the two definitions above, then (Exact) Annihilating Ideal Graph is defined as follows.

Definition 3. Annihilating Ideal Graph [10]
Annihilating Ideal graph of ring R denoted by $\mathbb{A}G(R)$ is a graph with vertices $\mathbb{A}(R)^* = \mathbb{A}(R)\backslash\{(0)\}$ and $(I, J) \in E(\mathbb{A}G(R))$ if and only if $IJ = (0)$.

Definition 4. Exact Annihilating Ideal Graph [11]
Exact Annihilating Ideal graph of ring R denoted by $\mathbb{EAG}(R)$ is a graph with vertices $\mathbb{EA}(R)^* = \mathbb{EA}(R)\backslash\{(0)\}$ and $(I, J) \in E(\mathbb{EA}G(R))$ if and only if $Ann(I) = J$ and $Ann(J) = I$.

Definition of (Exact) Annihilating Ideal Graph, this article further describes the properties of the graph with rings $\mathbb{Z}_n$. Comparison of the properties of annihilating ideal and exact annihilating ideal graph of ring $\mathbb{Z}_n$ is also presented in this article.

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3. RESULTS AND DISCUSSION

In this section, we will show some result of Annihilating ideal graph of ring $\mathbb{Z}_n$.

Theorem 1. Lower Bound of Cardinality $\mathbb{A}(\mathbb{Z}_n)^*$
Suppose $\mathbb{Z}_n$ ring of integer modulo n where n not prime.
1. If $n = p^2$, *where* p *is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.
2. If $n \neq p^2$, *where* p *is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$

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Proof.
(1) Suppose $n = p^2$ then there exists uniquely non zero proper ideal in di $\mathbb{Z}_n$, $\langle \bar{p} \rangle = \{\overline{pz} | \bar{z} \in \mathbb{Z}_n\}$. If $n = p^2$ its means $\overline{p^2} = \bar{n} = \bar{0}$ such that $\langle \bar{p} \rangle \langle \bar{p} \rangle = \langle \bar{0} \rangle$. Ideal $\langle \bar{p} \rangle$ is an annihilating ideal of $\mathbb{Z}_n$ by Definition 1. Since ideal $\langle \bar{p} \rangle$ is the only one of proper non zero ideal in $\mathbb{Z}_n$, hence $\mathbb{A}(\mathbb{Z}_n)^* = \{\langle \bar{p} \rangle\}$ or $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.
(2) Suppose n is nonprime, that is $n = ab$ for some $a, b \in \mathbb{Z}$ where $1 < a < n$, $1 < b < n$, and $a \neq b$. The product of two ideal, $\langle a \rangle \langle b \rangle = \{(za)(yb) | a, b \in \mathbb{Z}, y, z \in \mathbb{Z}_n\}$. Since $n = ab$, $zy(ab) = zy(n)$ then $\langle a \rangle \langle b \rangle = \{zyn | z, y \in \mathbb{Z}_n\} = \langle \bar{0} \rangle$. Clearly, $\langle a \rangle \neq \langle \bar{0} \rangle$ and $\langle b \rangle \neq \langle \bar{0} \rangle$. Ideals $\langle a \rangle$ and $\langle b \rangle$ are annihilating ideal by Definition 1. Hence, $\langle a \rangle, \langle b \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$. That is prove that for any nonprime n, $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$. ■

Theorem 2. Cardinality of $\mathbb{A}(\mathbb{Z}_n)^*$
Suppose $\mathbb{Z}_n$ ring of integer modulo n. The number of vertices of annihilating ideal graph $\mathbb{AG}(\mathbb{Z}_n)$ is $\varphi(n) - 2$, where $\varphi(n)$ is the number of positive factors of n.
Proof.
Suppose $n = (p_1)^{\alpha_1}(p_2)^{\alpha_2} \dots (p_n)^{\alpha_n}$ is prime factorization of n. If $x|n$ then $x = (p_1)^{\beta_1}(p_2)^{\beta_2} \dots (p_n)^{\beta_n}$ where $\beta_i \leq \alpha_i$ for all i. If $x|n$, also means that there exists integer y such that $xy = n$. Suppose $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \dots (p_n)^{\gamma_n}$ then $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \dots (p_n)^{\gamma_n}$, where $\alpha_i = \beta_i + \gamma_i$ for $1 \leq i \leq n$.

We construct principal ideal $\langle x \rangle = \{\overline{xz} | z \in \mathbb{Z}_n\}$ and $\langle y \rangle = \{\overline{yt} | t \in \mathbb{Z}_n\}$ of $\mathbb{Z}_n$. The product of these ideal $\langle x \rangle \langle y \rangle = \{(\overline{xz})(\overline{yt})\} = \{(\overline{xy})(\overline{zt})\}$. As $xy = n$ implies $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. For all $\langle x \rangle$, where x is a positive factor of n, there exists ideal $\langle y \rangle$ such that $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. The number of Ideal $\langle x \rangle$ that satisfied the condition is the number of positive factor of n, $\varphi(n)$. Suppose the set

$$\mathbb{I}(\mathbb{Z}_n) = \{\langle x \rangle \text{ ideal } \mathbb{Z}_n | \exists y \in \mathbb{Z} \text{ such that } xy = n\}$$

Based on the process above, we have $|\mathbb{I}(\mathbb{Z}_n)| = \varphi(n)$. All of elements $\mathbb{I}(\mathbb{Z}_n)$ is the elements of $\mathbb{A}(\mathbb{Z}_n)^*$ except $\langle 1 \rangle$ and $\langle n \rangle$. Hence $|\mathbb{A}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. ■

Theorem 3. Criteria vertex of $\mathbb{AG}(\mathbb{Z}_n)$
Suppose $\mathbb{Z}_n$ *ring of integer modulo* n. *If* $\langle \bar{a} \rangle$ *is a vertex of graph* $\mathbb{AG}(\mathbb{Z}_n)$ *then* a *is a factor of* n.
Proof.
Assume a isn't factor of n. We have $n = ax + y$, where x and y is integer and $0 < y < a$.
The product of ideal $\langle \bar{a} \rangle$ and $\langle \bar{x} \rangle$ is

$$\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\bar{a}r)(\bar{x}n)\} = \{\bar{a}(r\bar{x}n)\} = \{\bar{a}(\bar{x}rn)\} = \{(\bar{a}\bar{x})rn\} = \{(\overline{ax})rn\}$$

We have element $\bar{n} = \bar{y}$ because $n = ax + y$. Then $\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\overline{ax})rn\} = \{(\overline{axr})n\} = \langle \bar{n} \rangle = \langle \bar{y} \rangle$. In means $\langle \bar{a} \rangle$ isn't a ideal annihilator of $\mathbb{Z}_n$. Hence $\langle \bar{a} \rangle \notin \mathbb{A}(\mathbb{Z}_n)^*$. By the contraposition, we have if $\langle \bar{a} \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$ then a is a factor of n. ■

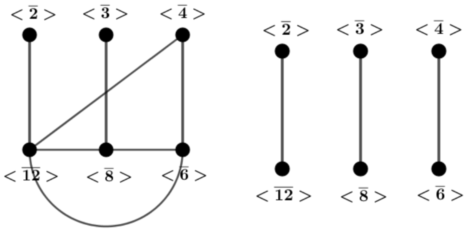
The converse of Theorem 3 is not true. For all $n \in \mathbb{Z}$, we have $1|n$, but clearly $\langle \bar{1} \rangle$ is not an ideal annihilator of $\mathbb{Z}_n$. It means $\langle \bar{1} \rangle$ is not a vertex ini $\mathbb{AG}(\mathbb{Z}_n)$.

Theorem 4. Adjacency of $\mathbb{AG}(\mathbb{Z}_n)$
Suppose $\langle \bar{p} \rangle$ *and* $\langle \bar{q} \rangle$ *are ideal in* $\mathbb{Z}_n$. *Vertex* $\langle \bar{p} \rangle$ *and* $\langle \bar{q} \rangle$ *are adjacent in* $\mathbb{AG}(\mathbb{Z}_n)$ *if and only if* $n|pq$.
Proof.
Suppose $\langle p \rangle = \{pa | a \in \mathbb{Z}_n\}$ and $\langle q \rangle = \{qb | r \in \mathbb{Z}_n\}$. The product $\langle p \rangle \langle q \rangle = \langle \overline{pq} \rangle$. If $n|pq$ then $\langle p \rangle \langle q \rangle = \langle \bar{0} \rangle = \{\bar{0}\}$. Hence $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ by Definition 3.
If $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent then $\langle p \rangle \langle q \rangle = \{\bar{0}\}$. It means $(pq)(ab) = nk$ for some integer a, b, and k. The equation $(pq)(ab) = nk$ implies $n|(pq)(ab)$, especially must be $n|pq$. ■

We will continue to discuss the relation some part of annihilating ideal and exact annihilating ideal graph of any commutative ring R.

Lemma 5. Relation Set of All Vertex $\mathbb{EAG}(\boldsymbol{R})$ and $\mathbb{AG}(\boldsymbol{R})$
For any commutative ring R, $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$
Proof.
Take any ideal $I \in \mathbb{EA}(R)^*$. It means there exist ideal J of R such that $Ann(I) = J$ and $Ann(J) = I$. Based on definition of annihilator, the product of ideal $IJ = 0$. Hence $I \in \mathbb{A}(R)^*$.
Now, take any ideal $I \in \mathbb{A}(R)^*$. It means there exist nonzero ideal J such that $IJ = 0$. Ideal I is annihilator ideal then $Ann(I) \neq 0$. Suppose $J = Ann(I)$ then J is nonzero ideal of R. We have $Ann(J) = Ann\big(Ann(I)\big) = I$. We conclude $Ann(I) = J$ and $Ann(J) = I$. Hence $I \in \mathbb{EA}(R)^*$. ■

Lemma 6.
For any commutative ring R, $\mathbb{EAG}(R)$ is a subgraph of $\mathbb{AG}(R)$.
Proof.
Lemma 5 show us that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. We will prove that for all $(I,J) \in E\big(\mathbb{EAG}(R)\big)$ then $(I,J) \in E(\mathbb{AG}(R))$. Adjacency of ideal I and J on $\mathbb{EAG}(R)$ means that $I = Ann(J)$ and $J = Ann(I)$. Based on properties of annihilator of ideal, we have $IJ = 0$. Based on definition of adjacency on $\mathbb{AG}(R)$, we have $(I,J) \in E(\mathbb{AG}(R))$. ■
The converse of Theorem 4 not valid for exact annihilating ideal graph. Figure 1 below show that vertex $\langle\bar{6}\rangle$ and $\langle\overline{12}\rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_{24})$ but not adjacent in $\mathbb{EAG}(\mathbb{Z}_{24})$ although $24|12 \times 6$.



(a) $\mathbb{AG}(\mathbb{Z}_{24})$ (b) $\mathbb{EAG}(\mathbb{Z}_{24})$

Figure 1. Representation of $\mathbb{AG}(\mathbb{Z}_{24})$ and $\mathbb{EAG}(\mathbb{Z}_{24})$

Based on the situation, we will construct the criteria of adjacency in exact annihilating ideal graph.

Theorem 7. Adjacency of $\mathbb{EAG}(\mathbb{Z}_n)$
Suppose commutative ring $\mathbb{Z}_n$ with identity $\bar{1}$. Ideals $\langle\bar{p}\rangle$ and $\langle\bar{q}\rangle$ are adjacent vertex of $\mathbb{EAG}(\mathbb{Z}_n)$ if and only if $n = pq$.
Diberikan ring komutatif $\mathbb{Z}_n$ dengan satuan $\bar{1}$. $\langle\bar{p}\rangle$ dan $\langle\bar{q}\rangle$ merupakan simpul yang bertetangga di $\mathbb{EAG}(\mathbb{Z}_n)$ jika dan hanya jika $n = pq$.
Proof.
($\leftarrow$). Assume $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent. We will proof $n \neq pq$. We have $\langle\bar{p}\rangle = \{\overline{pa}|\bar{a}, \bar{p} \in \mathbb{Z}_n\}$ and $\langle\bar{q}\rangle = \{\overline{qb}\big|\bar{b}, \bar{q} \in \mathbb{Z}_n\}$ are not adjacent. It means $Ann(\langle\bar{p}\rangle) \neq \langle\bar{q}\rangle$ and $Ann(\langle\bar{q}\rangle) \neq \langle\bar{p}\rangle$ such that $\langle\bar{p}\rangle\langle\bar{q}\rangle \neq \{\bar{0}\}$. We use commutative and associative property of $\mathbb{Z}_n$ to get form $\langle\bar{p}\rangle\langle\bar{q}\rangle = \{(\overline{pa})(\overline{qb})\} = \{(\overline{pq})(\overline{ab})\} \neq \{\bar{0}\}$. It imply $pq \nmid n$. Hence $pq \neq n$.
($\rightarrow$). Assume $n \neq pq$. We will proof vertex $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$. If $n \neq pq$ then $n = pq + a$ with a is non-zero integer. We construct two principal ideal generated by p and q on $\mathbb{Z}_n$. Now, we have the product of these ideal

$$\langle\bar{p}\rangle\langle\bar{q}\rangle = \{(\overline{pr})(\overline{qt})\} = \{(\overline{n-a})(\overline{rt})\} = \langle\overline{-a}\rangle$$

We have $(\langle\bar{p}\rangle, \langle\bar{q}\rangle) \notin E\big(\mathbb{AG}(\mathbb{Z}_n)\big)$. Based on Lemma 6, vertex $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$.■

Theorem 8. Isolated Vertex in $\mathbb{EAG}(\mathbb{Z}_n)$
Suppose commutative ring $\mathbb{Z}_n$ with identity $\bar{1}$. If $n = r^2$ then $\langle\bar{r}\rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$.
Proof.
Suppose $n = r^2$ and principal ideal $\langle\bar{r}\rangle$ of ring $\mathbb{Z}_n$. We have $Ann(\langle\bar{r}\rangle) = \langle\bar{r}\rangle$. Its means $\langle\bar{r}\rangle$ is a vertex of $\mathbb{EAG}(\mathbb{Z}_n)$. Assume there is a vertex $\langle\bar{a}\rangle$ (not equal to $\langle\bar{r}\rangle$) of $\mathbb{EAG}(\mathbb{Z}_n)$ such that $\langle\bar{r}\rangle$ and $\langle\bar{a}\rangle$ adjacent. The product of the ideals is $\langle\bar{a}\rangle\langle\bar{r}\rangle \neq \langle\bar{r}\rangle\langle\bar{r}\rangle = \langle\bar{0}\rangle$. Vertex $\langle\bar{r}\rangle$ and $\langle\bar{a}\rangle$ adjacent on $\mathbb{EAG}(\mathbb{Z}_n)$ means that $\langle\bar{r}\rangle =$

$Ann(\langle \bar{a} \rangle)$ and $\langle \bar{a} \rangle = Ann(\langle \bar{r} \rangle)$. Furthermore $\langle \bar{r} \rangle \langle \bar{a} \rangle = \langle \bar{0} \rangle$. Its contradiction with the product ideals $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$. Hence there is no vertex adjacent with $\langle \bar{r} \rangle$ on $\mathbb{EAG}(\mathbb{Z}_n)$. ■

In [11] showed that $diam(\mathbb{EAG}(R)) < 1$ and $g(\mathbb{EAG}(R)) \leq 4$ for any commutative ring R. In this paper, we will show more specific result about diameter, girth, cycle existence of $\mathbb{EAG}(R)$.

Theorem 9.

Suppose commutative ring R. If $\mathbb{EAG}(R)$ is connected graph then $diam(\mathbb{EAG}(R)) = 1$.

Proof.

Suppose I and J are two different vertex of $\mathbb{EAG}(R)$. Assume $d(I,J) = 2 > 1$, means that exist a vertex A of $\mathbb{EAG}(R)$ such that $I - A - J$ is a path in $\mathbb{EAG}(R)$. Based on Definition 4 we have $I = Ann(A), A = Ann(I), A = Ann(J),$ and $J = Ann(A)$. It imply $I = Ann(A) = Ann(Ann(J))$. Based on Lemma 2.1 on [3], we get $Ann(Ann(J)) = J$. Two last equation imply $I = J$. We have a contradiction with ideal I and J must be different. So, $d(I,J) = 1$ for all ideal I and J. It proved that $diam(\mathbb{EAG}(R)) = 1$. ■

Collorary 10.

Suppose commutative ring R. If $\mathbb{EAG}(R)$ contain a cycle then $g(\mathbb{EAG}(R)) \leq 3$.

Proof.

If graph G contain a cycle then $g(G) \leq 2diam(G) + 1$. Theorem 9 has shown that $diam(\mathbb{EAG}(R)) = 1$. Finally, we have $g(\mathbb{EAG}(R)) \leq 2diam(\mathbb{EAG}(R)) + 1 = 3$. ■

Theorem 3.9 in [11] showed that $\mathbb{EAG}(\mathbb{Z}_{p^n})$ where p is prime can be represented as union of some complete graph. Figure 1 below show that $\mathbb{EAG}(\mathbb{Z}_{24})$ can be represented as union of K_2 graph, although $24 \neq p^n$ for any prime p. Based on this fact, we construct a theorem to generalize properties of representation of $\mathbb{EAG}(\mathbb{Z}_n)$.

Theorem 11. Decomposition of $\mathbb{EAG}(\mathbb{Z}_n)$

The number of complete subgraph of Exact annihilating ideal graph of ring $\mathbb{Z}_{\boldsymbol{n}}$ is $\left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$.

Proof.

Lemma 5 showed that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. Based on Theorem 2, we have $|\mathbb{EA}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. Theorem 9 showed that $diam(\mathbb{EAG}(R)) = 1$ for any commutative ring R. We conclude that the maximum number of edges $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)}{2} - 1$. Its means the maximum complete subgraph of $\mathbb{EAG}(\mathbb{Z}_n)$ is also $\frac{\varphi(n)}{2} - 1$.

Case 1: $n = r^2$ for some integer r

Based on Theorem 8, $\langle \bar{r} \rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$. We have $\varphi(n) - 3$ other vertices of $\mathbb{EAG}(\mathbb{Z}_n)$. Obviously there is no positive integer a such that $n = a^2$. In another word, we just found exactly one isolated vertex on $\mathbb{EAG}(\mathbb{Z}_n)$. We can partition $\mathbb{EAG}(\mathbb{Z}_n)$ to be $\frac{\varphi(n)-3}{2}$ graph K_2. Isolated vertex can be represented as K_1. The total of number complete graph that contain in $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)-3}{2} + 1 = \frac{\varphi(n)+1}{2} - 1 = \left\lceil \frac{\varphi(n)}{2} \right\rceil - 1 = \left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$.

Case 2: $n \neq r^2$ for any integer r

If $n \neq r^2$ for any integer r then $n = ab$ where $a \neq b$. Ideal $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ are vertices in $\mathbb{EAG}(\mathbb{Z}_n)$. Based on theorem 7, $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ adjacent in $\mathbb{EAG}(\mathbb{Z}_n)$. This condition means there is no isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$. Graph $\mathbb{EAG}(\mathbb{Z}_n)$ is fully partition into complete graph K_2. Total number of K_2 is $\frac{\varphi(n)-2}{2} = \frac{\varphi(n)}{2} - 1 = \left\lceil \frac{\varphi(n)}{2} \right\rceil - 1 = \left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$. ■

4. CONCLUSIONS

Factorization on $\mathbb{Z}_n$ characterizes the (Exact) Annihilating Ideal Graph, especially in 1) the number of vertices in an annihilating ideal graph, 2) adjacency of the vertices, and 3) decomposition of exact annihilating ideal graph.

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ANNIHILATING IDEAL AND EXACT ANNIHILATING IDEAL GRAPH OF RING $\mathbb{Z}_n$

Anindito Wisnu Susanto [1], **Dewa Putu Wiadnyana Putra** [2*]

[1,2]*Department of Mathematics Education, Faculty of Teacher Training and Education, Sanata Dharma University*
Jl. Affandi, Mrican, Caturtunggal, Depok, Sleman, Yogyakarta, 55281, Indonesia

Corresponding author's e-mail: [2*] ***dewa@usd.ac.id***

ABSTRACT

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The existence of annihilator in the ring motivates the emergence of studies on Annihilating Ideal and Exact Annihilating Ideal Graphs. The purpose of this research is to describe the characteristics of an (exact) annihilating ideal of ring $\mathbb{Z}_n$. The method used in this research is literature study. The results of this study discuss finiteness, adjacency, connectedness, vertices, and types of $\mathbb{AG}(\mathbb{Z}_n)$ and $\mathbb{EAG}(\mathbb{Z}_n)$. Furthermore, the number of vertices of an Annihilating Ideal Graph is determined by the factorization of n. The adjacency of two vertices is determined by the divisibleness of n. The results also show that $\mathbb{EAG}(\mathbb{Z}_n)$ is a subgraph of $\mathbb{AG}(\mathbb{Z}_n)$. $\mathbb{EAG}(\mathbb{Z}_n)$ can be represented as a union of several complete graphs.

Keywords:

Annihilating Ideal;
Exact Annihilating Ideal;
Graph;
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1. INTRODUCTION

The use of graphs in representing algebraic structures has been carried out since at least 1878 in [1]. This representation starts from representing a group structure into a graph. The vertices of a graph are all elements of a group and changes to an element due to operations on the group are represented by directed edges. Furthermore, [2], [3] began to associate graphs with a broader structure, namely rings. Investigation of the ring structure is carried out through the colored representation of the graph. The representation of an algebraic structure on a graph opens up opportunities for visual investigation of the properties of a particular structure. An essential part in the process of representing a particular algebraic structure to a graph is how to define the connection between the vertices of the graph. Different ways of defining adjacent vertices can lead to different variations of properties as well.

One of the interesting things in the ring, which is about zero divisor. A non-zero element a is said to be a zero divisor if it can be found a non-zero element b such that $ab = 0$. From this structure, [4] proposed the origin of the zero divisor graph. The vertices of the graph are all zero divisors. Two vertices are adjacent if and only if the product of the two elements is zero. Many interesting properties result from this concept, one of which is about the combinatorics of a finite ring [5]–[7].

The concept is similar to zero divisor in the ring is the Annihilator. Badawi has started a study on annihilator graphs [8]. In its development, annihilating graphs are generalized into annihilating Ideal graphs. In [9], it is stated that Annihilator is an ideal I, namely $Ann(I) = \{r \in R|rl = 0 \,\forall\, l \in L\}$. If $Ann(I)$ is not a trivial set, then I is called an ideal annilator. In 2011, [10] started to represent a structure consisting of annihilator ideals into a graph. The graph that is formed is named Annihilating Ideal Graph. In line with the development of zero divisor graphs, [11] is continuing the study of Exact Annihilating Ideal graphs. The development of ideal annihilating and exact annihilating properties of graphs is studied separately. The general relationship between these two graphs began to be investigated by [12].

An integer modulo n, $\mathbb{Z}_n$ is a ring that has very interesting properties. This $\mathbb{Z}_n$ structure is widely used in graphs, for example in coloring Antimagic graphs [13] and Domination ratio [14]. The factorization theorem on integers is a motivation for developing graph studies involving a ring of integers modulo n. One of the graph studies carried out was a study on non-coprime for $\mathbb{Z}_n$ [15]. Based on this, the study of annihilating ideal graph for $\mathbb{Z}_n$ rings is interesting to do.

2. RESEARCH METHODS

This research is a literature research that examines the properties of annihilating ideal and exact annihilating ideal graphs on integer rings modulo n, $\mathbb{Z}_n$. The definition of (Exact) Annihilating Ideal based on [10], [11] is as follows.

Definition 1. Annihilating Ideal [10]
An Ideal I of commutative ring R with identity is a Annihilating Ideal if there exist non zero ideal J of R such that $\mathrm{IJ} = 0$. The set of all Annihilating Ideal of ring R is denoted by $\mathbb{A}(\mathrm{R})$.

Definition 2. Exact Annihilating Ideal [11]
An ideal I of commutative ring R with identity is Exact Annihilating Ideal if there exist non zero ideal J of R such that $\mathrm{Ann}(\mathrm{I}) = \mathrm{J}$ and $\mathrm{Ann}(\mathrm{J}) = \mathrm{I}$. The set of all Exact Annihilating Ideal of ring R denoted by $\mathbb{EA}(\mathrm{R})$.

Based on the two definitions above, then (Exact) Annihilating Ideal Graph is defined as follows.

Definition 3. Annihilating Ideal Graph [10]
Annihilating Ideal graph of ring R denoted by $\mathbb{A}\mathrm{G}(\mathrm{R})$ is a graph with vertices $\mathbb{A}(\mathrm{R})^* = \mathbb{A}(\mathrm{R})\backslash\{(0)\}$ and $(\mathrm{I},\mathrm{J}) \in \mathrm{E}(\mathbb{A}\mathrm{G}(\mathrm{R}))$ if and only if $\mathrm{IJ} = (0)$.

Definition 4. Exact Annihilating Ideal Graph [11]
Exact Annihilating Ideal graph of ring R denoted by $\mathbb{EAG}(\mathrm{R})$ is a graph with vertices $\mathbb{EA}(\mathrm{R})^* = \mathbb{EA}(\mathrm{R})\backslash\{(0)\}$ and $(\mathrm{I},\mathrm{J}) \in \mathrm{E}(\mathbb{EAG}(\mathrm{R}))$ if and only if $\mathrm{Ann}(\mathrm{I}) = \mathrm{J}$ and $\mathrm{Ann}(\mathrm{J}) = \mathrm{I}$.

Definition of (Exact) Annihilating Ideal Graph, this article further describes the properties of the graph with rings $\mathbb{Z}_n$. Comparison of the properties of annihilating ideal and exact annihilating ideal graph of ring $\mathbb{Z}_n$ is also presented in this article.

3. RESULTS AND DISCUSSION

In this section, we will show some result of Annihilating ideal graph of ring $\mathbb{Z}_n$.

Theorem 1. Lower Bound of Cardinality $\mathbb{A}(\mathbb{Z}_n)^*$
Suppose $\mathbb{Z}_n$ ring of integer modulo n where n not prime.

1. If $n = p^2$, *where p is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.
2. If $n \neq p^2$, *where p is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$

Proof.

(1) Suppose $n = p^2$ then there exists uniquely non zero proper ideal in di $\mathbb{Z}_n$, $\langle \bar{p} \rangle = \{\overline{pz} | \bar{z} \in \mathbb{Z}_n\}$. If $n = p^2$ its means $\overline{p^2} = \bar{n} = \bar{0}$ such that $\langle \bar{p} \rangle \langle \bar{p} \rangle = \langle \bar{0} \rangle$. Ideal $\langle \bar{p} \rangle$ is an annihilating ideal of $\mathbb{Z}_n$ by Definition 1. Since ideal $\langle \bar{p} \rangle$ is the only one of proper non zero ideal in $\mathbb{Z}_n$, hence $\mathbb{A}(\mathbb{Z}_n)^* = \{\langle \bar{p} \rangle\}$ or $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.

(2) Suppose n is nonprime, that is $n = ab$ for some $a, b \in \mathbb{Z}$ where $1 < a < n$, $1 < b < n$, and $a \neq b$. The product of two ideal, $\langle a \rangle \langle b \rangle = \{(za)(yb) | a, b \in \mathbb{Z}, y, z \in \mathbb{Z}_n\}$. Since $n = ab$, $zy(ab) = zy(n)$ then $\langle a \rangle \langle b \rangle = \{zyn | z, y \in \mathbb{Z}_n\} = \langle \bar{0} \rangle$. Clearly, $\langle a \rangle \neq \langle \bar{0} \rangle$ and $\langle b \rangle \neq \langle \bar{0} \rangle$. Ideals $\langle a \rangle$ and $\langle b \rangle$ are annihilating ideal by Definition 1. Hence, $\langle a \rangle, \langle b \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$. That is prove that for any nonprime n, $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$. ■

Theorem 2. Cardinality of $\mathbb{A}(\mathbb{Z}_n)^*$
Suppose $\mathbb{Z}_n$ ring of integer modulo n. The number of vertices of annihilating ideal graph $\mathbb{AG}(\mathbb{Z}_n)$ is $\varphi(n) - 2$, where $\varphi(n)$ is the number of positive factors of n.

Proof.

Suppose $n = (p_1)^{\alpha_1}(p_2)^{\alpha_2} \dots (p_n)^{\alpha_n}$ is prime factorization of n. If $x|n$ then $x = (p_1)^{\beta_1}(p_2)^{\beta_2} \dots (p_n)^{\beta_n}$ where $\beta_i \leq \alpha_i$ for all i. If $x|n$, also means that there exists integer y such that $xy = n$. Suppose $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \dots (p_n)^{\gamma_n}$ then $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \dots (p_n)^{\gamma_n}$, where $\alpha_i = \beta_i + \gamma_i$ for $1 \leq i \leq n$.

We construct principal ideal $\langle x \rangle = \{\overline{xz} | z \in \mathbb{Z}_n\}$ and $\langle y \rangle = \{\overline{yt} | t \in \mathbb{Z}_n\}$ of $\mathbb{Z}_n$. The product of these ideal $\langle x \rangle \langle y \rangle = \{(\overline{xz})(\overline{yt})\} = \{(\overline{xy})(\overline{zt})\}$. As $xy = n$ implies $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. For all $\langle x \rangle$, where x is a positive factor of n, there exists ideal $\langle y \rangle$ such that $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. The number of Ideal $\langle x \rangle$ that satisfied the condition is the number of positive factor of n, $\varphi(n)$. Suppose the set

$$\mathbb{I}(\mathbb{Z}_n) = \{\langle x \rangle \text{ ideal } \mathbb{Z}_n | \exists y \in \mathbb{Z} \text{ such that } xy = n\}$$

Based on the process above, we have $|\mathbb{I}(\mathbb{Z}_n)| = \varphi(n)$. All of elements $\mathbb{I}(\mathbb{Z}_n)$ is the elements of $\mathbb{A}(\mathbb{Z}_n)^*$ except $\langle 1 \rangle$ and $\langle n \rangle$. Hence $|\mathbb{A}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. ■

Theorem 3. Criteria vertex of $\mathbb{AG}(\mathbb{Z}_n)$
Suppose $\mathbb{Z}_n$ ring of integer modulo n. If $\langle \bar{a} \rangle$ is a vertex of graph $\mathbb{AG}(\mathbb{Z}_n)$ then a is a factor of n.

Proof.

Assume a isn't factor of n. We have $n = ax + y$, where x and y is integer and $0 < y < a$. The product of ideal $\langle \bar{a} \rangle$ and $\langle \bar{x} \rangle$ is

$$\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\bar{a}r)(\bar{x}n)\} = \{\bar{a}(r\bar{x}n)\} = \{\bar{a}(\bar{x}rn)\} = \{(\bar{a}\bar{x})rn\} = \{(\overline{ax})rn\}$$

We have element $\bar{n} = \bar{y}$ because $n = ax + y$. Then $\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\overline{ax})rn\} = \{(\overline{axr})n\} = \langle \bar{n} \rangle = \langle \bar{y} \rangle$. In means $\langle \bar{a} \rangle$ isn't a ideal annihilator of $\mathbb{Z}_n$. Hence $\langle \bar{a} \rangle \notin \mathbb{A}(\mathbb{Z}_n)^*$. By the contraposition, we have if $\langle \bar{a} \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$ then a is a factor of n. ■

The converse of Theorem 3 is not true. For all $n \in \mathbb{Z}$, we have $1|n$, but clearly $\langle \bar{1} \rangle$ is not an ideal annihilator of $\mathbb{Z}_n$. It means $\langle \bar{1} \rangle$ is not a vertex ini $\mathbb{AG}(\mathbb{Z}_n)$.

Theorem 4. Adjacency of $\mathbb{AG}(\mathbb{Z}_n)$
Suppose $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are ideal in $\mathbb{Z}_n$. Vertex $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ if and only if $n|pq$.

Proof.

Suppose $\langle p \rangle = \{pa | a \in \mathbb{Z}_n\}$ and $\langle q \rangle = \{qb | r \in \mathbb{Z}_n\}$. The product $\langle p \rangle \langle q \rangle = \langle \overline{pq} \rangle$. If $n|pq$ then $\langle p \rangle \langle q \rangle = \langle \bar{0} \rangle = \{\bar{0}\}$. Hence $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ by Definition 3.
If $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent then $\langle p \rangle \langle q \rangle = \{\bar{0}\}$. It means $(pq)(ab) = nk$ for some integer a, b, and k. The equation $(pq)(ab) = nk$ implies $n|(pq)(ab)$, especially must be $n|pq$. ■

We will continue to discuss the relation some part of annihilating ideal and exact annihilating ideal graph of any commutative ring R.

Lemma 5. Relation Set of All Vertex $\mathbb{EAG}(\boldsymbol{R})$ and $\mathbb{AG}(\boldsymbol{R})$

For any commutative ring R, $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$

Proof.

Take any ideal $I \in \mathbb{EA}(R)^*$. It means there exist ideal J of R such that $Ann(I) = J$ and $Ann(J) = I$. Based on definition of annihilator, the product of ideal $IJ = 0$. Hence $I \in \mathbb{A}(R)^*$.

Now, take any ideal $I \in \mathbb{A}(R)^*$. It means there exist nonzero ideal J such that $IJ = 0$. Ideal I is annihilator ideal then $Ann(I) \neq 0$. Suppose $J = Ann(I)$ then J is nonzero ideal of R. We have $Ann(J) = Ann\big(Ann(I)\big) = I$. We conclude $Ann(I) = J$ and $Ann(J) = I$. Hence $I \in \mathbb{EA}(R)^*$. ■

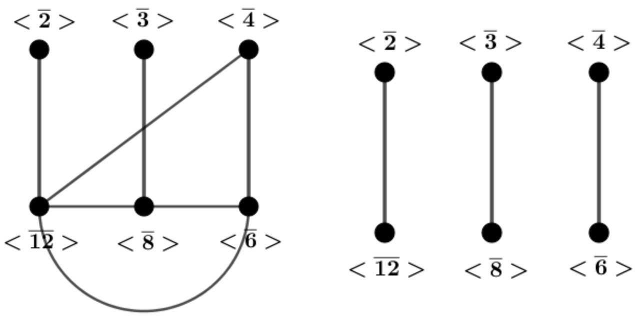
Lemma 6.

For any commutative ring R, $\mathbb{EAG}(R)$ is a subgraph of $\mathbb{AG}(R)$.

Proof.

Lemma 5 show us that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. We will prove that for all $(I,J) \in E\big(\mathbb{EAG}(R)\big)$ then $(I,J) \in E(\mathbb{AG}(R))$. Adjacency of ideal I and J on $\mathbb{EAG}(R)$ means that $I = Ann(J)$ and $J = Ann(I)$. Based on properties of annihilator of ideal, we have $IJ = 0$. Based on definition of adjacency on $\mathbb{AG}(R)$, we have $(I,J) \in E(\mathbb{AG}(R))$. ■

The converse of Theorem 4 not valid for exact annihilating ideal graph. Figure 1 below show that vertex $\langle\bar{6}\rangle$ and $\langle\overline{12}\rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_{24})$ but not adjacent in $\mathbb{EAG}(\mathbb{Z}_{24})$ although $24|12 \times 6$.



(a) $\mathbb{AG}(\mathbb{Z}_{24})$ (b) $\mathbb{EAG}(\mathbb{Z}_{24})$

Figure 1. Representation of $\mathbb{AG}(\mathbb{Z}_{24})$ and $\mathbb{EAG}(\mathbb{Z}_{24})$

Based on the situation, we will construct the criteria of adjacency in exact annihilating ideal graph.

Theorem 7. Adjacency of $\mathbb{EAG}(\mathbb{Z}_n)$

Suppose commutative ring $\mathbb{Z}_n$ with identity $\bar{1}$. Ideals $\langle\bar{p}\rangle$ and $\langle\bar{q}\rangle$ are adjacent vertex of $\mathbb{EAG}(\mathbb{Z}_n)$ if and only if $n = pq$.

Diberikan ring komutatif $\mathbb{Z}_n$ dengan satuan $\bar{1}$. $\langle\bar{p}\rangle$ dan $\langle\bar{q}\rangle$ merupakan simpul yang bertetangga di $\mathbb{EAG}(\mathbb{Z}_n)$ jika dan hanya jika $n = pq$.

Proof.

(←). Assume $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent. We will proof $n \neq pq$. We have $\langle\bar{p}\rangle = \{\overline{pa}|\bar{a},\bar{p} \in \mathbb{Z}_n\}$ and $\langle\bar{q}\rangle = \{\overline{qb}|\bar{b},\bar{q} \in \mathbb{Z}_n\}$ are not adjacent. It means $Ann(\langle\bar{p}\rangle) \neq \langle\bar{q}\rangle$ and $Ann(\langle\bar{q}\rangle) \neq \langle\bar{p}\rangle$ such that $\langle\bar{p}\rangle\langle\bar{q}\rangle \neq \{\bar{0}\}$. We use commutative and associative property of $\mathbb{Z}_n$ to get form $\langle\bar{p}\rangle\langle\bar{q}\rangle = \{(\overline{pa})(\overline{qb})\} = \{(\overline{pq})(\overline{ab})\} \neq \{\bar{0}\}$. It imply $pq \nmid n$. Hence $pq \neq n$.

(→). Assume $n \neq pq$. We will proof vertex $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$. If $n \neq pq$ then $n = pq + a$ with a is non-zero integer. We construct two principal ideal generated by p and q on $\mathbb{Z}_n$. Now, we have the product of these ideal

$$\langle\bar{p}\rangle\langle\bar{q}\rangle = \{(\overline{pr})(\overline{qt})\} = \{(\overline{n-a})(\overline{rt})\} = \langle\overline{-a}\rangle$$

We have $(\langle\bar{p}\rangle,\langle\bar{q}\rangle) \notin E\big(\mathbb{AG}(\mathbb{Z}_n)\big)$. Based on Lemma 6, vertex $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$.■

Theorem 8. Isolated Vertex in $\mathbb{EAG}(\mathbb{Z}_n)$

Suppose commutative ring $\mathbb{Z}_n$ with identity $\bar{1}$. If $n = r^2$ then $\langle \bar{r} \rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$.

Proof.

Suppose $n = r^2$ and principal ideal $\langle \bar{r} \rangle$ of ring $\mathbb{Z}_n$. We have $Ann(\langle \bar{r} \rangle) = \langle \bar{r} \rangle$. Its means $\langle \bar{r} \rangle$ is a vertex of $\mathbb{EAG}(\mathbb{Z}_n)$. Assume there is a vertex $\langle \bar{a} \rangle$ (not equal to $\langle \bar{r} \rangle$) of $\mathbb{EAG}(\mathbb{Z}_n)$ such that $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$ adjacent. The product of the ideals is $\langle \bar{a} \rangle\langle \bar{r} \rangle \neq \langle \bar{r} \rangle\langle \bar{r} \rangle = \langle \bar{0} \rangle$. Vertex $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$ adjacent on $\mathbb{EAG}(\mathbb{Z}_n)$ means that $\langle \bar{r} \rangle = Ann(\langle \bar{a} \rangle)$ and $\langle \bar{a} \rangle = Ann(\langle \bar{r} \rangle)$. Furthermore $\langle \bar{r} \rangle\langle \bar{a} \rangle = \langle \bar{0} \rangle$. Its contradiction with the product ideals $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$. Hence there is no vertex adjacent with $\langle \bar{r} \rangle$ on $\mathbb{EAG}(\mathbb{Z}_n)$. ■

In [11] showed that $diam\big(\mathbb{EAG}(R)\big) < 1$ and $g\big(\mathbb{EAG}(R)\big) \leq 4$ for any commutative ring R. In this paper, we will show more specific result about diameter, girth, cycle existence of $\mathbb{EAG}(R)$.

Theorem 9.

Suppose commutative ring R. If $\mathbb{EAG}(R)$ is connected graph then $diam\big(\mathbb{EAG}(R)\big) = 1$.

Proof.

Suppose I and J are two different vertex of $\mathbb{EAG}(R)$. Assume $d(I,J) = 2 > 1$, means that exist a vertex A of $\mathbb{EAG}(R)$ such that $I - A - J$ is a path in $\mathbb{EAG}(R)$. Based on Definition 4 we have $I = Ann(A), A = Ann(I), A = Ann(J),$ and $J = Ann(A)$. It imply $I = Ann(A) = Ann\big(Ann(J)\big)$. Based on Lemma 2.1 on [3], we get $Ann\big(Ann(J)\big) = J$. Two last equation imply $I = J$. We have a contradiction with ideal I and J must be different. So, $d(I,J) = 1$ for all ideal I and J. It proved that $diam\big(\mathbb{EAG}(R)\big) = 1$. ■

Collorary 10.

Suppose commutative ring R. If $\mathbb{EAG}(R)$ contain a cycle then $g\big(\mathbb{EAG}(R)\big) \leq 3$.

Proof.

If graph G contain a cycle then $g(G) \leq 2diam(G) + 1$. Theorem 9 has shown that $diam\big(\mathbb{EAG}(R)\big) = 1$. Finally, we have $g\big(\mathbb{EAG}(R)\big) \leq 2diam\big(\mathbb{EAG}(R)\big) + 1 = 3$. ■

Theorem 3.9 in [11] showed that $\mathbb{EAG}\big(\mathbb{Z}_{p^n}\big)$ where p is prime can be represented as union of some complete graph. Figure 1 below show that $\mathbb{EAG}(\mathbb{Z}_{24})$ can be represented as union of K_2 graph, although $24 \neq p^n$ for any prime p. Based on this fact, we construct a theorem to generalize properties of representation of $\mathbb{EAG}(\mathbb{Z}_n)$.

Theorem 11. Decomposition of $\mathbb{EAG}(\mathbb{Z}_n)$

The number of complete subgraph of Exact annihilating ideal graph of ring $\mathbb{Z}_n$ is $\left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$.

Proof.

Lemma 5 showed that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. Based on Theorem 2, we have $|\mathbb{EA}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. Theorem 9 showed that $diam\big(\mathbb{EAG}(R)\big) = 1$ for any commutative ring R. We conclude that the maximum number of edges $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)}{2} - 1$. Its means the maximum complete subgraph of $\mathbb{EAG}(\mathbb{Z}_n)$ is also $\frac{\varphi(n)}{2} - 1$.

Case 1: $n = r^2$ for some integer r

Based on Theorem 8, $\langle \bar{r} \rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$. We have $\varphi(n) - 3$ other vertices of $\mathbb{EAG}(\mathbb{Z}_n)$. Obviously there is no positive integer a such that $n = a^2$. In another word, we just found exactly one isolated vertex on $\mathbb{EAG}(\mathbb{Z}_n)$. We can partition $\mathbb{EAG}(\mathbb{Z}_n)$ to be $\frac{\varphi(n)-3}{2}$ graph K_2. Isolated vertex can be represented as K_1. The total of number complete graph that contain in $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)-3}{2} + 1 = \frac{\varphi(n)+1}{2} - 1 = \left\lceil \frac{\varphi(n)}{2} \right\rceil - 1 = \left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$.

Case 2: $n \neq r^2$ for any integer r

If $n \neq r^2$ for any integer r then $n = ab$ where $a \neq b$. Ideal $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ are vertices in $\mathbb{EAG}(\mathbb{Z}_n)$. Based on theorem 7, $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ adjacent in $\mathbb{EAG}(\mathbb{Z}_n)$. This condition means there is no isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$.

Graph $\mathbb{EAG}(\mathbb{Z}_n)$ is fully partition into complete graph K_2. Total number of K_2 is $\frac{\varphi(n)-2}{2} = \frac{\varphi(n)}{2} - 1 = \left\lceil\frac{\varphi(n)}{2}\right\rceil - 1 = \left\lceil\frac{\varphi(n)}{2} - 1\right\rceil$. ■

4. CONCLUSIONS

Factorization on $\mathbb{Z}_n$ characterizes the (Exact) Annihilating Ideal Graph, especially in 1) the number of vertices in an annihilating ideal graph, 2) adjacency of the vertices, and 3) decomposition of exact annihilating ideal graph.

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ANNIHILATING IDEAL AND EXACT ANNIHILATING IDEAL GRAPH OF RING $\mathbb{Z}_n$

Anindito Wisnu Susanto [1], **Dewa Putu Wiadnyana Putra** [2*]

[1,2] *Department of Mathematics Education, Faculty of Teacher Training and Education, Sanata Dharma University*
Jl. Affandi, Mrican, Caturtunggal, Depok, Sleman, Yogyakarta, 55281, Indonesia

Corresponding author's e-mail: [2*] ***dewa@usd.ac.id***

ABSTRACT

The existence of annihilator in the ring motivates the emergence of studies on Annihilating Ideal and Exact Annihilating Ideal Graphs. The purpose of this research is to describe the characteristics of an (exact) annihilating ideal of ring $\mathbb{Z}_n$. The method used in this research is literature study. The results of this study discuss finiteness, adjacency, connectedness, vertices, and types of $\mathbb{AG}(\mathbb{Z}_n)$ and $\mathbb{EAG}(\mathbb{Z}_n)$. Furthermore, the number of vertices of an Annihilating Ideal Graph is determined by the factorization of n. The adjacency of two vertices is determined by the divisibleness of n. The results also show that $\mathbb{EAG}(\mathbb{Z}_n)$ is a subgraph of $\mathbb{AG}(\mathbb{Z}_n)$. $\mathbb{EAG}(\mathbb{Z}_n)$ can be represented as a union of several complete graphs.

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1. INTRODUCTION

The use of graphs in representing algebraic structures has been carried out since at least 1878 in [1]. This representation starts from representing a group structure into a graph. The vertices of a graph are all elements of a group and changes to an element due to operations on the group are represented by directed edges. Furthermore, [2], [3] began to associate graphs with a broader structure, namely rings. Investigation of the ring structure is carried out through the colored representation of the graph. The representation of an algebraic structure on a graph opens up opportunities for visual investigation of the properties of a particular structure. An essential part in the process of representing a particular algebraic structure to a graph is how to define the connection between the vertices of the graph. Different ways of defining adjacent vertices can lead to different variations of properties as well.

One of the interesting things in the ring, which is about zero divisor. A non-zero element a is said to be a zero divisor if it can be found a non-zero element b such that $ab = 0$. From this structure, [4] proposed the origin of the zero divisor graph. The vertices of the graph are all zero divisors. Two vertices are adjacent if and only if the product of the two elements is zero. Many interesting properties result from this concept, one of which is about the combinatorics of a finite ring [5]–[7].

The concept is similar to zero divisor in the ring is the Annihilator. Badawi has started a study on annihilator graphs [8]. In its development, annihilating graphs are generalized into annihilating Ideal graphs. In [9], it is stated that Annihilator is an ideal I, namely $Ann(I) = \{r \in R | rl = 0 \, \forall \, l \in L\}$. If $Ann(I)$ is not a trivial set, then I is called an ideal annilator. In 2011, [10] started to represent a structure consisting of annihilator ideals into a graph. The graph that is formed is named Annihilating Ideal Graph. In line with the development of zero divisor graphs, [11] is continuing the study of Exact Annihilating Ideal graphs. The development of ideal annihilating and exact annihilating properties of graphs is studied separately. The general relationship between these two graphs began to be investigated by [12].

An integer modulo n, $\mathbb{Z}_n$ is a ring that has very interesting properties. This $\mathbb{Z}_n$ structure is widely used in graphs, for example in coloring Antimagic graphs [13] and Domination ratio [14]. The factorization theorem on integers is a motivation for developing graph studies involving a ring of integers modulo n. One of the graph studies carried out was a study on non-coprime for $\mathbb{Z}_n$ [15]. Based on this, the study of annihilating ideal graph for $\mathbb{Z}_n$ rings is interesting to do.

2. RESEARCH METHODS

This research is a literature research that examines the properties of annihilating ideal and exact annihilating ideal graphs on integer rings modulo n, $\mathbb{Z}_n$. The definition of (Exact) Annihilating Ideal based on [10], [11] is as follows.

Definition 1. Annihilating Ideal [10]
An Ideal I of commutative ring R with identity is a Annihilating Ideal if there exist non zero ideal J of R such that $\mathrm{IJ} = 0$. The set of all Annihilating Ideal of ring R is denoted by $\mathbb{A}(\mathrm{R})$.

Definition 2. Exact Annihilating Ideal [11]
An ideal I of commutative ring R with identity is Exact Annihilating Ideal if there exist non zero ideal J of R such that $\mathrm{Ann(I) = J}$ and $\mathrm{Ann(J) = I}$. The set of all Exact Annihilating Ideal of ring R denoted by $\mathbb{EA}(\mathrm{R})$.

Based on the two definitions above, then (Exact) Annihilating Ideal Graph is defined as follows.

Definition 3. Annihilating Ideal Graph [10]
Annihilating Ideal graph of ring R denoted by $\mathbb{AG}(\mathrm{R})$ is a graph with vertices $\mathbb{A}(\mathrm{R})^* = \mathbb{A}(\mathrm{R}) \backslash \{(0)\}$ and $(\mathrm{I}, \mathrm{J}) \in \mathrm{E}(\mathbb{AG}(\mathrm{R}))$ if and only if $\mathrm{IJ} = (0)$.

Definition 4. Exact Annihilating Ideal Graph [11]
Exact Annihilating Ideal graph of ring R denoted by $\mathbb{EAG}(\mathrm{R})$ is a graph with vertices $\mathbb{EA}(\mathrm{R})^* = \mathbb{EA}(\mathrm{R}) \backslash \{(0)\}$ and $(\mathrm{I}, \mathrm{J}) \in \mathrm{E}(\mathbb{EAG}(\mathrm{R}))$ if and only if $\mathrm{Ann(I) = J}$ and $\mathrm{Ann(J) = I}$.

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Definition of (Exact) Annihilating Ideal Graph, this article further describes the properties of the graph with rings $\mathbb{Z}_n$. Comparison of the properties of annihilating ideal and exact annihilating ideal graph of ring $\mathbb{Z}_n$ is also presented in this article.

3. RESULTS AND DISCUSSION

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In this section, we will show some result of Annihilating ideal graph of ring $\mathbb{Z}_n$.

Theorem 1. Lower Bound of Cardinality $\mathbb{A}(\mathbb{Z}_n)^*$

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So it is only written as "Theorem 1"

Suppose $\mathbb{Z}_n$ ring of integer modulo n where n not prime.

1. If $n = p^2$, *where p is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.
2. If $n \neq p^2$, *where p is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$

Proof.

(1) Suppose $n = p^2$ then there exists uniquely non zero proper ideal in di $\mathbb{Z}_n$, $\langle \bar{p} \rangle = \{\overline{pz} | \bar{z} \in \mathbb{Z}_n\}$. If $n = p^2$ its means $\overline{p^2} = \bar{n} = \bar{0}$ such that $\langle \bar{p} \rangle \langle \bar{p} \rangle = \langle \bar{0} \rangle$. Ideal $\langle \bar{p} \rangle$ is an annihilating ideal of $\mathbb{Z}_n$ by Definition 1. Since ideal $\langle \bar{p} \rangle$ is the only one of proper non zero ideal in $\mathbb{Z}_n$, hence $\mathbb{A}(\mathbb{Z}_n)^* = \{\langle \bar{p} \rangle\}$ or $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.

(2) Suppose n is nonprime, that is $n = ab$ for some $a, b \in \mathbb{Z}$ where $1 < a < n$, $1 < b < n$, and $a \neq b$. The product of two ideal, $\langle a \rangle \langle b \rangle = \{(za)(yb) | a, b \in \mathbb{Z}, y, z \in \mathbb{Z}_n\}$. Since $n = ab$, $zy(ab) = zy(n)$ then $\langle a \rangle \langle b \rangle = \{zyn | z, y \in \mathbb{Z}_n\} = \langle \bar{0} \rangle$. Clearly, $\langle a \rangle \neq \langle \bar{0} \rangle$ and $\langle b \rangle \neq \langle \bar{0} \rangle$. Ideals $\langle a \rangle$ and $\langle b \rangle$ are annihilating ideal by Definition 1. Hence, $\langle a \rangle, \langle b \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$. That is prove that for any nonprime n, $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$. ■

Theorem 2. Cardinality of $\mathbb{A}(\mathbb{Z}_n)^*$

Suppose $\mathbb{Z}_n$ ring of integer modulo n. The number of vertices of annihilating ideal graph $\mathbb{AG}(\mathbb{Z}_n)$ is $\varphi(n) - 2$, where $\varphi(n)$ is the number of positive factors of n.

Proof.

Suppose $n = (p_1)^{\alpha_1}(p_2)^{\alpha_2} \ldots (p_n)^{\alpha_n}$ is prime factorization of n. If $x|n$ then $x = (p_1)^{\beta_1}(p_2)^{\beta_2} \ldots (p_n)^{\beta_n}$ where $\beta_i \leq \alpha_i$ for all i. If $x|n$, also means that there exists integer y such that $xy = n$. Suppose $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \ldots (p_n)^{\gamma_n}$ then $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \ldots (p_n)^{\gamma_n}$, where $\alpha_i = \beta_i + \gamma_i$ for $1 \leq i \leq n$.

We construct principal ideal $\langle x \rangle = \{\overline{xz} | z \in \mathbb{Z}_n\}$ and $\langle y \rangle = \{\overline{yt} | t \in \mathbb{Z}_n\}$ of $\mathbb{Z}_n$. The product of these ideal $\langle x \rangle \langle y \rangle = \{(\overline{xz})(\overline{yt})\} = \{(\overline{xy})(\overline{zt})\}$. As $xy = n$ implies $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. For all $\langle x \rangle$, where x is a positive factor of n, there exists ideal $\langle y \rangle$ such that $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. The number of Ideal $\langle x \rangle$ that satisfied the condition is the number of positive factor of n, $\varphi(n)$. Suppose the set

$$\mathbb{I}(\mathbb{Z}_n) = \{\langle x \rangle \text{ ideal } \mathbb{Z}_n | \exists y \in \mathbb{Z} \text{ such that } xy = n\}$$

Based on the process above, we have $|\mathbb{I}(\mathbb{Z}_n)| = \varphi(n)$. All of elements $\mathbb{I}(\mathbb{Z}_n)$ is the elements of $\mathbb{A}(\mathbb{Z}_n)^*$ except $\langle 1 \rangle$ and $\langle n \rangle$. Hence $|\mathbb{A}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. ■

Theorem 3. Criteria vertex of $\mathbb{AG}(\mathbb{Z}_n)$

Suppose $\mathbb{Z}_n$ ring of integer modulo n. If $\langle \bar{a} \rangle$ is a vertex of graph $\mathbb{AG}(\mathbb{Z}_n)$ then a is a factor of .

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Proof.

Assume a isn't factor of n. We have $n = ax + y$, where x and y is integer and $0 < y < a$.
The product of ideal $\langle \bar{a} \rangle$ and $\langle \bar{x} \rangle$ is

$$\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\bar{a}r)(\bar{x}n)\} = \{\bar{a}(r\bar{x}n)\} = \{\bar{a}(\bar{x}rn)\} = \{(\bar{a}\bar{x})rn\} = \{(\overline{ax})rn\}$$

We have element $\bar{n} = \bar{y}$ because $n = ax + y$. Then $\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\overline{ax})rn\} = \{(\overline{axr})n\} = \langle \bar{n} \rangle = \langle \bar{y} \rangle$. In means $\langle \bar{a} \rangle$ isn't a ideal annihilator of $\mathbb{Z}_n$. Hence $\langle \bar{a} \rangle \notin \mathbb{A}(\mathbb{Z}_n)^*$. By the contraposition, we have if $\langle \bar{a} \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$ then a is a factor of n. ■

The converse of Theorem 3 is not true. For all $n \in \mathbb{Z}$, we have $1|n$, but clearly $\langle \bar{1} \rangle$ is not an ideal annihilator of $\mathbb{Z}_n$. It means $\langle \bar{1} \rangle$ is not a vertex ini $\mathbb{AG}(\mathbb{Z}_n)$.

Theorem 4. Adjacency of $\mathbb{AG}(\mathbb{Z}_n)$

Suppose $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are ideal in $\mathbb{Z}_n$. Vertex $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ if and only if $n|pq$.

Proof.

Suppose $\langle p \rangle = \{pa | a \in \mathbb{Z}_n\}$ and $\langle q \rangle = \{qb | r \in \mathbb{Z}_n\}$. The product $\langle p \rangle \langle q \rangle = \langle \overline{pq} \rangle$. If $n|pq$ then $\langle p \rangle \langle q \rangle = \langle \bar{0} \rangle = \{\bar{0}\}$. Hence $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ by Definition 3.
If $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent then $\langle p \rangle \langle q \rangle = \{\bar{0}\}$. It means $(pq)(ab) = nk$ for some integer a, b, and k. The equation $(pq)(ab) = nk$ implies $n|(pq)(ab)$, especially must be $n|pq$. ■

We will continue to discuss the relation some part of annihilating ideal and exact annihilating ideal graph of any commutative ring R.

Lemma 5. Relation Set of All Vertex $\mathbb{EAG}(\boldsymbol{R})$ and $\mathbb{AG}(\boldsymbol{R})$

For any commutative ring R, $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$

Proof.

Take any ideal $I \in \mathbb{EA}(R)^*$. It means there exist ideal J of R such that $Ann(I) = J$ and $Ann(J) = I$. Based on definition of annihilator, the product of ideal $IJ = 0$. Hence $I \in \mathbb{A}(R)^*$.

Now, take any ideal $I \in \mathbb{A}(R)^*$. It means there exist nonzero ideal J such that $IJ = 0$. Ideal I is annihilator ideal then $Ann(I) \neq 0$. Suppose $J = Ann(I)$ then J is nonzero ideal of R. We have $Ann(J) = Ann\big(Ann(I)\big) = I$. We conclude $Ann(I) = J$ and $Ann(J) = I$. Hence $I \in \mathbb{EA}(R)^*$. ■

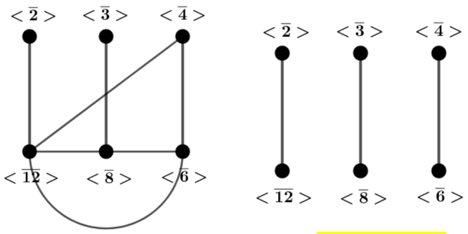
Lemma 6.

For any commutative ring R, $\mathbb{EAG}(R)$ is a subgraph of $\mathbb{AG}(R)$.

Proof.

Lemma 5 show us that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. We will prove that for all $(I,J) \in E\big(\mathbb{EAG}(R)\big)$ then $(I,J) \in E(\mathbb{AG}(R))$. Adjacency of ideal I and J on $\mathbb{EAG}(R)$ means that $I = Ann(J)$ and $J = Ann(I)$. Based on properties of annihilator of ideal, we have $IJ = 0$. Based on definition of adjacency on $\mathbb{AG}(R)$, we have $(I,J) \in E(\mathbb{AG}(R))$. ■

The converse of Theorem 4 not valid for exact annihilating ideal graph. Figure 1 below show that vertex $\langle\bar{6}\rangle$ and $\langle\overline{12}\rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_{24})$ but not adjacent in $\mathbb{EAG}(\mathbb{Z}_{24})$ although $24|12 \times 6$.



(a) $\mathbb{AG}(\mathbb{Z}_{24})$ (b) $\mathbb{EAG}(\mathbb{Z}_{24})$

Figure 1. Representation of $\mathbb{AG}(\mathbb{Z}_{24})$ and $\mathbb{EAG}(\mathbb{Z}_{24})$

Based on the situation, we will construct the criteria of adjacency in exact annihilating ideal graph.

Theorem 7. Adjacency of $\mathbb{EAG}(\mathbb{Z}_n)$

Suppose commutative ring $\mathbb{Z}_n$ with identity $\bar{1}$. Ideals $\langle\bar{p}\rangle$ and $\langle\bar{q}\rangle$ are adjacent vertex of $\mathbb{EAG}(\mathbb{Z}_n)$ if and only if $n = pq$.

Diberikan ring komutatif $\mathbb{Z}_n$ dengan satuan $\bar{1}$. $\langle\bar{p}\rangle$ dan $\langle\bar{q}\rangle$ merupakan simpul yang bertetangga di $\mathbb{EAG}(\mathbb{Z}_n)$ jika dan hanya jika $n = pq$.

Proof.

($\leftarrow$). Assume $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent. We will proof $n \neq pq$. We have $\langle\bar{p}\rangle = \{\overline{pa}|\bar{a}, \bar{p} \in \mathbb{Z}_n\}$ and $\langle\bar{q}\rangle = \{\overline{qb}|\bar{b}, \bar{q} \in \mathbb{Z}_n\}$ are not adjacent. It means $Ann(\langle\bar{p}\rangle) \neq \langle\bar{q}\rangle$ and $Ann(\langle\bar{q}\rangle) \neq \langle\bar{p}\rangle$ such that $\langle\bar{p}\rangle\langle\bar{q}\rangle \neq \{\bar{0}\}$. We use commutative and associative property of $\mathbb{Z}_n$ to get form $\langle\bar{p}\rangle\langle\bar{q}\rangle = \{(\overline{pa})(\overline{qb})\} = \{(\overline{pq})(\overline{ab})\} \neq \{\bar{0}\}$. It imply $pq \nmid n$. Hence $pq \neq n$.

($\rightarrow$). Assume $n \neq pq$. We will proof vertex $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$. If $n \neq pq$ then $n = pq + a$ with a is non-zero integer. We construct two principal ideal generated by p and q on $\mathbb{Z}_n$. Now, we have the product of these ideal

$$\langle\bar{p}\rangle\langle\bar{q}\rangle = \{(\overline{pr})(\overline{qt})\} = \{(\overline{n-a})(\overline{rt})\} = \langle\overline{-a}\rangle$$

We have $(\langle\bar{p}\rangle, \langle\bar{q}\rangle) \notin E\big(\mathbb{AG}(\mathbb{Z}_n)\big)$. Based on Lemma 6, vertex $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$.■

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Theorem 8. Isolated Vertex in $\mathbb{EAG}(\mathbb{Z}_n)$

Suppose commutative ring $\mathbb{Z}_n$ with identity $\bar{1}$. If $n = r^2$ then $\langle \bar{r} \rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$.

Proof.

Suppose $n = r^2$ and principal ideal $\langle \bar{r} \rangle$ of ring $\mathbb{Z}_n$. We have $Ann(\langle \bar{r} \rangle) = \langle \bar{r} \rangle$. Its means $\langle \bar{r} \rangle$ is a vertex of $\mathbb{EAG}(\mathbb{Z}_n)$. Assume there is a vertex $\langle \bar{a} \rangle$ (not equal to $\langle \bar{r} \rangle$) of $\mathbb{EAG}(\mathbb{Z}_n)$ such that $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$ adjacent. The product of the ideals is $\langle \bar{a} \rangle\langle \bar{r} \rangle \neq \langle \bar{r} \rangle\langle \bar{r} \rangle = \langle \bar{0} \rangle$. Vertex $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$ adjacent on $\mathbb{EAG}(\mathbb{Z}_n)$ means that $\langle \bar{r} \rangle = Ann(\langle \bar{a} \rangle)$ and $\langle \bar{a} \rangle = Ann(\langle \bar{r} \rangle)$. Furthermore $\langle \bar{r} \rangle\langle \bar{a} \rangle = \langle \bar{0} \rangle$. Its contradiction with the product ideals $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$. Hence there is no vertex adjacent with $\langle \bar{r} \rangle$ on $\mathbb{EAG}(\mathbb{Z}_n)$. ■

In [11] showed that $diam(\mathbb{EAG}(R)) < 1$ and $g(\mathbb{EAG}(R)) \leq 4$ for any commutative ring R. In this paper, we will show more specific result about diameter, girth, cycle existence of $\mathbb{EAG}(R)$.

Theorem 9.

Suppose commutative ring R. If $\mathbb{EAG}(R)$ is connected graph then $diam(\mathbb{EAG}(R)) = 1$.

Proof.

Suppose I and J are two different vertex of $\mathbb{EAG}(R)$. Assume $d(I,J) = 2 > 1$, means that exist a vertex A of $\mathbb{EAG}(R)$ such that $I - A - J$ is a path in $\mathbb{EAG}(R)$. Based on Definition 4 we have $I = Ann(A), A = Ann(I), A = Ann(J)$, and $J = Ann(A)$. It imply $I = Ann(A) = Ann(Ann(J))$. Based on Lemma 2.1 on [3], we get $Ann(Ann(J)) = J$. Two last equation imply $I = J$. We have a contradiction with ideal I and J must be different. So, $d(I,J) = 1$ for all ideal I and J. It proved that $diam(\mathbb{EAG}(R)) = 1$. ■

Collorary 10.

Suppose commutative ring R. If $\mathbb{EAG}(R)$ contain a cycle then $g(\mathbb{EAG}(R)) \leq 3$.

Proof.

If graph G contain a cycle then $g(G) \leq 2diam(G) + 1$. Theorem 9 has shown that $diam(\mathbb{EAG}(R)) = 1$. Finally, we have $g(\mathbb{EAG}(R)) \leq 2diam(\mathbb{EAG}(R)) + 1 = 3$. ■

Theorem 3.9 in [11] showed that $\mathbb{EAG}(\mathbb{Z}_{p^n})$ where p is prime can be represented as union of some complete graph. Figure 1 below show that $\mathbb{EAG}(\mathbb{Z}_{24})$ can be represented as union of K_2 graph, although $24 \neq p^n$ for any prime p. Based on this fact, we construct a theorem to generalize properties of representation of $\mathbb{EAG}(\mathbb{Z}_n)$.

Theorem 11. Decomposition of $\mathbb{EAG}(\mathbb{Z}_n)$

The number of complete subgraph of Exact annihilating ideal graph of ring $\mathbb{Z}_n$ is $\left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$.

Proof.

Lemma 5 showed that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. Based on Theorem 2, we have $|\mathbb{EA}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. Theorem 9 showed that $diam(\mathbb{EAG}(R)) = 1$ for any commutative ring R. We conclude that the maximum number of edges $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)}{2} - 1$. Its means the maximum complete subgraph of $\mathbb{EAG}(\mathbb{Z}_n)$ is also $\frac{\varphi(n)}{2} - 1$.

Case 1: $n = r^2$ for some integer r

Based on Theorem 8, $\langle \bar{r} \rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$. We have $\varphi(n) - 3$ other vertices of $\mathbb{EAG}(\mathbb{Z}_n)$. Obviously there is no positive integer a such that $n = a^2$. In another word, we just found exactly one isolated vertex on $\mathbb{EAG}(\mathbb{Z}_n)$. We can partition $\mathbb{EAG}(\mathbb{Z}_n)$ to be $\frac{\varphi(n)-3}{2}$ graph K_2. Isolated vertex can be represented as K_1. The total of number complete graph that contain in $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)-3}{2} + 1 = \frac{\varphi(n)+1}{2} - 1 = \left\lceil \frac{\varphi(n)}{2} \right\rceil - 1 = \left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$.

Case 2: $n \neq r^2$ for any integer r

If $n \neq r^2$ for any integer r then $n = ab$ where $a \neq b$. Ideal $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ are vertices in $\mathbb{EAG}(\mathbb{Z}_n)$. Based on theorem 7, $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ adjacent in $\mathbb{EAG}(\mathbb{Z}_n)$. This condition means there is no isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$. Graph $\mathbb{EAG}(\mathbb{Z}_n)$ is fully partition into complete graph K_2. Total number of K_2 is $\frac{\varphi(n)-2}{2} = \frac{\varphi(n)}{2} - 1 = \left\lceil \frac{\varphi(n)}{2} \right\rceil - 1 = \left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$. ■

4. CONCLUSIONS

Factorization on $\mathbb{Z}_n$ characterizes the (Exact) Annihilating Ideal Graph, especially in 1) the number of vertices in an annihilating ideal graph, 2) adjacency of the vertices, and 3) decomposition of exact annihilating ideal graph.

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ANNIHILATING IDEAL AND EXACT ANNIHILATING IDEAL GRAPH OF RING $\mathbb{Z}_n$

Anindito Wisnu Susanto [1], **Dewa Putu Wiadnyana Putra** [2*]

[1,2] *Department of Mathematics Education, Faculty of Teacher Training and Education, Sanata Dharma University Jl. Affandi, Mrican, Caturtunggal, Depok, Sleman, Yogyakarta, 55281, Indonesia*

Corresponding author's e-mail: [2*] **dewa@usd.ac.id**

ABSTRACT

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The existence of annihilator in the ring motivates the emergence of studies on Annihilating Ideal and Exact Annihilating Ideal Graphs. The purpose of this research is to describe the characteristics of an (exact) annihilating ideal of ring $\mathbb{Z}_n$. The method used in this research is literature study. The results of this study discuss finiteness, adjacency, connectedness, vertices, and types of $\mathbb{AG}(\mathbb{Z}_n)$ and $\mathbb{EAG}(\mathbb{Z}_n)$. Furthermore, the number of vertices of an Annihilating Ideal Graph is determined by the factorization of n. The adjacency of two vertices is determined by the divisibleness of n. The results also show that $\mathbb{EAG}(\mathbb{Z}_n)$ is a subgraph of $\mathbb{AG}(\mathbb{Z}_n)$. $\mathbb{EAG}(\mathbb{Z}_n)$ can be represented as a union of several complete graphs.

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1. INTRODUCTION

The use of graphs in representing algebraic structures has been carried out since at least 1878 in [1]. This representation starts from representing a group structure into a graph. The vertices of a graph are all elements of a group and changes to an element due to operations on the group are represented by directed edges. Furthermore, [2], [3] began to associate graphs with a broader structure, namely rings. Investigation of the ring structure is carried out through the colored representation of the graph. The representation of an algebraic structure on a graph opens up opportunities for visual investigation of the properties of a particular structure. An essential part in the process of representing a particular algebraic structure to a graph is how to define the connection between the vertices of the graph. Different ways of defining adjacent vertices can lead to different variations of properties as well.

One of the interesting things in the ring, which is about zero divisor. A non-zero element a is said to be a zero divisor if it can be found a non-zero element b such that $ab = 0$. From this structure, [4] proposed the origin of the zero divisor graph. The vertices of the graph are all zero divisors. Two vertices are adjacent if and only if the product of the two elements is zero. Many interesting properties result from this concept, one of which is about the combinatorics of a finite ring [5], [6], [7].

The concept is similar to zero divisor in the ring is the Annihilator. Badawi has started a study on annihilator graphs [8]. In its development, annihilating graphs are generalized into annihilating Ideal graphs. In [9], it is stated that Annihilator is an ideal I, namely $Ann(I) = \{r \in R | rl = 0 \; \forall \, l \in L\}$. If $Ann(I)$ is not a trivial set, then I is called an ideal annilator. In 2011, [10] started to represent a structure consisting of annihilator ideals into a graph. The graph that is formed is named Annihilating Ideal Graph. In line with the development of zero divisor graphs, [11] is continuing the study of Exact Annihilating Ideal graphs. The development of ideal annihilating and exact annihilating properties of graphs is studied separately. The general relationship between these two graphs began to be investigated by [12].

An integer modulo n, $\mathbb{Z}_n$ is a ring that has very interesting properties. This $\mathbb{Z}_n$ structure is widely used in graphs, for example in coloring Antimagic graphs [13] and Domination ratio [14]. The factorization theorem on integers is a motivation for developing graph studies involving a ring of integers modulo n. One of the graph studies carried out was a study on non-coprime for $\mathbb{Z}_n$ [15]. Based on this, the study of annihilating ideal graph for $\mathbb{Z}_n$ rings is interesting to do.

2. RESEARCH METHODS

This research is a literature research that examines the properties of annihilating ideal and exact annihilating ideal graphs on integer rings modulo n, $\mathbb{Z}_n$. The definition of (Exact) Annihilating Ideal based on [10], [11] is as follows.

Definition 1. [10]
An Ideal I of commutative ring R with identity is a Annihilating Ideal if there exist non zero ideal J of R such that $\mathrm{IJ} = 0$. The set of all Annihilating Ideal of ring R is denoted by $\mathbb{A}(\mathrm{R})$.

Definition 2. [11]
An ideal I of commutative ring R with identity is Exact Annihilating Ideal if there exist non zero ideal J of R such that $\mathrm{Ann}(\mathrm{I}) = \mathrm{J}$ and $\mathrm{Ann}(\mathrm{J}) = \mathrm{I}$. The set of all Exact Annihilating Ideal of ring R denoted by $\mathbb{EA}(\mathrm{R})$.

Based on the two definitions above, then (Exact) Annihilating Ideal Graph is defined as follows.

Definition 3. [10]
Annihilating Ideal graph of ring R denoted by $\mathbb{A}\mathrm{G}(\mathrm{R})$ is a graph with vertices $\mathbb{A}(\mathrm{R})^* = \mathbb{A}(\mathrm{R}) \backslash \{(0)\}$ and $(\mathrm{I}, \mathrm{J}) \in \mathrm{E}(\mathbb{A}\mathrm{G}(\mathrm{R}))$ if and only if $\mathrm{IJ} = (0)$.

Definition 4. [11]
Exact Annihilating Ideal graph of ring R denoted by $\mathbb{EAG}(\mathrm{R})$ is a graph with vertices $\mathbb{EA}(\mathrm{R})^* = \mathbb{EA}(\mathrm{R}) \backslash \{(0)\}$ and $(\mathrm{I}, \mathrm{J}) \in \mathrm{E}(\mathbb{EAG}(\mathrm{R}))$ if and only if $\mathrm{Ann}(\mathrm{I}) = \mathrm{J}$ and $\mathrm{Ann}(\mathrm{J}) = \mathrm{I}$.

Definition of (Exact) Annihilating Ideal Graph, this article further describes the properties of the graph with rings $\mathbb{Z}_n$. Comparison of the properties of annihilating ideal and exact annihilating ideal graph of ring $\mathbb{Z}_n$ is also presented in this article.

3. RESULTS AND DISCUSSION

In this section, we will show some result of Annihilating ideal graph of ring $\mathbb{Z}_n$.

Theorem 1.
Suppose $\mathbb{Z}_n$ ring of integer modulo n where n not prime.

1. If $n = p^2$, *where p is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.
2. If $n \neq p^2$, *where p is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$

Proof.

(1) Suppose $n = p^2$ then there exists uniquely non zero proper ideal in di $\mathbb{Z}_n$, $\langle \bar{p} \rangle = \{\overline{pz} | \bar{z} \in \mathbb{Z}_n\}$. If $n = p^2$ its means $\overline{p^2} = \bar{n} = \bar{0}$ such that $\langle \bar{p} \rangle \langle \bar{p} \rangle = \langle \bar{0} \rangle$. Ideal $\langle \bar{p} \rangle$ is an annihilating ideal of $\mathbb{Z}_n$ by Definition 1. Since ideal $\langle \bar{p} \rangle$ is the only one of proper non zero ideal in $\mathbb{Z}_n$, hence $\mathbb{A}(\mathbb{Z}_n)^* = \{\langle \bar{p} \rangle\}$ or $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.

(2) Suppose n is nonprime, that is $n = ab$ for some $a, b \in \mathbb{Z}$ where $1 < a < n$, $1 < b < n$, and $a \neq b$. The product of two ideal, $\langle a \rangle \langle b \rangle = \{(za)(yb) | a, b \in \mathbb{Z}, y, z \in \mathbb{Z}_n\}$. Since $n = ab$, $zy(ab) = zy(n)$ then $\langle a \rangle \langle b \rangle = \{zyn | z, y \in \mathbb{Z}_n\} = \langle \bar{0} \rangle$. Clearly, $\langle a \rangle \neq \langle \bar{0} \rangle$ and $\langle b \rangle \neq \langle \bar{0} \rangle$. Ideals $\langle a \rangle$ and $\langle b \rangle$ are annihilating ideal by Definition 1. Hence, $\langle a \rangle, \langle b \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$. That is prove that for any nonprime n, $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$. ■

Theorem 2.
Suppose $\mathbb{Z}_n$ ring of integer modulo n. The number of vertices of annihilating ideal graph $\mathbb{AG}(\mathbb{Z}_n)$ is $\varphi(n) - 2$, where $\varphi(n)$ is the number of positive factors of n.

Proof.
Suppose $n = (p_1)^{\alpha_1}(p_2)^{\alpha_2} \dots (p_n)^{\alpha_n}$ is prime factorization of n. If $x|n$ then $x = (p_1)^{\beta_1}(p_2)^{\beta_2} \dots (p_n)^{\beta_n}$ where $\beta_i \leq \alpha_i$ for all i. If $x|n$, also means that there exists integer y such that $xy = n$. Suppose $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \dots (p_n)^{\gamma_n}$ then $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \dots (p_n)^{\gamma_n}$, where $\alpha_i = \beta_i + \gamma_i$ for $1 \leq i \leq n$.

We construct principal ideal $\langle x \rangle = \{\overline{xz} | z \in \mathbb{Z}_n\}$ and $\langle y \rangle = \{\overline{yt} | t \in \mathbb{Z}_n\}$ of $\mathbb{Z}_n$. The product of these ideal $\langle x \rangle \langle y \rangle = \{(\overline{xz})(\overline{yt})\} = \{(\overline{xy})(\overline{zt})\}$. As $xy = n$ implies $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. For all $\langle x \rangle$, where x is a positive factor of n, there exists ideal $\langle y \rangle$ such that $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. The number of Ideal $\langle x \rangle$ that satisfied the condition is the number of positive factor of n, $\varphi(n)$. Suppose the set

$$\mathbb{I}(\mathbb{Z}_n) = \{\langle x \rangle \text{ ideal } \mathbb{Z}_n | \exists y \in \mathbb{Z} \text{ such that } xy = n\}$$

Based on the process above, we have $|\mathbb{I}(\mathbb{Z}_n)| = \varphi(n)$. All of elements $\mathbb{I}(\mathbb{Z}_n)$ is the elements of $\mathbb{A}(\mathbb{Z}_n)^*$ except $\langle 1 \rangle$ and $\langle n \rangle$. Hence $|\mathbb{A}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. ■

Theorem 3.
Suppose $\mathbb{Z}_n$ ring of integer modulo n. If $\langle \bar{a} \rangle$ is a vertex of graph $\mathbb{AG}(\mathbb{Z}_n)$ then a is a factor of n.

Proof.
Assume a isn't factor of n. We have $n = ax + y$, where x and y is integer and $0 < y < a$.
The product of ideal $\langle \bar{a} \rangle$ and $\langle \bar{x} \rangle$ is

$$\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\bar{a}r)(\bar{x}n)\} = \{\bar{a}(r\bar{x}n)\} = \{\bar{a}(\bar{x}rn)\} = \{(\bar{a}\bar{x})rn\} = \{(\overline{ax})rn\}$$

We have element $\bar{n} = \bar{y}$ because $n = ax + y$. Then $\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\overline{ax})rn\} = \{(\overline{axr})n\} = \langle \bar{n} \rangle = \langle \bar{y} \rangle$. In means $\langle \bar{a} \rangle$ isn't a ideal annihilator of $\mathbb{Z}_n$. Hence $\langle \bar{a} \rangle \notin \mathbb{A}(\mathbb{Z}_n)^*$. By the contraposition, we have if $\langle \bar{a} \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$ then a is a factor of n. ■

The converse of Theorem 3 is not true. For all $n \in \mathbb{Z}$, we have $1|n$, but clearly $\langle \bar{1} \rangle$ is not an ideal annihilator of $\mathbb{Z}_n$. It means $\langle \bar{1} \rangle$ is not a vertex ini $\mathbb{AG}(\mathbb{Z}_n)$.

Theorem 4.
Suppose $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are ideal in $\mathbb{Z}_n$. Vertex $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ if and only if $n|pq$.

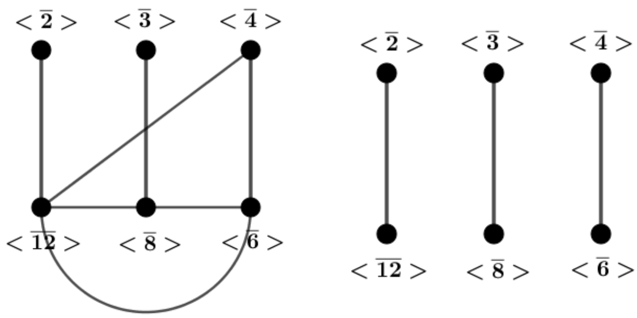
Proof.
Suppose $\langle p \rangle = \{pa | a \in \mathbb{Z}_n\}$ and $\langle q \rangle = \{qb | r \in \mathbb{Z}_n\}$. The product $\langle p \rangle \langle q \rangle = \langle \overline{pq} \rangle$. If $n|pq$ then $\langle p \rangle \langle q \rangle = \langle \bar{0} \rangle = \{\bar{0}\}$. Hence $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ by Definition 3.

If $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent then $\langle p \rangle \langle q \rangle = \{\bar{0}\}$. It means $(pq)(ab) = nk$ for some integer $a, b,$ and k. The equation $(pq)(ab) = nk$ implies $n|(pq)(ab)$, especially must be $n|pq$. ■

We will continue to discuss the relation some part of annihilating ideal and exact annihilating ideal graph of any commutative ring R.

Lemma 5.
For any commutative ring R, $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$
Proof.
Take any ideal $I \in \mathbb{EA}(R)^*$. It means there exist ideal J of R such that $Ann(I) = J$ and $Ann(J) = I$. Based on definition of annihilator, the product of ideal $IJ = 0$. Hence $I \in \mathbb{A}(R)^*$.
Now, take any ideal $I \in \mathbb{A}(R)^*$. It means there exist nonzero ideal J such that $IJ = 0$. Ideal I is annihilator ideal then $Ann(I) \neq 0$. Suppose $J = Ann(I)$ then J is nonzero ideal of R. We have $Ann(J) = Ann\big(Ann(I)\big) = I$. We conclude $Ann(I) = J$ and $Ann(J) = I$. Hence $I \in \mathbb{EA}(R)^*$. ■

Lemma 6.
For any commutative ring R, $\mathbb{EAG}(R)$ is a subgraph of $\mathbb{AG}(R)$.
Proof.
Lemma 5 show us that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. We will prove that for all $(I, J) \in E\big(\mathbb{EAG}(R)\big)$ then $(I, J) \in E(\mathbb{AG}(R))$. Adjacency of ideal I and J on $\mathbb{EAG}(R)$ means that $I = Ann(J)$ and $J = Ann(I)$. Based on properties of annihilator of ideal, we have $IJ = 0$. Based on definition of adjacency on $\mathbb{AG}(R)$, we have $(I, J) \in E(\mathbb{AG}(R))$. ■
The converse of Theorem 4 not valid for exact annihilating ideal graph. Figure 1 below show that vertex $\langle \overline{6} \rangle$ and $\langle \overline{12} \rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_{24})$ but not adjacent in $\mathbb{EAG}(\mathbb{Z}_{24})$ although $24|12 \times 6$.



(a) **(b)**

Figure 1. Representation of Annihilating Ideal and Exact Annihilating Ideal Graph
(a)$\mathbb{AG}(\mathbb{Z}_{24})$, **(b)**$\mathbb{EAG}(\mathbb{Z}_{24})$

Based on the situation, we will construct the criteria of adjacency in exact annihilating ideal graph.
Theorem 7.
Suppose commutative ring $\mathbb{Z}_n$ with identity $\bar{1}$. Ideals $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent vertex of $\mathbb{EAG}(\mathbb{Z}_n)$ if and only if $n = pq$.
Proof.
$(\leftarrow)$. Assume $\langle \bar{p} \rangle$ *and* $\langle \bar{q} \rangle$ are not adjacent. We will proof $n \neq pq$. We have $\langle \bar{p} \rangle = \{\overline{pa} | \bar{a}, \bar{p} \in \mathbb{Z}_n\}$ and $\langle \bar{q} \rangle = \{\overline{qb} \big| \bar{b}, \bar{q} \in \mathbb{Z}_n\}$ are not adjacent. It means $Ann(\langle \bar{p} \rangle) \neq \langle \bar{q} \rangle$ and $Ann(\langle \bar{q} \rangle) \neq \langle \bar{p} \rangle$ such that $\langle \bar{p} \rangle \langle \bar{q} \rangle \neq \{\bar{0}\}$. We use commutative and associative property of $\mathbb{Z}_n$ to get form $\langle \bar{p} \rangle \langle \bar{q} \rangle = \big\{(\overline{pa})(\overline{qb})\big\} = \big\{(\overline{pq})(\overline{ab})\big\} \neq \{\bar{0}\}$. It imply $pq \nmid n$. Hence $pq \neq n$.
$(\rightarrow)$. Assume $n \neq pq$. We will proof vertex $\langle \bar{p} \rangle$ *and* $\langle \bar{q} \rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$. If $n \neq pq$ then $n = pq + a$ with a is non-zero integer. We construct two principal ideal generated by p and q on $\mathbb{Z}_n$. Now, we have the product of these ideal

$$\langle \bar{p} \rangle \langle \bar{q} \rangle = \{(\overline{pr})(\overline{qt})\} = \{(\overline{n - a})(\overline{rt})\} = \langle \overline{-a} \rangle$$

We have $(\langle \bar{p} \rangle, \langle \bar{q} \rangle) \notin E\big(\mathbb{AG}(\mathbb{Z}_n)\big)$. Based on Lemma 6, vertex $\langle \bar{p} \rangle$ *and* $\langle \bar{q} \rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$.■

Theorem 8.
Suppose commutative ring $\mathbb{Z}_n$ with identity $\bar{1}$. If $n = r^2$ then $\langle \bar{r} \rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$.
Proof.
Suppose $n = r^2$ and principal ideal $\langle \bar{r} \rangle$ of ring $\mathbb{Z}_n$. We have $Ann(\langle \bar{r} \rangle) = \langle \bar{r} \rangle$. Its means $\langle \bar{r} \rangle$ is a vertex of $\mathbb{EAG}(\mathbb{Z}_n)$. Assume there is a vertex $\langle \bar{a} \rangle$ (not equal to $\langle \bar{r} \rangle$) of $\mathbb{EAG}(\mathbb{Z}_n)$ such that $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$ adjacent. The product of the ideals is $\langle \bar{a} \rangle \langle \bar{r} \rangle \neq \langle \bar{r} \rangle \langle \bar{r} \rangle = \langle \bar{0} \rangle$. Vertex $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$ adjacent on $\mathbb{EAG}(\mathbb{Z}_n)$ means that $\langle \bar{r} \rangle = Ann(\langle \bar{a} \rangle)$ and $\langle \bar{a} \rangle = Ann(\langle \bar{r} \rangle)$. Furthermore $\langle \bar{r} \rangle \langle \bar{a} \rangle = \langle \bar{0} \rangle$. Its contradiction with the product ideals $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$. Hence there is no vertex adjacent with $\langle \bar{r} \rangle$ on $\mathbb{EAG}(\mathbb{Z}_n)$. ■

In [11] showed that $diam(\mathbb{EAG}(R)) < 1$ and $g(\mathbb{EAG}(R)) \leq 4$ for any commutative ring R. In this paper, we will show more specific result about diameter, girth, cycle existence of $\mathbb{EAG}(R)$.

Theorem 9.
Suppose commutative ring R. If $\mathbb{EAG}(R)$ is connected graph then $diam(\mathbb{EAG}(R)) = 1$.
Proof.
Suppose I and J are two different vertex of $\mathbb{EAG}(R)$. Assume $d(I,J) = 2 > 1$, means that exist a vertex A of $\mathbb{EAG}(R)$ such that $I - A - J$ is a path in $\mathbb{EAG}(R)$. Based on Definition 4 we have $I = Ann(A), A = Ann(I), A = Ann(J),$ and $J = Ann(A)$. It imply $I = Ann(A) = Ann(Ann(J))$. Based on Lemma 2.1 on [3], we get $Ann(Ann(J)) = J$. Two last equation imply $I = J$. We have a contradiction with ideal I and J must be different. So, $d(I,J) = 1$ for all ideal I and J. It proved that $diam(\mathbb{EAG}(R)) = 1$. ■

Collorary 10.
Suppose commutative ring R. If $\mathbb{EAG}(R)$ contain a cycle then $g(\mathbb{EAG}(R)) \leq 3$.

Proof.
If graph G contain a cycle then $g(G) \leq 2diam(G) + 1$. Theorem 9 has shown that $diam(\mathbb{EAG}(R)) = 1$. Finally, we have $g(\mathbb{EAG}(R)) \leq 2diam(\mathbb{EAG}(R)) + 1 = 3$. ■

Theorem 3.9 in [11] showed that $\mathbb{EAG}(\mathbb{Z}_{p^n})$ where p is prime can be represented as union of some complete graph. Figure 1 below show that $\mathbb{EAG}(\mathbb{Z}_{24})$ can be represented as union of K_2 graph, although $24 \neq p^n$ for any prime p. Based on this fact, we construct a theorem to generalize properties of representation of $\mathbb{EAG}(\mathbb{Z}_n)$.

Theorem 11.
The number of complete subgraph of Exact annihilating ideal graph of ring $\mathbb{Z}_{\boldsymbol{n}}$ is $\left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$.
Proof.
Lemma 5 showed that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. Based on Theorem 2, we have $|\mathbb{EA}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. Theorem 9 showed that $diam(\mathbb{EAG}(R)) = 1$ for any commutative ring R. We conclude that the maximum number of edges $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)}{2} - 1$. Its means the maximum complete subgraph of $\mathbb{EAG}(\mathbb{Z}_n)$ is also $\frac{\varphi(n)}{2} - 1$.
Case 1: $n = r^2$ for some integer r
Based on Theorem 8, $\langle \bar{r} \rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$. We have $\varphi(n) - 3$ other vertices of $\mathbb{EAG}(\mathbb{Z}_n)$. Obviously there is no positive integer a such that $n = a^2$. In another word, we just found exactly one isolated vertex on $\mathbb{EAG}(\mathbb{Z}_n)$. We can partition $\mathbb{EAG}(\mathbb{Z}_n)$ to be $\frac{\varphi(n)-3}{2}$ graph K_2. Isolated vertex can be represented as K_1. The total of number complete graph that contain in $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)-3}{2} + 1 = \frac{\varphi(n)+1}{2} - 1 = \left\lceil \frac{\varphi(n)}{2} \right\rceil - 1 = \left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$.
Case 2: $n \neq r^2$ for any integer r
If $n \neq r^2$ for any integer r then $n = ab$ where $a \neq b$. Ideal $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ are vertices in $\mathbb{EAG}(\mathbb{Z}_n)$. Based on theorem 7, $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ adjacent in $\mathbb{EAG}(\mathbb{Z}_n)$. This condition means there is no isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$. Graph $\mathbb{EAG}(\mathbb{Z}_n)$ is fully partition into complete graph K_2. Total number of K_2 is $\frac{\varphi(n)-2}{2} = \frac{\varphi(n)}{2} - 1 = \left\lceil \frac{\varphi(n)}{2} \right\rceil - 1 = \left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$. ■

4. CONCLUSIONS

Factorization on $\mathbb{Z}_n$ characterizes the (Exact) Annihilating Ideal Graph, especially in 1) the number of vertices in an annihilating ideal graph, 2) adjacency of the vertices, and 3) decomposition of exact annihilating ideal graph.

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ANNIHILATING IDEAL AND EXACT ANNIHILATING IDEAL GRAPH OF RING $\mathbb{Z}_n$

Anindito Wisnu Susanto [1], **Dewa Putu Wiadnyana Putra** [2*]

[1,2] *Department of Mathematics Education, Faculty of Teacher Training and Education, Sanata Dharma University Jl. Affandi, Mrican, Caturtunggal, Depok, Sleman, Yogyakarta, 55281, Indonesia*

Corresponding author's e-mail: [2*] ***dewa@usd.ac.id***

ABSTRACT

The existence of annihilator in the ring motivates the emergence of studies on Annihilating Ideal and Exact Annihilating Ideal Graphs. The purpose of this research is to describe the characteristics of an (exact) annihilating ideal of ring $\mathbb{Z}_n$. The method used in this research is literature study. The results of this study discuss finiteness, adjacency, connectedness, vertices, and types of $\mathbb{AG}(\mathbb{Z}_n)$ and $\mathbb{EAG}(\mathbb{Z}_n)$. Furthermore, the number of vertices of an Annihilating Ideal Graph is determined by the factorization of n. The adjacency of two vertices is determined by the divisibleness of n. The results also show that $\mathbb{EAG}(\mathbb{Z}_n)$ is a subgraph of $\mathbb{AG}(\mathbb{Z}_n)$. $\mathbb{EAG}(\mathbb{Z}_n)$ can be represented as a union of several complete graphs.

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1. INTRODUCTION

The use of graphs in representing algebraic structures has been carried out since at least 1878 in [1]. This representation starts from representing a group structure into a graph. The vertices of a graph are all elements of a group and changes to an element due to operations on the group are represented by directed edges. Furthermore, [2], [3] began to associate graphs with a broader structure, namely rings. Investigation of the ring structure is carried out through the colored representation of the graph. The representation of an algebraic structure on a graph opens up opportunities for visual investigation of the properties of a particular structure. An essential part in the process of representing a particular algebraic structure to a graph is how to define the connection between the vertices of the graph. Different ways of defining adjacent vertices can lead to different variations of properties as well.

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One of the interesting things in the ring, which is about zero divisor. A non-zero element a is said to be a zero divisor if it can be found a non-zero element b such that $ab = 0$. From this structure, [4] proposed the origin of the zero divisor graph. The vertices of the graph are all zero divisors. Two vertices are adjacent if and only if the product of the two elements is zero. Many interesting properties result from this concept, one of which is about the combinatorics of a finite ring [5], [6], [7].

The concept is similar to zero divisor in the ring is the Annihilator. Badawi has started a study on annihilator graphs [8]. In its development, annihilating graphs are generalized into annihilating Ideal graphs. In [9], it is stated that Annihilator is an ideal I, namely $Ann(I) = \{r \in R | rl = 0 \, \forall \, l \in L\}$. If $Ann(I)$ is not a trivial set, then I is called an ideal annilator. In 2011, [10] started to represent a structure consisting of annihilator ideals into a graph. The graph that is formed is named Annihilating Ideal Graph. In line with the development of zero divisor graphs, [11] is continuing the study of Exact Annihilating Ideal graphs. The development of ideal annihilating and exact annihilating properties of graphs is studied separately. The general relationship between these two graphs began to be investigated by [12].

An integer modulo n, $\mathbb{Z}_n$ is a ring that has very interesting properties. This $\mathbb{Z}_n$ structure is widely used in graphs, for example in coloring Antimagic graphs [13] and Domination ratio [14]. The factorization theorem on integers is a motivation for developing graph studies involving a ring of integers modulo n. One of the graph studies carried out was a study on non-coprime for $\mathbb{Z}_n$ [15]. Based on this, the study of annihilating ideal graph for $\mathbb{Z}_n$ rings is interesting to do.

2. RESEARCH METHODS

This research is a literature research that examines the properties of annihilating ideal and exact annihilating ideal graphs on integer rings modulo n, $\mathbb{Z}_n$. The definition of (Exact) Annihilating Ideal based on [10], [11] is as follows.

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Definition 1. [10]
An Ideal I of commutative ring R with identity is a Annihilating Ideal if there exist non zero ideal J of R such that $\mathrm{IJ} = 0$. The set of all Annihilating Ideal of ring R is denoted by $\mathbb{A}(\mathrm{R})$.

Definition 2. [11]
An ideal I of commutative ring R with identity is Exact Annihilating Ideal if there exist non zero ideal J of R such that $\mathrm{Ann(I) = J}$ and $\mathrm{Ann(J) = I}$. The set of all Exact Annihilating Ideal of ring R denoted by $\mathbb{EA}(\mathrm{R})$.

Based on the two definitions above, then (Exact) Annihilating Ideal Graph is defined as follows.

Definition 3. [10]
Annihilating Ideal graph of ring R denoted by $\mathbb{AG}(\mathrm{R})$ is a graph with vertices $\mathbb{A}(\mathrm{R})^* = \mathbb{A}(\mathrm{R})\backslash\{(0)\}$ and $(\mathrm{I}, \mathrm{J}) \in \mathrm{E}(\mathbb{AG}(\mathrm{R}))$ if and only if $\mathrm{IJ} = (0)$.

Definition 4. [11]
Exact Annihilating Ideal graph of ring R denoted by $\mathbb{EAG}(\mathrm{R})$ is a graph with vertices $\mathbb{EA}(\mathrm{R})^* = \mathbb{EA}(\mathrm{R})\backslash\{(0)\}$ and $(\mathrm{I}, \mathrm{J}) \in \mathrm{E}(\mathbb{EAG}(\mathrm{R}))$ if and only if $\mathrm{Ann(I) = J}$ and $\mathrm{Ann(J) = I}$.

Definition of (Exact) Annihilating Ideal Graph, this article further describes the properties of the graph with rings $\mathbb{Z}_n$. Comparison of the properties of annihilating ideal and exact annihilating ideal graph of ring $\mathbb{Z}_n$ is also presented in this article.

3. RESULTS AND DISCUSSION

In this section, we will show some result of Annihilating ideal graph of ring $\mathbb{Z}_n$.

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Theorem 1.

Suppose $\mathbb{Z}_n$ ring of integer modulo n where n not prime.

1. If $n = p^2$, *where p is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.
2. If $n \neq p^2$, *where p is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$

Proof.

(1) Suppose $n = p^2$ then there exists uniquely non zero proper ideal in di $\mathbb{Z}_n$, $\langle \bar{p} \rangle = \{\overline{pz} | \bar{z} \in \mathbb{Z}_n\}$. If $n = p^2$ its means $\overline{p^2} = \bar{n} = \bar{0}$ such that $\langle \bar{p} \rangle \langle \bar{p} \rangle = \langle \bar{0} \rangle$. Ideal $\langle \bar{p} \rangle$ is an annihilating ideal of $\mathbb{Z}_n$ by Definition 1. Since ideal $\langle \bar{p} \rangle$ is the only one of proper non zero ideal in $\mathbb{Z}_n$, hence $\mathbb{A}(\mathbb{Z}_n)^* = \{\langle \bar{p} \rangle\}$ or $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.

(2) Suppose n is nonprime, that is $n = ab$ for some $a, b \in \mathbb{Z}$ where $1 < a < n$, $1 < b < n$, and $a \neq b$. The product of two ideal, $\langle a \rangle \langle b \rangle = \{(za)(yb) | a, b \in \mathbb{Z}, y, z \in \mathbb{Z}_n\}$. Since $n = ab$, $zy(ab) = zy(n)$ then $\langle a \rangle \langle b \rangle = \{zyn | z, y \in \mathbb{Z}_n\} = \langle \bar{0} \rangle$. Clearly, $\langle a \rangle \neq \langle \bar{0} \rangle$ and $\langle b \rangle \neq \langle \bar{0} \rangle$. Ideals $\langle a \rangle$ and $\langle b \rangle$ are annihilating ideal by Definition 1. Hence, $\langle a \rangle, \langle b \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$. That is prove that for any nonprime n, $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$. ■

Theorem 2.

Suppose $\mathbb{Z}_n$ ring of integer modulo n. The number of vertices of annihilating ideal graph $\mathbb{AG}(\mathbb{Z}_n)$ is $\varphi(n) - 2$, where $\varphi(n)$ is the number of positive factors of n.

Proof.

Suppose $n = (p_1)^{\alpha_1}(p_2)^{\alpha_2} \dots (p_n)^{\alpha_n}$ is prime factorization of n. If $x|n$ then $x = (p_1)^{\beta_1}(p_2)^{\beta_2} \dots (p_n)^{\beta_n}$ where $\beta_i \leq \alpha_i$ for all i. If $x|n$, also means that there exists integer y such that $xy = n$. Suppose $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \dots (p_n)^{\gamma_n}$ then $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \dots (p_n)^{\gamma_n}$, where $\alpha_i = \beta_i + \gamma_i$ for $1 \leq i \leq n$.

We construct principal ideal $\langle x \rangle = \{\overline{xz} | z \in \mathbb{Z}_n\}$ and $\langle y \rangle = \{\overline{yt} | t \in \mathbb{Z}_n\}$ of $\mathbb{Z}_n$. The product of these ideal $\langle x \rangle \langle y \rangle = \{(\overline{xz})(\overline{yt})\} = \{(\overline{xy})(\overline{zt})\}$. As $xy = n$ implies $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. For all $\langle x \rangle$, where x is a positive factor of n, there exists ideal $\langle y \rangle$ such that $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. The number of Ideal $\langle x \rangle$ that satisfied the condition is the number of positive factor of n, $\varphi(n)$. Suppose the set

$$\mathbb{I}(\mathbb{Z}_n) = \{\langle x \rangle \text{ ideal } \mathbb{Z}_n | \exists y \in \mathbb{Z} \text{ such that } xy = n\}$$

Based on the process above, we have $|\mathbb{I}(\mathbb{Z}_n)| = \varphi(n)$. All of elements $\mathbb{I}(\mathbb{Z}_n)$ is the elements of $\mathbb{A}(\mathbb{Z}_n)^*$ except $\langle 1 \rangle$ and $\langle n \rangle$. Hence $|\mathbb{A}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. ■

Theorem 3.

Suppose $\mathbb{Z}_n$ ring of integer modulo n. If $\langle \bar{a} \rangle$ is a vertex of graph $\mathbb{AG}(\mathbb{Z}_n)$ then a is a factor of n.

Proof.

Assume a isn't factor of n. We have $n = ax + y$, where x and y is integer and $0 < y < a$.
The product of ideal $\langle \bar{a} \rangle$ and $\langle \bar{x} \rangle$ is

$$\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\bar{a}r)(\bar{x}n)\} = \{\bar{a}(r\bar{x}n)\} = \{\bar{a}(\bar{x}rn)\} = \{(\bar{a}\bar{x})rn\} = \{(\overline{ax})rn\}$$

We have element $\bar{n} = \bar{y}$ because $n = ax + y$. Then $\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\overline{ax})rn\} = \{(\overline{axr})n\} = \langle \bar{n} \rangle = \langle \bar{y} \rangle$. In means $\langle \bar{a} \rangle$ isn't a ideal annihilator of $\mathbb{Z}_n$. Hence $\langle \bar{a} \rangle \notin \mathbb{A}(\mathbb{Z}_n)^*$. By the contraposition, we have if $\langle \bar{a} \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$ then a is a factor of n. ■

The converse of Theorem 3 is not true. For all $n \in \mathbb{Z}$, we have $1|n$, but clearly $\langle \bar{1} \rangle$ is not an ideal annihilator of $\mathbb{Z}_n$. It means $\langle \bar{1} \rangle$ is not a vertex ini $\mathbb{AG}(\mathbb{Z}_n)$.

Theorem 4.

Suppose $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are ideal in $\mathbb{Z}_n$. Vertex $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ if and only if $n|pq$.

Proof.

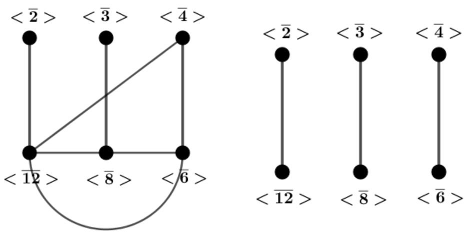
Suppose $\langle p \rangle = \{pa | a \in \mathbb{Z}_n\}$ and $\langle q \rangle = \{qb | r \in \mathbb{Z}_n\}$. The product $\langle p \rangle \langle q \rangle = \langle \overline{pq} \rangle$. If $n|pq$ then $\langle p \rangle \langle q \rangle = \langle \bar{0} \rangle = \{\bar{0}\}$. Hence $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ by Definition 3.

If $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent then $\langle p \rangle \langle q \rangle = \{\bar{0}\}$. It means $(pq)(ab) = nk$ for some integer $a, b,$ and k. The equation $(pq)(ab) = nk$ implies $n|(pq)(ab)$, especially must be $n|pq$. ■

We will continue to discuss the relation some part of annihilating ideal and exact annihilating ideal graph of any commutative ring R.

Lemma 5.
For any commutative ring R, $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$
Proof.
Take any ideal $I \in \mathbb{EA}(R)^*$. It means there exist ideal J of R such that $Ann(I) = J$ and $Ann(J) = I$. Based on definition of annihilator, the product of ideal $IJ = 0$. Hence $I \in \mathbb{A}(R)^*$.
Now, take any ideal $I \in \mathbb{A}(R)^*$. It means there exist nonzero ideal J such that $IJ = 0$. Ideal I is annihilator ideal then $Ann(I) \neq 0$. Suppose $J = Ann(I)$ then J is nonzero ideal of R. We have $Ann(J) = Ann\big(Ann(I)\big) = I$. We conclude $Ann(I) = J$ and $Ann(J) = I$. Hence $I \in \mathbb{EA}(R)^*$. ■

Lemma 6.
For any commutative ring R, $\mathbb{EAG}(R)$ is a subgraph of $\mathbb{AG}(R)$.
Proof.
Lemma 5 show us that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. We will prove that for all $(I, J) \in E\big(\mathbb{EAG}(R)\big)$ then $(I, J) \in E(\mathbb{AG}(R))$. Adjacency of ideal I and J on $\mathbb{EAG}(R)$ means that $I = Ann(J)$ and $J = Ann(I)$. Based on properties of annihilator of ideal, we have $IJ = 0$. Based on definition of adjacency on $\mathbb{AG}(R)$, we have $(I, J) \in E(\mathbb{AG}(R))$. ■
The converse of Theorem 4 not valid for exact annihilating ideal graph. Figure 1 below show that vertex $\langle \bar{6} \rangle$ and $\langle \overline{12} \rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_{24})$ but not adjacent in $\mathbb{EAG}(\mathbb{Z}_{24})$ although $24|12 \times 6$.



(a) (b)

Figure 1. Representation of Annihilating Ideal and Exact Annihilating Ideal Graph (a)$\mathbb{AG}(\mathbb{Z}_{24})$, (b)$\mathbb{EAG}(\mathbb{Z}_{24})$

Based on the situation, we will construct the criteria of adjacency in exact annihilating ideal graph.
Theorem 7.
Suppose commutative ring $\mathbb{Z}_n$ with identity $\bar{1}$. Ideals $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent vertex of $\mathbb{EAG}(\mathbb{Z}_n)$ if and only if $n = pq$.
Proof.
($\leftarrow$). Assume $\langle \bar{p} \rangle$ *and* $\langle \bar{q} \rangle$ are not adjacent. We will proof $n \neq pq$. We have $\langle \bar{p} \rangle = \{\overline{pa} | \bar{a}, \bar{p} \in \mathbb{Z}_n\}$ and $\langle \bar{q} \rangle = \{\overline{qb} | \bar{b}, \bar{q} \in \mathbb{Z}_n\}$ are not adjacent. It means $Ann(\langle \bar{p} \rangle) \neq \langle \bar{q} \rangle$ and $Ann(\langle \bar{q} \rangle) \neq \langle \bar{p} \rangle$ such that $\langle \bar{p} \rangle \langle \bar{q} \rangle \neq \{\bar{0}\}$. We use commutative and associative property of $\mathbb{Z}_n$ to get form $\langle \bar{p} \rangle \langle \bar{q} \rangle = \{(\overline{pa})(\overline{qb})\} = \{(\overline{pq})(\overline{ab})\} \neq \{\bar{0}\}$. It imply $pq \nmid n$. Hence $pq \neq n$.
($\rightarrow$). Assume $n \neq pq$. We will proof vertex $\langle \bar{p} \rangle$ *and* $\langle \bar{q} \rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$. If $n \neq pq$ then $n = pq + a$ with a is non-zero integer. We construct two principal ideal generated by p and q on $\mathbb{Z}_n$. Now, we have the product of these ideal

$$\langle \bar{p} \rangle \langle \bar{q} \rangle = \{(\overline{pr})(\overline{qt})\} = \{(\overline{n-a})(\overline{rt})\} = \langle \overline{-a} \rangle$$

We have $(\langle \bar{p} \rangle, \langle \bar{q} \rangle) \notin E\big(\mathbb{AG}(\mathbb{Z}_n)\big)$. Based on Lemma 6, vertex $\langle \bar{p} \rangle$ *and* $\langle \bar{q} \rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$. ■

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Theorem 8.
Suppose commutative ring $\mathbb{Z}_n$ *with identity* $\bar{1}$. *If* $n = r^2$ *then* $\langle \bar{r} \rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$.
Proof.
Suppose $n = r^2$ and principal ideal $\langle \bar{r} \rangle$ of ring $\mathbb{Z}_n$. We have $Ann(\langle \bar{r} \rangle) = \langle \bar{r} \rangle$. Its means $\langle \bar{r} \rangle$ is a vertex of $\mathbb{EAG}(\mathbb{Z}_n)$. Assume there is a vertex $\langle \bar{a} \rangle$ (not equal to $\langle \bar{r} \rangle$) of $\mathbb{EAG}(\mathbb{Z}_n)$ such that $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$ adjacent. The product of the ideals is $\langle \bar{a} \rangle\langle \bar{r} \rangle \neq \langle \bar{r} \rangle\langle \bar{r} \rangle = \langle \bar{0} \rangle$. Vertex $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$ adjacent on $\mathbb{EAG}(\mathbb{Z}_n)$ means that $\langle \bar{r} \rangle = Ann(\langle \bar{a} \rangle)$ and $\langle \bar{a} \rangle = Ann(\langle \bar{r} \rangle)$. Furthermore $\langle \bar{r} \rangle\langle \bar{a} \rangle = \langle \bar{0} \rangle$. Its contradiction with the product ideals $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$. Hence there is no vertex adjacent with $\langle \bar{r} \rangle$ on $\mathbb{EAG}(\mathbb{Z}_n)$. ∎

In [11] showed that $diam(\mathbb{EAG}(R)) < 1$ and $g(\mathbb{EAG}(R)) \leq 4$ for any commutative ring R. In this paper, we will show more specific result about diameter, girth, cycle existence of $\mathbb{EAG}(R)$.

Theorem 9.
Suppose commutative ring R. *If* $\mathbb{EAG}(R)$ *is connected graph then* $diam(\mathbb{EAG}(R)) = 1$.
Proof.
Suppose I and J are two different vertex of $\mathbb{EAG}(R)$. Assume $d(I,J) = 2 > 1$, means that exist a vertex A of $\mathbb{EAG}(R)$ such that $I - A - J$ is a path in $\mathbb{EAG}(R)$. Based on Definition 4 we have $I = Ann(A), A = Ann(I), A = Ann(J)$, and $J = Ann(A)$. It imply $I = Ann(A) = Ann(Ann(J))$. Based on Lemma 2.1 on [3], we get $Ann(Ann(J)) = J$. Two last equation imply $I = J$. We have a contradiction with ideal I and J must be different. So, $d(I,J) = 1$ for all ideal I and J. It proved that $diam(\mathbb{EAG}(R)) = 1$. ∎

Collorary 10.
Suppose commutative ring R. *If* $\mathbb{EAG}(R)$ *contain a cycle then* $g(\mathbb{EAG}(R)) \leq 3$.

Proof.
If graph G contain a cycle then $g(G) \leq 2diam(G) + 1$. Theorem 9 has shown that $diam(\mathbb{EAG}(R)) = 1$. Finally, we have $g(\mathbb{EAG}(R)) \leq 2diam(\mathbb{EAG}(R)) + 1 = 3$. ∎

Theorem 3.9 in [11] showed that $\mathbb{EAG}(\mathbb{Z}_{p^n})$ where p is prime can be represented as union of some complete graph. Figure 1 below show that $\mathbb{EAG}(\mathbb{Z}_{24})$ can be represented as union of K_2 graph, although $24 \neq p^n$ for any prime p. Based on this fact, we construct a theorem to generalize properties of representation of $\mathbb{EAG}(\mathbb{Z}_n)$.

Theorem 11.
The number of complete subgraph of Exact annihilating ideal graph of ring $\mathbb{Z}_n$ *is* $\left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$.
Proof.
Lemma 5 showed that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. Based on Theorem 2, we have $|\mathbb{EA}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. Theorem 9 showed that $diam(\mathbb{EAG}(R)) = 1$ for any commutative ring R. We conclude that the maximum number of edges $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)}{2} - 1$. Its means the maximum complete subgraph of $\mathbb{EAG}(\mathbb{Z}_n)$ is also $\frac{\varphi(n)}{2} - 1$.
Case 1: $n = r^2$ for some integer r
Based on Theorem 8, $\langle \bar{r} \rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$. We have $\varphi(n) - 3$ other vertices of $\mathbb{EAG}(\mathbb{Z}_n)$. Obviously there is no positive integer a such that $n = a^2$. In another word, we just found exactly one isolated vertex on $\mathbb{EAG}(\mathbb{Z}_n)$. We can partition $\mathbb{EAG}(\mathbb{Z}_n)$ to be $\frac{\varphi(n)-3}{2}$ graph K_2. Isolated vertex can be represented as K_1. The total of number complete graph that contain in $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)-3}{2} + 1 = \frac{\varphi(n)+1}{2} - 1 = \left\lceil \frac{\varphi(n)}{2} \right\rceil - 1 = \left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$.
Case 2: $n \neq r^2$ for any integer r
If $n \neq r^2$ for any integer r then $n = ab$ where $a \neq b$. Ideal $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ are vertices in $\mathbb{EAG}(\mathbb{Z}_n)$. Based on theorem 7, $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ adjacent in $\mathbb{EAG}(\mathbb{Z}_n)$. This condition means there is no isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$. Graph $\mathbb{EAG}(\mathbb{Z}_n)$ is fully partition into complete graph K_2. Total number of K_2 is $\frac{\varphi(n)-2}{2} = \frac{\varphi(n)}{2} - 1 = \left\lceil \frac{\varphi(n)}{2} \right\rceil - 1 = \left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$. ∎

4. CONCLUSIONS

Factorization on $\mathbb{Z}_n$ characterizes the (Exact) Annihilating Ideal Graph, especially in 1) the number of vertices in an annihilating ideal graph, 2) adjacency of the vertices, and 3) decomposition of exact annihilating ideal graph.

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ANNIHILATING IDEAL AND EXACT ANNIHILATING IDEAL GRAPH OF RING $\mathbb{Z}_n$

Anindito Wisnu Susanto [1], **Dewa Putu Wiadnyana Putra** [2*]

[1,2] *Department of Mathematics Education, Faculty of Teacher Training and Education, Sanata Dharma University*
Jl. Affandi, Mrican, Caturtunggal, Depok, Sleman, Yogyakarta, 55281, Indonesia

Corresponding author's e-mail: [2*] ***dewa@usd.ac.id***

ABSTRACT

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The existence of annihilator in the ring motivates the emergence of studies on Annihilating Ideal and Exact Annihilating Ideal Graphs. The purpose of this research is to describe the characteristics of an (exact) annihilating ideal of ring $\mathbb{Z}_n$. The method used in this research is literature study. The results of this study discuss finiteness, adjacency, connectedness, vertices, and types of $\mathbb{AG}(\mathbb{Z}_n)$ and $\mathbb{EAG}(\mathbb{Z}_n)$. Furthermore, the number of vertices of an Annihilating Ideal Graph is determined by the factorization of n. The adjacency of two vertices is determined by the divisibleness of n. The results also show that $\mathbb{EAG}(\mathbb{Z}_n)$ is a subgraph of $\mathbb{AG}(\mathbb{Z}_n)$. $\mathbb{EAG}(\mathbb{Z}_n)$ can be represented as a union of several complete graphs.

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1. INTRODUCTION

The use of graphs in representing algebraic structures has been carried out since at least 1878 in [1]. This representation starts from representing a group structure into a graph. The vertices of a graph are all elements of a group and changes to an element due to operations on the group are represented by directed edges. Furthermore, [2], [3] began to associate graphs with a broader structure, namely rings. Investigation of the ring structure is carried out through the colored representation of the graph. The representation of an algebraic structure on a graph opens up opportunities for visual investigation of the properties of a particular structure. An essential part in the process of representing a particular algebraic structure to a graph is how to define the connection between the vertices of the graph. Different ways of defining adjacent vertices can lead to different variations of properties as well.

One of the interesting things in the ring, which is about zero divisor. A non-zero element a is said to be a zero divisor if it can be found a non-zero element b such that $ab = 0$. From this structure, [4] proposed the origin of the zero divisor graph. The vertices of the graph are all zero divisors. Two vertices are adjacent if and only if the product of the two elements is zero. Many interesting properties result from this concept, one of which is about the combinatorics of a finite ring [5], [6], [7].

The concept is similar to zero divisor in the ring is the Annihilator. Badawi has started a study on annihilator graphs [8]. In its development, annihilating graphs are generalized into annihilating Ideal graphs. In [9], it is stated that Annihilator is an ideal I, namely $Ann(I) = \{r \in R | rl = 0 \,\forall\, l \in L\}$. If $Ann(I)$ is not a trivial set, then I is called an ideal annilator. In 2011, [10] started to represent a structure consisting of annihilator ideals into a graph. The graph that is formed is named Annihilating Ideal Graph. In line with the development of zero divisor graphs, [11] is continuing the study of Exact Annihilating Ideal graphs. The development of ideal annihilating and exact annihilating properties of graphs is studied separately. The general relationship between these two graphs began to be investigated by [12].

An integer modulo n, $\mathbb{Z}_n$ is a ring that has very interesting properties. This $\mathbb{Z}_n$ structure is widely used in graphs, for example in coloring Antimagic graphs [13] and Domination ratio [14]. The factorization theorem on integers is a motivation for developing graph studies involving a ring of integers modulo n. One of the graph studies carried out was a study on non-coprime for $\mathbb{Z}_n$ [15]. Based on this, the study of annihilating ideal graph for $\mathbb{Z}_n$ rings is interesting to do.

2. RESEARCH METHODS

This research is a literature research that examines the properties of annihilating ideal and exact annihilating ideal graphs on integer rings modulo n, $\mathbb{Z}_n$. The properties studied are the relationship between the factorization of integer n and the vertex of an ideal annihilating graph, the adjacency of vertices, and the relationship between integer decomposition and graph decomposition. The definition of (Exact) Annihilating Ideal based on [10], [11] is as follows.

Definition 1. [10]
An Ideal I of commutative ring R with identity is a Annihilating Ideal if there exist non zero ideal J of R such that $\mathrm{IJ} = 0$. The set of all Annihilating Ideal of ring R is denoted by $\mathbb{A}(\mathrm{R})$.

Definition 2. [11]
An ideal I of commutative ring R with identity is Exact Annihilating Ideal if there exist non zero ideal J of R such that $\mathrm{Ann(I)} = \mathrm{J}$ and $\mathrm{Ann(J)} = \mathrm{I}$. The set of all Exact Annihilating Ideal of ring R denoted by $\mathbb{EA}(\mathrm{R})$.

Based on the two definitions above, then (Exact) Annihilating Ideal Graph is defined as follows.

Definition 3. [10]
Annihilating Ideal graph of ring R denoted by $\mathbb{AG}(\mathrm{R})$ is a graph with vertices $\mathbb{A}(\mathrm{R})^* = \mathbb{A}(\mathrm{R})\backslash\{(0)\}$ and $(\mathrm{I},\mathrm{J}) \in \mathrm{E}(\mathbb{AG}(\mathrm{R}))$ if and only if $\mathrm{IJ} = (0)$.

Definition 4. [11]
Exact Annihilating Ideal graph of ring R denoted by $\mathbb{EAG}(\mathrm{R})$ is a graph with vertices $\mathbb{EA}(\mathrm{R})^* = \mathbb{EA}(\mathrm{R})\backslash\{(0)\}$ and $(\mathrm{I},\mathrm{J}) \in \mathrm{E}(\mathbb{EAG}(\mathrm{R}))$ if and only if $\mathrm{Ann(I)} = \mathrm{J}$ and $\mathrm{Ann(J)} = \mathrm{I}$.

Definition of (Exact) Annihilating Ideal Graph, this article further describes the properties of the graph with rings $\mathbb{Z}_n$. Comparison of the properties of annihilating ideal and exact annihilating ideal graph of ring $\mathbb{Z}_n$ is also presented in this article.

3. RESULTS AND DISCUSSION

Integers are partitioned into prime numbers and composite numbers. Integer factorization affects the cardinality of the set of all vertices of an ideal annihilating graph. Conversely, can also be observed from the ideal annihilating graph, the characteristics of these integers can be determined. The following theorem shows the relationship between integer factorization and vertex cardinality of an ideal annihilating graph.

Theorem 1.
Suppose $\mathbb{Z}_n$ ring of integer modulo n where n not prime.
1. If $n = p^2$, *where p is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.
2. If $n \neq p^2$, *where p is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$

Proof.
(1) Suppose $n = p^2$ then there exists uniquely non zero proper ideal in di $\mathbb{Z}_n$, $\langle \bar{p} \rangle = \{\overline{pz} | \bar{z} \in \mathbb{Z}_n\}$. If $n = p^2$ its means $\overline{p^2} = \bar{n} = \bar{0}$ such that $\langle \bar{p} \rangle \langle \bar{p} \rangle = \langle \bar{0} \rangle$. Ideal $\langle \bar{p} \rangle$ is an annihilating ideal of $\mathbb{Z}_n$ by Definition 1. Since ideal $\langle \bar{p} \rangle$ is the only one of proper non zero ideal in $\mathbb{Z}_n$, hence $\mathbb{A}(\mathbb{Z}_n)^* = \{\langle \bar{p} \rangle\}$ or $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.
(2) Suppose n is nonprime, that is $n = ab$ for some $a, b \in \mathbb{Z}$ where $1 < a < n$, $1 < b < n$, and $a \neq b$. The product of two ideal, $\langle a \rangle \langle b \rangle = \{(za)(yb) | a, b \in \mathbb{Z}, y, z \in \mathbb{Z}_n\}$. Since $n = ab$, $zy(ab) = zy(n)$ then $\langle a \rangle \langle b \rangle = \{zyn | z, y \in \mathbb{Z}_n\} = \langle \bar{0} \rangle$. Clearly, $\langle a \rangle \neq \langle \bar{0} \rangle$ and $\langle b \rangle \neq \langle \bar{0} \rangle$. Ideals $\langle a \rangle$ and $\langle b \rangle$ are annihilating ideal by Definition 1. Hence, $\langle a \rangle, \langle b \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$. That is prove that for any nonprime n, $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$. ∎

Theorem 2.
Suppose $\mathbb{Z}_n$ ring of integer modulo n. The number of vertices of annihilating ideal graph $\mathbb{AG}(\mathbb{Z}_n)$ is $\varphi(n) - 2$, where $\varphi(n)$ is the number of positive factors of n.
Proof.
Suppose $n = (p_1)^{\alpha_1}(p_2)^{\alpha_2} \dots (p_n)^{\alpha_n}$ is prime factorization of n. If $x|n$ then $x = (p_1)^{\beta_1}(p_2)^{\beta_2} \dots (p_n)^{\beta_n}$ where $\beta_i \leq \alpha_i$ for all i. If $x|n$, also means that there exists integer y such that $xy = n$. Suppose $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \dots (p_n)^{\gamma_n}$ then $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \dots (p_n)^{\gamma_n}$, where $\alpha_i = \beta_i + \gamma_i$ for $1 \leq i \leq n$.

We construct principal ideal $\langle x \rangle = \{\overline{xz} | z \in \mathbb{Z}_n\}$ and $\langle y \rangle = \{\overline{yt} | t \in \mathbb{Z}_n\}$ of $\mathbb{Z}_n$. The product of these ideal $\langle x \rangle \langle y \rangle = \{(\overline{xz})(\overline{yt})\} = \{(\overline{xy})(\overline{zt})\}$. As $xy = n$ implies $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. For all $\langle x \rangle$, where x is a positive factor of n, there exists ideal $\langle y \rangle$ such that $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. The number of Ideal $\langle x \rangle$ that satisfied the condition is the number of positive factor of n, $\varphi(n)$. Suppose the set

$$\mathbb{I}(\mathbb{Z}_n) = \{\langle x \rangle \text{ ideal } \mathbb{Z}_n | \exists y \in \mathbb{Z} \text{ such that } xy = n\}$$

Based on the process above, we have $|\mathbb{I}(\mathbb{Z}_n)| = \varphi(n)$. All of elements $\mathbb{I}(\mathbb{Z}_n)$ is the elements of $\mathbb{A}(\mathbb{Z}_n)^*$ except $\langle 1 \rangle$ and $\langle n \rangle$. Hence $|\mathbb{A}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. ∎

Theorem 3.
Suppose $\mathbb{Z}_n$ ring of integer modulo n. If $\langle \bar{a} \rangle$ is a vertex of graph $\mathbb{AG}(\mathbb{Z}_n)$ then a is a factor of n.
Proof.
Assume a isn't factor of n. We have $n = ax + y$, where x and y is integer and $0 < y < a$.
The product of ideal $\langle \bar{a} \rangle$ and $\langle \bar{x} \rangle$ is

$$\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\bar{a}r)(\bar{x}n)\} = \{\bar{a}(r\bar{x}n)\} = \{\bar{a}(\bar{x}rn)\} = \{(\bar{a}\bar{x})rn\} = \{(\overline{ax})rn\}$$

We have element $\bar{n} = \bar{y}$ because $n = ax + y$. Then $\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\overline{ax})rn\} = \{(\overline{axr})n\} = \langle \bar{n} \rangle = \langle \bar{y} \rangle$. In means $\langle \bar{a} \rangle$ isn't a ideal annihilator of $\mathbb{Z}_n$. Hence $\langle \bar{a} \rangle \notin \mathbb{A}(\mathbb{Z}_n)^*$. By the contraposition, we have if $\langle \bar{a} \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$ then a is a factor of n. ∎

The converse of Theorem 3 is not true. For all $n \in \mathbb{Z}$, we have $1|n$, but clearly $\langle \bar{1} \rangle$ is not an ideal annihilator of $\mathbb{Z}_n$. It means $\langle \bar{1} \rangle$ is not a vertex ini $\mathbb{AG}(\mathbb{Z}_n)$.

Theorem 4.
Suppose $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are ideal in $\mathbb{Z}_n$. Vertex $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ if and only if $n|pq$.
Proof.

Suppose $\langle p\rangle = \{pa|a \in \mathbb{Z}_n\}$ and $\langle q\rangle = \{qb|r \in \mathbb{Z}_n\}$. The product $\langle p\rangle\langle q\rangle = \langle \overline{pq}\rangle$. If $n|pq$ then $\langle p\rangle\langle q\rangle = \langle \bar{0}\rangle = \{\bar{0}\}$. Hence $\langle \bar{p}\rangle$ and $\langle \bar{q}\rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ by Definition 3.
If $\langle \bar{p}\rangle$ and $\langle \bar{q}\rangle$ are adjacent then $\langle p\rangle\langle q\rangle = \{\bar{0}\}$. It means $(pq)(ab) = nk$ for some integer $a, b,$ and k. The equation $(pq)(ab) = nk$ implies $n|(pq)(ab)$, especially must be $n|pq$. ■

We will continue to discuss the relation some part of annihilating ideal and exact annihilating ideal graph of any commutative ring R.

Lemma 5.
For any commutative ring R, $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$
Proof.
Take any ideal $I \in \mathbb{EA}(R)^*$. It means there exist ideal J of R such that $Ann(I) = J$ and $Ann(J) = I$. Based on definition of annihilator, the product of ideal $IJ = 0$. Hence $I \in \mathbb{A}(R)^*$.
Now, take any ideal $I \in \mathbb{A}(R)^*$. It means there exist nonzero ideal J such that $IJ = 0$. Ideal I is annihilator ideal then $Ann(I) \neq 0$. Suppose $J = Ann(I)$ then J is nonzero ideal of R. We have $Ann(J) = Ann\big(Ann(I)\big) = I$. We conclude $Ann(I) = J$ and $Ann(J) = I$. Hence $I \in \mathbb{EA}(R)^*$. ■

Lemma 6.
For any commutative ring R, $\mathbb{EAG}(R)$ is a subgraph of $\mathbb{AG}(R)$.
Proof.
Lemma 5 show us that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. We will prove that for all $(I,J) \in E\big(\mathbb{EAG}(R)\big)$ then $(I,J) \in E(\mathbb{AG}(R))$. Adjacency of ideal I and J on $\mathbb{EAG}(R)$ means that $I = Ann(J)$ and $J = Ann(I)$. Based on properties of annihilator of ideal, we have $IJ = 0$. Based on definition of adjacency on $\mathbb{AG}(R)$, we have $(I,J) \in E(\mathbb{AG}(R))$. ■
The converse of Theorem 4 not valid for exact annihilating ideal graph. Figure 1 below show that vertex $\langle \bar{6}\rangle$ and $\langle \overline{12}\rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_{24})$ but not adjacent in $\mathbb{EAG}(\mathbb{Z}_{24})$ although $24|12 \times 6$.

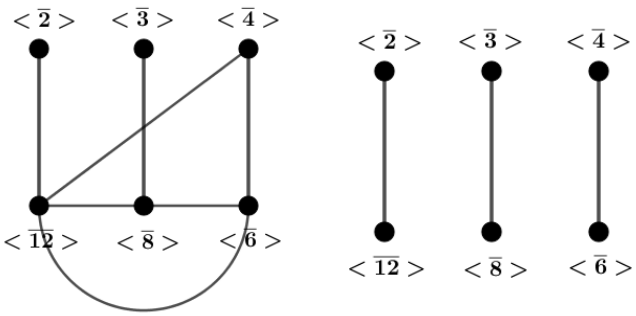

Figure 1. Representation of Annihilating Ideal and Exact Annihilating Ideal Graph (a)$\mathbb{AG}(\mathbb{Z}_{24})$, (b)$\mathbb{EAG}(\mathbb{Z}_{24})$

Based on the situation, we will construct the criteria of adjacency in exact annihilating ideal graph.

Theorem 7.
Suppose commutative ring $\mathbb{Z}_n$ with identity $\bar{1}$. Ideals $\langle \bar{p}\rangle$ and $\langle \bar{q}\rangle$ are adjacent vertex of $\mathbb{EAG}(\mathbb{Z}_n)$ if and only if $n = pq$.
Proof.
$(\leftarrow)$. Assume $\langle \bar{p}\rangle$ *and* $\langle \bar{q}\rangle$ are not adjacent. We will proof $n \neq pq$. We have $\langle \bar{p}\rangle = \{\overline{pa}|\bar{a}, \bar{p} \in \mathbb{Z}_n\}$ and $\langle \bar{q}\rangle = \left\{\overline{qb}\middle|\bar{b}, \bar{q} \in \mathbb{Z}_n\right\}$ are not adjacent. It means $Ann(\langle \bar{p}\rangle) \neq \langle \bar{q}\rangle$ and $Ann(\langle \bar{q}\rangle) \neq \langle \bar{p}\rangle$ such that $\langle \bar{p}\rangle\langle \bar{q}\rangle \neq \{\bar{0}\}$. We use commutative and associative property of $\mathbb{Z}_n$ to get form $\langle \bar{p}\rangle\langle \bar{q}\rangle = \{(\overline{pa})(\overline{qb})\} = \{(\overline{pq})(\overline{ab})\} \neq \{\bar{0}\}$. It imply $pq \nmid n$. Hence $pq \neq n$.
$(\rightarrow)$. Assume $n \neq pq$. We will proof vertex $\langle \bar{p}\rangle$ *and* $\langle \bar{q}\rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$. If $n \neq pq$ then $n = pq + a$ with a is non-zero integer. We construct two principal ideal generated by p and q on $\mathbb{Z}_n$. Now, we have the product of these ideal

$$\langle \bar{p}\rangle\langle \bar{q}\rangle = \{(\overline{pr})(\overline{qt})\} = \{(\overline{n-a})(\overline{rt})\} = \langle \overline{-a}\rangle$$

We have $(\langle \bar{p}\rangle, \langle \bar{q}\rangle) \notin E\big(\mathbb{AG}(\mathbb{Z}_n)\big)$. Based on Lemma 6, vertex $\langle \bar{p}\rangle$ *and* $\langle \bar{q}\rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$.■

Theorem 8.
Suppose commutative ring $\mathbb{Z}_n$ *with identity* $\bar{1}$. *If* $n = r^2$ *then* $\langle \bar{r} \rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$.
Proof.
Suppose $n = r^2$ and principal ideal $\langle \bar{r} \rangle$ of ring $\mathbb{Z}_n$. We have $Ann(\langle \bar{r} \rangle) = \langle \bar{r} \rangle$. Its means $\langle \bar{r} \rangle$ is a vertex of $\mathbb{EAG}(\mathbb{Z}_n)$. Assume there is a vertex $\langle \bar{a} \rangle$ (not equal to $\langle \bar{r} \rangle$) of $\mathbb{EAG}(\mathbb{Z}_n)$ such that $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$ adjacent. The product of the ideals is $\langle \bar{a} \rangle\langle \bar{r} \rangle \neq \langle \bar{r} \rangle\langle \bar{r} \rangle = \langle \bar{0} \rangle$. Vertex $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$ adjacent on $\mathbb{EAG}(\mathbb{Z}_n)$ means that $\langle \bar{r} \rangle = Ann(\langle \bar{a} \rangle)$ and $\langle \bar{a} \rangle = Ann(\langle \bar{r} \rangle)$. Furthermore $\langle \bar{r} \rangle\langle \bar{a} \rangle = \langle \bar{0} \rangle$. Its contradiction with the product ideals $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$. Hence there is no vertex adjacent with $\langle \bar{r} \rangle$ on $\mathbb{EAG}(\mathbb{Z}_n)$. ∎

In [11] showed that $diam(\mathbb{EAG}(R)) < 1$ and $g(\mathbb{EAG}(R)) \leq 4$ for any commutative ring R. In this paper, we will show more specific result about diameter, girth, cycle existence of $\mathbb{EAG}(R)$.

Theorem 9.
Suppose commutative ring R. *If* $\mathbb{EAG}(R)$ *is connected graph then* $diam(\mathbb{EAG}(R)) = 1$.
Proof.
Suppose I and J are two different vertex of $\mathbb{EAG}(R)$. Assume $d(I,J) = 2 > 1$, means that exist a vertex A of $\mathbb{EAG}(R)$ such that $I - A - J$ is a path in $\mathbb{EAG}(R)$. Based on Definition 4 we have $I = Ann(A), A = Ann(I), A = Ann(J)$, and $J = Ann(A)$. It imply $I = Ann(A) = Ann(Ann(J))$. Based on Lemma 2.1 on [3], we get $Ann(Ann(J)) = J$. Two last equation imply $I = J$. We have a contradiction with ideal I and J must be different. So, $d(I,J) = 1$ for all ideal I and J. It proved that $diam(\mathbb{EAG}(R)) = 1$. ∎

Collorary 10.
Suppose commutative ring R. *If* $\mathbb{EAG}(R)$ *contain a cycle then* $g(\mathbb{EAG}(R)) \leq 3$.

Proof.
If graph G contain a cycle then $g(G) \leq 2diam(G) + 1$. Theorem 9 has shown that $diam(\mathbb{EAG}(R)) = 1$. Finally, we have $g(\mathbb{EAG}(R)) \leq 2diam(\mathbb{EAG}(R)) + 1 = 3$. ∎

Theorem 3.9 in [11] showed that $\mathbb{EAG}(\mathbb{Z}_{p^n})$ where p is prime can be represented as union of some complete graph. Figure 1 below show that $\mathbb{EAG}(\mathbb{Z}_{24})$ can be represented as union of K_2 graph, although $24 \neq p^n$ for any prime p. Based on this fact, we construct a theorem to generalize properties of representation of $\mathbb{EAG}(\mathbb{Z}_n)$.

Theorem 11.
The number of complete subgraph of Exact annihilating ideal graph of ring $\mathbb{Z}_{\boldsymbol{n}}$ *is* $\left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$.
Proof.
Lemma 5 showed that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. Based on Theorem 2, we have $|\mathbb{EA}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. Theorem 9 showed that $diam(\mathbb{EAG}(R)) = 1$ for any commutative ring R. We conclude that the maximum number of edges $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)}{2} - 1$. Its means the maximum complete subgraph of $\mathbb{EAG}(\mathbb{Z}_n)$ is also $\frac{\varphi(n)}{2} - 1$.
Case 1: $n = r^2$ for some integer r
Based on Theorem 8, $\langle \bar{r} \rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$. We have $\varphi(n) - 3$ other vertices of $\mathbb{EAG}(\mathbb{Z}_n)$. Obviously there is no positive integer a such that $n = a^2$. In another word, we just found exactly one isolated vertex on $\mathbb{EAG}(\mathbb{Z}_n)$. We can partition $\mathbb{EAG}(\mathbb{Z}_n)$ to be $\frac{\varphi(n)-3}{2}$ graph K_2. Isolated vertex can be represented as K_1. The total of number complete graph that contain in $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)-3}{2} + 1 = \frac{\varphi(n)+1}{2} - 1 = \left\lceil \frac{\varphi(n)}{2} \right\rceil - 1 = \left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$.
Case 2: $n \neq r^2$ for any integer r
If $n \neq r^2$ for any integer r then $n = ab$ where $a \neq b$. Ideal $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ are vertices in $\mathbb{EAG}(\mathbb{Z}_n)$. Based on theorem 7, $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ adjacent in $\mathbb{EAG}(\mathbb{Z}_n)$. This condition means there is no isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$.

Graph $\mathbb{EAG}(\mathbb{Z}_n)$ is fully partition into complete graph K_2. Total number of K_2 is $\frac{\varphi(n)-2}{2} = \frac{\varphi(n)}{2} - 1 = \left\lceil \frac{\varphi(n)}{2} \right\rceil - 1 = \left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$. ■

4. CONCLUSIONS

Factorization on $\mathbb{Z}_n$ characterizes the (Exact) Annihilating Ideal Graph, especially in 1) the number of vertices in an annihilating ideal graph, 2) adjacency of the vertices, and 3) decomposition of exact annihilating ideal graph.

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ANNIHILATING IDEAL AND EXACT ANNIHILATING IDEAL GRAPH OF RING $\mathbb{Z}_n$

Anindito Wisnu Susanto [1], Dewa Putu Wiadnyana Putra[2*]

[1,2]*Department of Mathematics Education, Faculty of Teacher Training and Education, Sanata Dharma University*
Jl. Affandi, Mrican, Caturtunggal, Depok, Sleman, Yogyakarta, 55281, Indonesia

Corresponding author's e-mail: [2] **dewa@usd.ac.id***

ABSTRACT

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The existence of annihilator in the ring motivates the emergence of studies on Annihilating Ideal and Exact Annihilating Ideal Graphs. The purpose of this research is to describe the characteristics of an (exact) annihilating ideal of ring $\mathbb{Z}_n$. The method used in this research is literature study. The results of this study discuss finiteness, adjacency, connectedness, vertices, and types of $\mathbb{AG}(\mathbb{Z}_n)$ and $\mathbb{EAG}(\mathbb{Z}_n)$. Furthermore, the number of vertices of an Annihilating Ideal Graph is determined by the factorization of n. The adjacency of two vertices is determined by the divisibleness of n. The results also show that $\mathbb{EAG}(\mathbb{Z}_n)$ is a subgraph of $\mathbb{AG}(\mathbb{Z}_n)$. $\mathbb{EAG}(\mathbb{Z}_n)$ can be represented as a union of several complete graphs.

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1. INTRODUCTION

The use of graphs in representing algebraic structures has been carried out since at least 1878 in [1]. This representation starts from representing a group structure into a graph. The vertices of a graph are all elements of a group and changes to an element due to operations on the group are represented by directed edges. Furthermore, [2], [3] began to associate graphs with a broader structure, namely rings. Investigation of the ring structure is carried out through the colored representation of the graph. The representation of an algebraic structure on a graph opens up opportunities for visual investigation of the properties of a particular structure. An essential part in the process of representing a particular algebraic structure to a graph is how to define the connection between the vertices of the graph. Different ways of defining adjacent vertices can lead to different variations of properties as well.

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One of the interesting things in the ring, which is about zero divisor. A non-zero element a is said to be a zero divisor if it can be found a non-zero element b such that $ab = 0$. From this structure, [4] proposed the origin of the zero divisor graph. The vertices of the graph are all zero divisors. Two vertices are adjacent if and only if the product of the two elements is zero. Many interesting properties result from this concept, one of which is about the combinatorics of a finite ring [5], [6], [7].

The concept is similar to zero divisor in the ring is the Annihilator. Badawi has started a study on annihilator graphs [8]. In its development, annihilating graphs are generalized into annihilating Ideal graphs. In [9], it is stated that Annihilator is an ideal I, namely $Ann(I) = \{r \in R | rl = 0 \,\forall\, l \in L\}$. If $Ann(I)$ is not a trivial set, then I is called an ideal annilator. In 2011, [10] started to represent a structure consisting of annihilator ideals into a graph. The graph that is formed is named Annihilating Ideal Graph. In line with the development of zero divisor graphs, [11] is continuing the study of Exact Annihilating Ideal graphs. The development of ideal annihilating and exact annihilating properties of graphs is studied separately. The general relationship between these two graphs began to be investigated by [12].

An integer modulo n, $\mathbb{Z}_n$ is a ring that has very interesting properties. This $\mathbb{Z}_n$ structure is widely used in graphs, for example in coloring Antimagic graphs [13] and Domination ratio [14]. The factorization theorem on integers is a motivation for developing graph studies involving a ring of integers modulo n. One of the graph studies carried out was a study on non-coprime for $\mathbb{Z}_n$ [15]. Based on this, the study of annihilating ideal graph for $\mathbb{Z}_n$ rings is interesting to do.

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2. RESEARCH METHODS

This research is a literature research that examines the properties of annihilating ideal and exact annihilating ideal graphs on integer rings modulo n, $\mathbb{Z}_n$. The properties studied are the relationship between the factorization of integer n and the vertex of an ideal annihilating graph, the adjacency of vertices, and the relationship between integer decomposition and graph decomposition. The definition of (Exact) Annihilating Ideal based on [10], [11] is as follows.

Definition 1. [10]

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An Ideal I of commutative ring R with identity is a Annihilating Ideal if there exist non zero ideal J of R such that $\mathrm{IJ} = 0$. The set of all Annihilating Ideal of ring R is denoted by $\mathbb{A}(\mathrm{R})$.

Definition 2. [11]

An ideal I of commutative ring R with identity is Exact Annihilating Ideal if there exist non zero ideal J of R such that $\mathrm{Ann(I)} = \mathrm{J}$ and $\mathrm{Ann(J)} = \mathrm{I}$. The set of all Exact Annihilating Ideal of ring R denoted by $\mathbb{EA}(\mathrm{R})$.

Based on the two definitions above, then (Exact) Annihilating Ideal Graph is defined as follows.

Definition 3. [10]

Annihilating Ideal graph of ring R denoted by $\mathbb{AG}(\mathrm{R})$ is a graph with vertices $\mathbb{A}(\mathrm{R})^* = \mathbb{A}(\mathrm{R})\backslash\{(0)\}$ and $(\mathrm{I},\mathrm{J}) \in \mathrm{E}(\mathbb{AG}(\mathrm{R}))$ if and only if $\mathrm{IJ} = (0)$.

Definition 4. [11]

Exact Annihilating Ideal graph of ring R denoted by $\mathbb{EAG}(\mathrm{R})$ is a graph with vertices $\mathbb{EA}(\mathrm{R})^* = \mathbb{EA}(\mathrm{R})\backslash\{(0)\}$ and $(\mathrm{I},\mathrm{J}) \in \mathrm{E}(\mathbb{EAG}(\mathrm{R}))$ if and only if $\mathrm{Ann(I)} = \mathrm{J}$ and $\mathrm{Ann(J)} = \mathrm{I}$.

Definition of (Exact) Annihilating Ideal Graph, this article further describes the properties of the graph with rings $\mathbb{Z}_n$. Comparison of the properties of annihilating ideal and exact annihilating ideal graph of ring $\mathbb{Z}_n$ is also presented in this article.

3. RESULTS AND DISCUSSION

Integers are partitioned into prime numbers and composite numbers. Integer factorization affects the cardinality of the set of all vertices of an ideal annihilating graph. Conversely, can also be observed from the ideal annihilating graph, the characteristics of these integers can be determined. The following theorem shows the relationship between integer factorization and vertex cardinality of an ideal annihilating graph.

Theorem 1.

Suppose $\mathbb{Z}_n$ ring of integer modulo n where n not prime.

1. If $n = p^2$, *where p is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.
2. If $n \neq p^2$, *where p is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$

Proof.

(1) Suppose $n = p^2$ then there exists uniquely non zero proper ideal in di $\mathbb{Z}_n$, $\langle \bar{p} \rangle = \{\overline{pz} | \bar{z} \in \mathbb{Z}_n\}$. If $n = p^2$ its means $\overline{p^2} = \bar{n} = \bar{0}$ such that $\langle \bar{p} \rangle \langle \bar{p} \rangle = \langle \bar{0} \rangle$. Ideal $\langle \bar{p} \rangle$ is an annihilating ideal of $\mathbb{Z}_n$ by Definition 1. Since ideal $\langle \bar{p} \rangle$ is the only one of proper non zero ideal in $\mathbb{Z}_n$, hence $\mathbb{A}(\mathbb{Z}_n)^* = \{\langle \bar{p} \rangle\}$ or $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.

(2) Suppose n is nonprime, that is $n = ab$ for some $a, b \in \mathbb{Z}$ where $1 < a < n$, $1 < b < n$, and $a \neq b$. The product of two ideal, $\langle a \rangle \langle b \rangle = \{(za)(yb) | a, b \in \mathbb{Z}, y, z \in \mathbb{Z}_n\}$. Since $n = ab$, $zy(ab) = zy(n)$ then $\langle a \rangle \langle b \rangle = \{zyn | z, y \in \mathbb{Z}_n\} = \langle \bar{0} \rangle$. Clearly, $\langle a \rangle \neq \langle \bar{0} \rangle$ and $\langle b \rangle \neq \langle \bar{0} \rangle$. Ideals $\langle a \rangle$ and $\langle b \rangle$ are annihilating ideal by Definition 1. Hence, $\langle a \rangle, \langle b \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$. That is prove that for any nonprime n, $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$. ■

Theorem 2.

Suppose $\mathbb{Z}_n$ ring of integer modulo n. The number of vertices of annihilating ideal graph $\mathbb{AG}(\mathbb{Z}_n)$ is $\varphi(n) - 2$, where $\varphi(n)$ is the number of positive factors of n.

Proof.

Suppose $n = (p_1)^{\alpha_1}(p_2)^{\alpha_2} \dots (p_n)^{\alpha_n}$ is prime factorization of n. If $x|n$ then $x = (p_1)^{\beta_1}(p_2)^{\beta_2} \dots (p_n)^{\beta_n}$ where $\beta_i \leq \alpha_i$ for all i. If $x|n$, also means that there exists integer y such that $xy = n$. Suppose $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \dots (p_n)^{\gamma_n}$ then $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \dots (p_n)^{\gamma_n}$, where $\alpha_i = \beta_i + \gamma_i$ for $1 \leq i \leq n$.

We construct principal ideal $\langle x \rangle = \{\overline{xz} | z \in \mathbb{Z}_n\}$ and $\langle y \rangle = \{\overline{yt} | t \in \mathbb{Z}_n\}$ of $\mathbb{Z}_n$. The product of these ideal $\langle x \rangle \langle y \rangle = \{(\overline{xz})(\overline{yt})\} = \{(\overline{xy})(\overline{zt})\}$. As $xy = n$ implies $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. For all $\langle x \rangle$, where x is a positive factor of n, there exists ideal $\langle y \rangle$ such that $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. The number of Ideal $\langle x \rangle$ that satisfied the condition is the number of positive factor of n, $\varphi(n)$. Suppose the set

$$\mathbb{I}(\mathbb{Z}_n) = \{\langle x \rangle \text{ ideal } \mathbb{Z}_n | \exists y \in \mathbb{Z} \text{ such that } xy = n\}$$

Based on the process above, we have $|\mathbb{I}(\mathbb{Z}_n)| = \varphi(n)$. All of elements $\mathbb{I}(\mathbb{Z}_n)$ is the elements of $\mathbb{A}(\mathbb{Z}_n)^*$ except $\langle 1 \rangle$ and $\langle n \rangle$. Hence $|\mathbb{A}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. ■

Theorem 3.

Suppose $\mathbb{Z}_n$ ring of integer modulo n. If $\langle \bar{a} \rangle$ is a vertex of graph $\mathbb{AG}(\mathbb{Z}_n)$ then a is a factor of n.

Proof.

Assume a isn't factor of n. We have $n = ax + y$, where x and y is integer and $0 < y < a$.
The product of ideal $\langle \bar{a} \rangle$ and $\langle \bar{x} \rangle$ is

$$\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\bar{a}r)(\bar{x}n)\} = \{\bar{a}(r\bar{x}n)\} = \{\bar{a}(\bar{x}rn)\} = \{(\bar{a}\bar{x})rn\} = \{(\overline{ax})rn\}$$

We have element $\bar{n} = \bar{y}$ because $n = ax + y$. Then $\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\overline{ax})rn\} = \{(\overline{axr})n\} = \langle \bar{n} \rangle = \langle \bar{y} \rangle$. In means $\langle \bar{a} \rangle$ isn't a ideal annihilator of $\mathbb{Z}_n$. Hence $\langle \bar{a} \rangle \notin \mathbb{A}(\mathbb{Z}_n)^*$. By the contraposition, we have if $\langle \bar{a} \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$ then a is a factor of n. ■

The converse of Theorem 3 is not true. For all $n \in \mathbb{Z}$, we have $1|n$, but clearly $\langle \bar{1} \rangle$ is not an ideal annihilator of $\mathbb{Z}_n$. It means $\langle \bar{1} \rangle$ is not a vertex ini $\mathbb{AG}(\mathbb{Z}_n)$.

Theorem 4.

Suppose $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are ideal in $\mathbb{Z}_n$. Vertex $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ if and only if $n|pq$.

Proof.

Suppose $\langle p\rangle = \{pa|a \in \mathbb{Z}_n\}$ and $\langle q\rangle = \{qb|r \in \mathbb{Z}_n\}$. The product $\langle p\rangle\langle q\rangle = \langle\overline{pq}\rangle$. If $n|pq$ then $\langle p\rangle\langle q\rangle = \langle\bar{0}\rangle = \{\bar{0}\}$. Hence $\langle\bar{p}\rangle$ and $\langle\bar{q}\rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ by Definition 3.
If $\langle\bar{p}\rangle$ and $\langle\bar{q}\rangle$ are adjacent then $\langle p\rangle\langle q\rangle = \{\bar{0}\}$. It means $(pq)(ab) = nk$ for some integer $a, b,$ and k. The equation $(pq)(ab) = nk$ implies $n|(pq)(ab)$, especially must be $n|pq$. ■

We will continue to discuss the relation some part of annihilating ideal and exact annihilating ideal graph of any commutative ring R.

Lemma 5.
For any commutative ring R, $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$
Proof.
Take any ideal $I \in \mathbb{EA}(R)^*$. It means there exist ideal J of R such that $Ann(I) = J$ and $Ann(J) = I$. Based on definition of annihilator, the product of ideal $IJ = 0$. Hence $I \in \mathbb{A}(R)^*$.
Now, take any ideal $I \in \mathbb{A}(R)^*$. It means there exist nonzero ideal J such that $IJ = 0$. Ideal I is annihilator ideal then $Ann(I) \neq 0$. Suppose $J = Ann(I)$ then J is nonzero ideal of R. We have $Ann(J) = Ann(Ann(I)) = I$. We conclude $Ann(I) = J$ and $Ann(J) = I$. Hence $I \in \mathbb{EA}(R)^*$. ■

Lemma 6.
For any commutative ring R, $\mathbb{EAG}(R)$ is a subgraph of $\mathbb{AG}(R)$.
Proof.
Lemma 5 show us that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. We will prove that for all $(I, J) \in E(\mathbb{EAG}(R))$ then $(I, J) \in E(\mathbb{AG}(R))$. Adjacency of ideal I and J on $\mathbb{EAG}(R)$ means that $I = Ann(J)$ and $J = Ann(I)$. Based on properties of annihilator of ideal, we have $IJ = 0$. Based on definition of adjacency on $\mathbb{AG}(R)$, we have $(I, J) \in E(\mathbb{AG}(R))$. ■
The converse of Theorem 4 not valid for exact annihilating ideal graph. Figure 1 below show that vertex $\langle\bar{6}\rangle$ and $\langle\overline{12}\rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_{24})$ but not adjacent in $\mathbb{EAG}(\mathbb{Z}_{24})$ although $24|12 \times 6$.

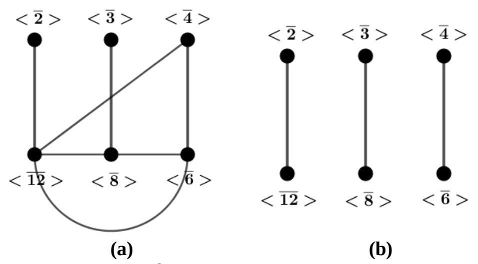

Figure 1. Representation of Annihilating Ideal and Exact Annihilating Ideal Graph (a)$\mathbb{AG}(\mathbb{Z}_{24})$, (b)$\mathbb{EAG}(\mathbb{Z}_{24})$

Based on the situation, we will construct the criteria of adjacency in exact annihilating ideal graph.

Theorem 7.
Suppose commutative ring $\mathbb{Z}_n$ with identity $\bar{1}$. Ideals $\langle\bar{p}\rangle$ and $\langle\bar{q}\rangle$ are adjacent vertex of $\mathbb{EAG}(\mathbb{Z}_n)$ if and only if $n = pq$.
Proof.
($\leftarrow$). Assume $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent. We will proof $n \neq pq$. We have $\langle\bar{p}\rangle = \{\overline{pa}|\bar{a}, \bar{p} \in \mathbb{Z}_n\}$ and $\langle\bar{q}\rangle = \{\overline{qb}|\bar{b}, \bar{q} \in \mathbb{Z}_n\}$ are not adjacent. It means $Ann(\langle\bar{p}\rangle) \neq \langle\bar{q}\rangle$ and $Ann(\langle\bar{q}\rangle) \neq \langle\bar{p}\rangle$ such that $\langle\bar{p}\rangle\langle\bar{q}\rangle \neq \{\bar{0}\}$. We use commutative and associative property of $\mathbb{Z}_n$ to get form $\langle\bar{p}\rangle\langle\bar{q}\rangle = \{(\overline{pa})(\overline{qb})\} = \{(\overline{pq})(\overline{ab})\} \neq \{\bar{0}\}$. It imply $pq \nmid n$. Hence $pq \neq n$.
($\rightarrow$). Assume $n \neq pq$. We will proof vertex $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$. If $n \neq pq$ then $n = pq + a$ with a is non-zero integer. We construct two principal ideal generated by p and q on $\mathbb{Z}_n$. Now, we have the product of these ideal

$$\langle\bar{p}\rangle\langle\bar{q}\rangle = \{(\overline{pr})(\overline{qt})\} = \{(\overline{n-a})(\overline{rt})\} = \langle\overline{-a}\rangle$$

We have $(\langle\bar{p}\rangle, \langle\bar{q}\rangle) \notin E(\mathbb{AG}(\mathbb{Z}_n))$. Based on Lemma 6, vertex $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$.■

Theorem 8.
Suppose commutative ring $\mathbb{Z}_n$ with identity $\bar{1}$. If $n = r^2$ then $\langle \bar{r} \rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$.
Proof.
Suppose $n = r^2$ and principal ideal $\langle \bar{r} \rangle$ of ring $\mathbb{Z}_n$. We have $Ann(\langle \bar{r} \rangle) = \langle \bar{r} \rangle$. Its means $\langle \bar{r} \rangle$ is a vertex of $\mathbb{EAG}(\mathbb{Z}_n)$. Assume there is a vertex $\langle \bar{a} \rangle$ (not equal to $\langle \bar{r} \rangle$) of $\mathbb{EAG}(\mathbb{Z}_n)$ such that $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$ adjacent. The product of the ideals is $\langle \bar{a} \rangle\langle \bar{r} \rangle \neq \langle \bar{r} \rangle\langle \bar{r} \rangle = \langle \bar{0} \rangle$. Vertex $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$ adjacent on $\mathbb{EAG}(\mathbb{Z}_n)$ means that $\langle \bar{r} \rangle = Ann(\langle \bar{a} \rangle)$ and $\langle \bar{a} \rangle = Ann(\langle \bar{r} \rangle)$. Furthermore $\langle \bar{r} \rangle\langle \bar{a} \rangle = \langle \bar{0} \rangle$. Its contradiction with the product ideals $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$. Hence there is no vertex adjacent with $\langle \bar{r} \rangle$ on $\mathbb{EAG}(\mathbb{Z}_n)$. ∎

In [11] showed that $diam\big(\mathbb{EAG}(R)\big) < 1$ and $g\big(\mathbb{EAG}(R)\big) \leq 4$ for any commutative ring R. In this paper, we will show more specific result about diameter, girth, cycle existence of $\mathbb{EAG}(R)$.

Theorem 9.
Suppose commutative ring R. If $\mathbb{EAG}(R)$ is connected graph then $diam\big(\mathbb{EAG}(R)\big) = 1$.
Proof.
Suppose I and J are two different vertex of $\mathbb{EAG}(R)$. Assume $d(I,J) = 2 > 1$, means that exist a vertex A of $\mathbb{EAG}(R)$ such that $I - A - J$ is a path in $\mathbb{EAG}(R)$. Based on Definition 4 we have $I = Ann(A), A = Ann(I), A = Ann(J)$, and $J = Ann(A)$. It imply $I = Ann(A) = Ann\big(Ann(J)\big)$. Based on Lemma 2.1 on [3], we get $Ann\big(Ann(J)\big) = J$. Two last equation imply $I = J$. We have a contradiction with ideal I and J must be different. So, $d(I,J) = 1$ for all ideal I and J. It proved that $diam\big(\mathbb{EAG}(R)\big) = 1$. ∎

Collorary 10.
Suppose commutative ring R. If $\mathbb{EAG}(R)$ contain a cycle then $g\big(\mathbb{EAG}(R)\big) \leq 3$.

Proof.
If graph G contain a cycle then $g(G) \leq 2diam(G) + 1$. Theorem 9 has shown that $diam\big(\mathbb{EAG}(R)\big) = 1$. Finally, we have $g\big(\mathbb{EAG}(R)\big) \leq 2diam\big(\mathbb{EAG}(R)\big) + 1 = 3$. ∎

Theorem 3.9 in [11] showed that $\mathbb{EAG}\big(\mathbb{Z}_{p^n}\big)$ where p is prime can be represented as union of some complete graph. Figure 1 below show that $\mathbb{EAG}(\mathbb{Z}_{24})$ can be represented as union of K_2 graph, although $24 \neq p^n$ for any prime p. Based on this fact, we construct a theorem to generalize properties of representation of $\mathbb{EAG}(\mathbb{Z}_n)$.

Theorem 11.
The number of complete subgraph of Exact annihilating ideal graph of ring $\mathbb{Z}_{\boldsymbol{n}}$ is $\left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$.
Proof.
Lemma 5 showed that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. Based on Theorem 2, we have $|\mathbb{EA}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. Theorem 9 showed that $diam\big(\mathbb{EAG}(R)\big) = 1$ for any commutative ring R. We conclude that the maximum number of edges $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)}{2} - 1$. Its means the maximum complete subgraph of $\mathbb{EAG}(\mathbb{Z}_n)$ is also $\frac{\varphi(n)}{2} - 1$.
Case 1: $n = r^2$ for some integer r
Based on Theorem 8, $\langle \bar{r} \rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$. We have $\varphi(n) - 3$ other vertices of $\mathbb{EAG}(\mathbb{Z}_n)$. Obviously there is no positive integer a such that $n = a^2$. In another word, we just found exactly one isolated vertex on $\mathbb{EAG}(\mathbb{Z}_n)$. We can partition $\mathbb{EAG}(\mathbb{Z}_n)$ to be $\frac{\varphi(n)-3}{2}$ graph K_2. Isolated vertex can be represented as K_1. The total of number complete graph that contain in $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)-3}{2} + 1 = \frac{\varphi(n)+1}{2} - 1 = \left\lceil \frac{\varphi(n)}{2} \right\rceil - 1 = \left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$.
Case 2: $n \neq r^2$ for any integer r
If $n \neq r^2$ for any integer r then $n = ab$ where $a \neq b$. Ideal $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ are vertices in $\mathbb{EAG}(\mathbb{Z}_n)$. Based on theorem 7, $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ adjacent in $\mathbb{EAG}(\mathbb{Z}_n)$. This condition means there is no isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$.

Graph $\mathbb{EAG}(\mathbb{Z}_n)$ is fully partition into complete graph K_2. Total number of K_2 is $\frac{\varphi(n)-2}{2} = \frac{\varphi(n)}{2} - 1 = \left\lceil \frac{\varphi(n)}{2} \right\rceil - 1 = \left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$. ■

4. CONCLUSIONS

Factorization on $\mathbb{Z}_n$ characterizes the (Exact) Annihilating Ideal Graph, especially in 1) the number of vertices in an annihilating ideal graph, 2) adjacency of the vertices, and 3) decomposition of exact annihilating ideal graph.

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ANNIHILATING IDEAL AND EXACT ANNIHILATING IDEAL GRAPH OF RING $\mathbb{Z}_n$

Anindito Wisnu Susanto [1], **Dewa Putu Wiadnyana Putra**[2*]

[1,2]*Department of Mathematics Education, Faculty of Teacher Training and Education, Sanata Dharma University*
Jl. Affandi, Mrican, Caturtunggal, Depok, Sleman, Yogyakarta, 55281, Indonesia

Corresponding author's e-mail: [2*] **dewa@usd.ac.id**

ABSTRACT

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The existence of annihilator in the ring motivates the emergence of studies on Annihilating Ideal and Exact Annihilating Ideal Graphs. The purpose of this research is to describe the characteristics of an (exact) annihilating ideal of ring $\mathbb{Z}_n$. The method used in this research is literature study. The results of this study discuss finiteness, adjacency, connectedness, vertices, and types of $\mathbb{AG}(\mathbb{Z}_n)$ and $\mathbb{EAG}(\mathbb{Z}_n)$. Furthermore, the number of vertices of an Annihilating Ideal Graph is determined by the factorization of n. The adjacency of two vertices is determined by the divisibleness of n. The results also show that $\mathbb{EAG}(\mathbb{Z}_n)$ is a subgraph of $\mathbb{AG}(\mathbb{Z}_n)$. $\mathbb{EAG}(\mathbb{Z}_n)$ can be represented as a union of several complete graphs.

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1. INTRODUCTION

The use of graphs in representing algebraic structures has been carried out since at least 1878 in [1]. This representation starts from representing a group structure into a graph. The vertices of a graph are all elements of a group and changes to an element due to operations on the group are represented by directed edges. Furthermore, [2], [3] began to associate graphs with a broader structure, namely rings. Investigation of the ring structure is carried out through the colored representation of the graph. The representation of an algebraic structure on a graph opens up opportunities for visual investigation of the properties of a particular structure. An essential part in the process of representing a particular algebraic structure to a graph is how to define the connection between the vertices of the graph. Different ways of defining adjacent vertices can lead to different variations of properties as well.

One of the interesting things in the ring, which is about zero divisor. A non-zero element a is said to be a zero divisor if it can be found a non-zero element b such that $ab = 0$. From this structure, [4] proposed the origin of the zero divisor graph. The vertices of the graph are all zero divisors. Two vertices are adjacent if and only if the product of the two elements is zero. Many interesting properties result from this concept, one of which is about the combinatorics of a finite ring [5], [6], [7].

The concept is similar to zero divisor in the ring is the Annihilator. Badawi has started a study on annihilator graphs [8]. In its development, annihilating graphs are generalized into annihilating Ideal graphs. In [9], it is stated that Annihilator is an ideal I, namely $Ann(I) = \{r \in R|rl = 0 \,\forall\, l \in L\}$. If $Ann(I)$ is not a trivial set, then I is called an ideal annilator. In 2011, [10] started to represent a structure consisting of annihilator ideals into a graph. The graph that is formed is named Annihilating Ideal Graph. In line with the development of zero divisor graphs, [11] is continuing the study of Exact Annihilating Ideal graphs. The development of ideal annihilating and exact annihilating properties of graphs is studied separately. The general relationship between these two graphs began to be investigated by [12].

An integer modulo n, $\mathbb{Z}_n$ is a ring that has very interesting properties. This $\mathbb{Z}_n$ structure is widely used in graphs, for example in coloring Antimagic graphs [13] and Domination ratio [14]. The factorization theorem on integers is a motivation for developing graph studies involving a ring of integers modulo n. One of the graph studies carried out was a study on non-coprime for $\mathbb{Z}_n$ [15]. In this research, we combine the properties of (Exact) Annihilating Ideal Graph of arbitrary ring with factorization of ring integer modulo n. These properties will be used to represent integer factors in a graph.

2. RESEARCH METHODS

This research is a literature research that examines the properties of annihilating ideal and exact annihilating ideal graphs on integer rings modulo n, $\mathbb{Z}_n$. The properties studied are the relationship between the factorization of integer n and the vertex of an ideal annihilating graph, the adjacency of vertices, and the relationship between integer decomposition and graph decomposition. The definition of (Exact) Annihilating Ideal based on [10], [11] is as follows.

Definition 1. [10]

An Ideal I of commutative ring R with identity is a Annihilating Ideal if there exist non zero ideal J of R such that $\mathrm{IJ} = 0$. The set of all Annihilating Ideal of ring R is denoted by $\mathbb{A}(\mathrm{R})$.

Definition 2. [11]

An ideal I of commutative ring R with identity is Exact Annihilating Ideal if there exist non zero ideal J of R such that $\mathrm{Ann}(\mathrm{I}) = \mathrm{J}$ and $\mathrm{Ann}(\mathrm{J}) = \mathrm{I}$. The set of all Exact Annihilating Ideal of ring R denoted by $\mathbb{EA}(\mathrm{R})$.

Based on the two definitions above, then (Exact) Annihilating Ideal Graph is defined as follows.

Definition 3. [10]

Annihilating Ideal graph of ring R denoted by $\mathbb{AG}(\mathrm{R})$ is a graph with vertices $\mathbb{A}(\mathrm{R})^* = \mathbb{A}(\mathrm{R})\backslash\{(0)\}$ and $(\mathrm{I},\mathrm{J}) \in \mathrm{E}(\mathbb{AG}(\mathrm{R}))$ if and only if $\mathrm{IJ} = (0)$.

Definition 4. [11]

Exact Annihilating Ideal graph of ring R denoted by $\mathbb{EAG}(\mathrm{R})$ is a graph with vertices $\mathbb{EA}(\mathrm{R})^* = \mathbb{EA}(\mathrm{R})\backslash\{(0)\}$ and $(\mathrm{I},\mathrm{J}) \in \mathrm{E}(\mathbb{EAG}(\mathrm{R}))$ if and only if $\mathrm{Ann}(\mathrm{I}) = \mathrm{J}$ and $\mathrm{Ann}(\mathrm{J}) = \mathrm{I}$.

Definition of (Exact) Annihilating Ideal Graph, this article further describes the properties of the graph with rings $\mathbb{Z}_n$. Comparison of the properties of annihilating ideal and exact annihilating ideal graph of ring $\mathbb{Z}_n$ is also presented in this article.

3. RESULTS AND DISCUSSION

Integers are partitioned into prime numbers and composite numbers. Integer factorization affects the cardinality of the set of all vertices of an ideal annihilating graph. Conversely, can also be observed from the ideal annihilating graph, the characteristics of these integers can be determined. The following theorem shows the relationship between integer factorization and vertex cardinality of an ideal annihilating graph.

Theorem 1.
Suppose $\mathbb{Z}_n$ ring of integer modulo n where n not prime.

1. If $n = p^2$, *where p is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.
2. If $n \neq p^2$, *where p is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$

Proof.

(1) Suppose $n = p^2$ then there exists uniquely non zero proper ideal in di $\mathbb{Z}_n$, $\langle \bar{p} \rangle = \{\overline{pz} | \bar{z} \in \mathbb{Z}_n\}$. If $n = p^2$ its means $\overline{p^2} = \bar{n} = \bar{0}$ such that $\langle \bar{p} \rangle \langle \bar{p} \rangle = \langle \bar{0} \rangle$. Ideal $\langle \bar{p} \rangle$ is an annihilating ideal of $\mathbb{Z}_n$ by Definition 1. Since ideal $\langle \bar{p} \rangle$ is the only one of proper non zero ideal in $\mathbb{Z}_n$, hence $\mathbb{A}(\mathbb{Z}_n)^* = \{\langle \bar{p} \rangle\}$ or $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.

(2) Suppose n is nonprime, that is $n = ab$ for some $a, b \in \mathbb{Z}$ where $1 < a < n$, $1 < b < n$, and $a \neq b$. The product of two ideal, $\langle a \rangle \langle b \rangle = \{(za)(yb) | a, b \in \mathbb{Z}, y, z \in \mathbb{Z}_n\}$. Since $n = ab$, $zy(ab) = zy(n)$ then $\langle a \rangle \langle b \rangle = \{zyn | z, y \in \mathbb{Z}_n\} = \langle \bar{0} \rangle$. Clearly, $\langle a \rangle \neq \langle \bar{0} \rangle$ and $\langle b \rangle \neq \langle \bar{0} \rangle$. Ideals $\langle a \rangle$ and $\langle b \rangle$ are annihilating ideal by Definition 1. Hence, $\langle a \rangle, \langle b \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$. That is prove that for any nonprime n, $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$. ■

Theorem 2.
Suppose $\mathbb{Z}_n$ ring of integer modulo n. The number of vertices of annihilating ideal graph $\mathbb{AG}(\mathbb{Z}_n)$ is $\varphi(n) - 2$, where $\varphi(n)$ is the number of positive factors of n.

Proof.

Suppose $n = (p_1)^{\alpha_1}(p_2)^{\alpha_2} \dots (p_n)^{\alpha_n}$ is prime factorization of n. If $x | n$ then $x = (p_1)^{\beta_1}(p_2)^{\beta_2} \dots (p_n)^{\beta_n}$ where $\beta_i \leq \alpha_i$ for all i. If $x | n$, also means that there exists integer y such that $xy = n$. Suppose $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \dots (p_n)^{\gamma_n}$ then $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \dots (p_n)^{\gamma_n}$, where $\alpha_i = \beta_i + \gamma_i$ for $1 \leq i \leq n$.

We construct principal ideal $\langle x \rangle = \{\overline{xz} | z \in \mathbb{Z}_n\}$ and $\langle y \rangle = \{\overline{yt} | t \in \mathbb{Z}_n\}$ of $\mathbb{Z}_n$. The product of these ideal $\langle x \rangle \langle y \rangle = \{(\overline{xz})(\overline{yt})\} = \{(\overline{xy})(\overline{zt})\}$. As $xy = n$ implies $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. For all $\langle x \rangle$, where x is a positive factor of n, there exists ideal $\langle y \rangle$ such that $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. The number of Ideal $\langle x \rangle$ that satisfied the condition is the number of positive factor of n, $\varphi(n)$. Suppose the set

$$\mathbb{I}(\mathbb{Z}_n) = \{\langle x \rangle \text{ ideal } \mathbb{Z}_n | \exists y \in \mathbb{Z} \text{ such that } xy = n\}$$

Based on the process above, we have $|\mathbb{I}(\mathbb{Z}_n)| = \varphi(n)$. All of elements $\mathbb{I}(\mathbb{Z}_n)$ is the elements of $\mathbb{A}(\mathbb{Z}_n)^*$ except $\langle 1 \rangle$ and $\langle n \rangle$. Hence $|\mathbb{A}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. ■

Theorem 3.
Suppose $\mathbb{Z}_n$ ring of integer modulo n. If $\langle \bar{a} \rangle$ is a vertex of graph $\mathbb{AG}(\mathbb{Z}_n)$ then a is a factor of n.

Proof.

Assume a isn't factor of n. We have $n = ax + y$, where x and y is integer and $0 < y < a$. The product of ideal $\langle \bar{a} \rangle$ and $\langle \bar{x} \rangle$ is

$$\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\bar{a}r)(\bar{x}n)\} = \{\bar{a}(r\bar{x}n)\} = \{\bar{a}(\bar{x}rn)\} = \{(\bar{a}\bar{x})rn\} = \{(\overline{ax})rn\}$$

We have element $\bar{n} = \bar{y}$ because $n = ax + y$. Then $\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\overline{ax})rn\} = \{(\overline{axr})n\} = \langle \bar{n} \rangle = \langle \bar{y} \rangle$. In means $\langle \bar{a} \rangle$ isn't a ideal annihilator of $\mathbb{Z}_n$. Hence $\langle \bar{a} \rangle \notin \mathbb{A}(\mathbb{Z}_n)^*$. By the contraposition, we have if $\langle \bar{a} \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$ then a is a factor of n. ■

The converse of Theorem 3 is not true. For all $n \in \mathbb{Z}$, we have $1 | n$, but clearly $\langle \bar{1} \rangle$ is not an ideal annihilator of $\mathbb{Z}_n$. It means $\langle \bar{1} \rangle$ is not a vertex ini $\mathbb{AG}(\mathbb{Z}_n)$.

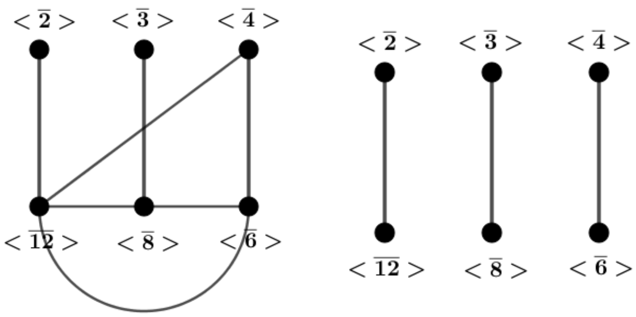
Theorem 4.
Suppose $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are ideal in $\mathbb{Z}_n$. Vertex $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ if and only if $n | pq$.

Proof.
Suppose $\langle p\rangle = \{pa|a \in \mathbb{Z}_n\}$ and $\langle q\rangle = \{qb|r \in \mathbb{Z}_n\}$. The product $\langle p\rangle\langle q\rangle = \langle\overline{pq}\rangle$. If $n|pq$ then $\langle p\rangle\langle q\rangle = \langle\bar{0}\rangle = \{\bar{0}\}$. Hence $\langle\bar{p}\rangle$ and $\langle\bar{q}\rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ by Definition 3.
If $\langle\bar{p}\rangle$ and $\langle\bar{q}\rangle$ are adjacent then $\langle p\rangle\langle q\rangle = \{\bar{0}\}$. It means $(pq)(ab) = nk$ for some integer $a, b,$ and k. The equation $(pq)(ab) = nk$ implies $n|(pq)(ab)$, especially must be $n|pq$. ■

We will continue to discuss the relation some part of annihilating ideal and exact annihilating ideal graph of any commutative ring R.

Lemma 5.
For any commutative ring R, $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$
Proof.
Take any ideal $I \in \mathbb{EA}(R)^*$. It means there exist ideal J of R such that $Ann(I) = J$ and $Ann(J) = I$. Based on definition of annihilator, the product of ideal $IJ = 0$. Hence $I \in \mathbb{A}(R)^*$.
Now, take any ideal $I \in \mathbb{A}(R)^*$. It means there exist nonzero ideal J such that $IJ = 0$. Ideal I is annihilator ideal then $Ann(I) \neq 0$. Suppose $J = Ann(I)$ then J is nonzero ideal of R. We have $Ann(J) = Ann\big(Ann(I)\big) = I$. We conclude $Ann(I) = J$ and $Ann(J) = I$. Hence $I \in \mathbb{EA}(R)^*$. ■

Lemma 6.
For any commutative ring R, $\mathbb{EAG}(R)$ is a subgraph of $\mathbb{AG}(R)$.
Proof.
Lemma 5 show us that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. We will prove that for all $(I,J) \in E\big(\mathbb{EAG}(R)\big)$ then $(I,J) \in E(\mathbb{AG}(R))$. Adjacency of ideal I and J on $\mathbb{EAG}(R)$ means that $I = Ann(J)$ and $J = Ann(I)$. Based on properties of annihilator of ideal, we have $IJ = 0$. Based on definition of adjacency on $\mathbb{AG}(R)$, we have $(I,J) \in E(\mathbb{AG}(R))$. ■
The converse of Theorem 4 not valid for exact annihilating ideal graph. Figure 1 below show that vertex $\langle\bar{6}\rangle$ and $\langle\overline{12}\rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_{24})$ but not adjacent in $\mathbb{EAG}(\mathbb{Z}_{24})$ although $24|12 \times 6$.



(a) **(b)**

Figure 1. **Representation of Annihilating Ideal and Exact Annihilating Ideal Graph (a)$\mathbb{AG}(\mathbb{Z}_{24})$, (b)$\mathbb{EAG}(\mathbb{Z}_{24})$**

Based on the situation, we will construct the criteria of adjacency in exact annihilating ideal graph.

Theorem 7.
Suppose commutative ring $\mathbb{Z}_n$ with identity $\bar{1}$. Ideals $\langle\bar{p}\rangle$ and $\langle\bar{q}\rangle$ are adjacent vertex of $\mathbb{EAG}(\mathbb{Z}_n)$ if and only if $n = pq$.
Proof.
$(\leftarrow)$. Assume $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent. We will proof $n \neq pq$. We have $\langle\bar{p}\rangle = \{\overline{pa}|\bar{a}, \bar{p} \in \mathbb{Z}_n\}$ and $\langle\bar{q}\rangle = \{\overline{qb}|\bar{b}, \bar{q} \in \mathbb{Z}_n\}$ are not adjacent. It means $Ann(\langle\bar{p}\rangle) \neq \langle\bar{q}\rangle$ and $Ann(\langle\bar{q}\rangle) \neq \langle\bar{p}\rangle$ such that $\langle\bar{p}\rangle\langle\bar{q}\rangle \neq \{\bar{0}\}$. We use commutative and associative property of $\mathbb{Z}_n$ to get form $\langle\bar{p}\rangle\langle\bar{q}\rangle = \{(\overline{pa})(\overline{qb})\} = \{(\overline{pq})(\overline{ab})\} \neq \{\bar{0}\}$. It imply $pq \nmid n$. Hence $pq \neq n$.
$(\rightarrow)$. Assume $n \neq pq$. We will proof vertex $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$. If $n \neq pq$ then $n = pq + a$ with a is non-zero integer. We construct two principal ideal generated by p and q on $\mathbb{Z}_n$. Now, we have the product of these ideal

$$\langle\bar{p}\rangle\langle\bar{q}\rangle = \{(\overline{pr})(\overline{qt})\} = \{(\overline{n-a})(\overline{rt})\} = \langle\overline{-a}\rangle$$

We have $(\langle\bar{p}\rangle, \langle\bar{q}\rangle) \notin E\big(\mathbb{AG}(\mathbb{Z}_n)\big)$. Based on Lemma 6, vertex $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$.■

Theorem 8.
Suppose commutative ring $\mathbb{Z}_n$ with identity $\bar{1}$. If $n = r^2$ then $\langle \bar{r} \rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$.
Proof.
Suppose $n = r^2$ and principal ideal $\langle \bar{r} \rangle$ of ring $\mathbb{Z}_n$. We have $Ann(\langle \bar{r} \rangle) = \langle \bar{r} \rangle$. Its means $\langle \bar{r} \rangle$ is a vertex of $\mathbb{EAG}(\mathbb{Z}_n)$. Assume there is a vertex $\langle \bar{a} \rangle$ (not equal to $\langle \bar{r} \rangle$) of $\mathbb{EAG}(\mathbb{Z}_n)$ such that $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$ adjacent. The product of the ideals is $\langle \bar{a} \rangle \langle \bar{r} \rangle \neq \langle \bar{r} \rangle \langle \bar{r} \rangle = \langle \bar{0} \rangle$. Vertex $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$ adjacent on $\mathbb{EAG}(\mathbb{Z}_n)$ means that $\langle \bar{r} \rangle = Ann(\langle \bar{a} \rangle)$ and $\langle \bar{a} \rangle = Ann(\langle \bar{r} \rangle)$. Furthermore $\langle \bar{r} \rangle \langle \bar{a} \rangle = \langle \bar{0} \rangle$. Its contradiction with the product ideals $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$. Hence there is no vertex adjacent with $\langle \bar{r} \rangle$ on $\mathbb{EAG}(\mathbb{Z}_n)$. ■

In [11] showed that $diam\left(\mathbb{EAG}(R)\right) < 1$ and $g\left(\mathbb{EAG}(R)\right) \leq 4$ for any commutative ring R. In this paper, we will show more specific result about diameter, girth, cycle existence of $\mathbb{EAG}(R)$.

Theorem 9.
Suppose commutative ring R. If $\mathbb{EAG}(R)$ is connected graph then $diam\left(\mathbb{EAG}(R)\right) = 1$.
Proof.
Suppose I and J are two different vertex of $\mathbb{EAG}(R)$. Assume $d(I,J) = 2 > 1$, means that exist a vertex A of $\mathbb{EAG}(R)$ such that $I - A - J$ is a path in $\mathbb{EAG}(R)$. Based on Definition 4 we have $I = Ann(A), A = Ann(I), A = Ann(J),$ and $J = Ann(A)$. It imply $I = Ann(A) = Ann\left(Ann(J)\right)$. Based on Lemma 2.1 on [3], we get $Ann\left(Ann(J)\right) = J$. Two last equation imply $I = J$. We have a contradiction with ideal I and J must be different. So, $d(I,J) = 1$ for all ideal I and J. It proved that $diam\left(\mathbb{EAG}(R)\right) = 1$. ■

Collorary 10.
Suppose commutative ring R. If $\mathbb{EAG}(R)$ contain a cycle then $g\left(\mathbb{EAG}(R)\right) \leq 3$.

Proof.
If graph G contain a cycle then $g(G) \leq 2diam(G) + 1$. Theorem 9 has shown that $diam\left(\mathbb{EAG}(R)\right) = 1$. Finally, we have $g\left(\mathbb{EAG}(R)\right) \leq 2diam\left(\mathbb{EAG}(R)\right) + 1 = 3$. ■

Theorem 3.9 in [11] showed that $\mathbb{EAG}\left(\mathbb{Z}_{p^n}\right)$ where p is prime can be represented as union of some complete graph. Figure 1 below show that $\mathbb{EAG}(\mathbb{Z}_{24})$ can be represented as union of K_2 graph, although $24 \neq p^n$ for any prime p. Based on this fact, we construct a theorem to generalize properties of representation of $\mathbb{EAG}(\mathbb{Z}_n)$.

Theorem 11.
The number of complete subgraph of Exact annihilating ideal graph of ring $\mathbb{Z}_n$ is $\left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$.
Proof.
Lemma 5 showed that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. Based on Theorem 2, we have $|\mathbb{EA}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. Theorem 9 showed that $diam\left(\mathbb{EAG}(R)\right) = 1$ for any commutative ring R. We conclude that the maximum number of edges $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)}{2} - 1$. Its means the maximum complete subgraph of $\mathbb{EAG}(\mathbb{Z}_n)$ is also $\frac{\varphi(n)}{2} - 1$.
Case 1: $n = r^2$ for some integer r
Based on Theorem 8, $\langle \bar{r} \rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$. We have $\varphi(n) - 3$ other vertices of $\mathbb{EAG}(\mathbb{Z}_n)$. Obviously there is no positive integer a such that $n = a^2$. In another word, we just found exactly one isolated vertex on $\mathbb{EAG}(\mathbb{Z}_n)$. We can partition $\mathbb{EAG}(\mathbb{Z}_n)$ to be $\frac{\varphi(n)-3}{2}$ graph K_2. Isolated vertex can be represented as K_1. The total of number complete graph that contain in $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)-3}{2} + 1 = \frac{\varphi(n)+1}{2} - 1 = \left\lceil \frac{\varphi(n)}{2} \right\rceil - 1 = \left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$.
Case 2: $n \neq r^2$ for any integer r
If $n \neq r^2$ for any integer r then $n = ab$ where $a \neq b$. Ideal $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ are vertices in $\mathbb{EAG}(\mathbb{Z}_n)$. Based on theorem 7, $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ adjacent in $\mathbb{EAG}(\mathbb{Z}_n)$. This condition means there is no isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$.

Graph $\mathbb{EAG}(\mathbb{Z}_n)$ is fully partition into complete graph K_2. Total number of K_2 is $\frac{\varphi(n)-2}{2} = \frac{\varphi(n)}{2} - 1 = \left\lceil\frac{\varphi(n)}{2}\right\rceil - 1 = \left\lceil\frac{\varphi(n)}{2} - 1\right\rceil$. ■

4. CONCLUSIONS

Factorization on $\mathbb{Z}_n$ characterizes the (Exact) Annihilating Ideal Graph, especially in 1) the number of vertices in an annihilating ideal graph, 2) adjacency of the vertices, and 3) decomposition of exact annihilating ideal graph.

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ANNIHILATING IDEAL AND EXACT ANNIHILATING IDEAL GRAPH OF RING $\mathbb{Z}_n$

Anindito Wisnu Susanto [1], Dewa Putu Wiadnyana Putra[2*]

[1,2]Department of Mathematics Education, Faculty of Teacher Training and Education, Sanata Dharma University
Jl. Affandi, Mrican, Caturtunggal, Depok, Sleman, Yogyakarta, 55281, Indonesia

Corresponding author's e-mail: [2] **dewa@usd.ac.id***

ABSTRACT

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The existence of annihilator in the ring motivates the emergence of studies on Annihilating Ideal and Exact Annihilating Ideal Graphs. The purpose of this research is to describe the characteristics of an (exact) annihilating ideal of ring $\mathbb{Z}_n$. The method used in this research is literature study. The results of this study discuss finiteness, adjacency, connectedness, vertices, and types of $\mathbb{AG}(\mathbb{Z}_n)$ and $\mathbb{EAG}(\mathbb{Z}_n)$. Furthermore, the number of vertices of an Annihilating Ideal Graph is determined by the factorization of n. The adjacency of two vertices is determined by the divisibleness of n. The results also show that $\mathbb{EAG}(\mathbb{Z}_n)$ is a subgraph of $\mathbb{AG}(\mathbb{Z}_n)$. $\mathbb{EAG}(\mathbb{Z}_n)$ can be represented as a union of several complete graphs.

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1. INTRODUCTION

The use of graphs in representing algebraic structures has been carried out since at least 1878 in [1]. This representation starts from representing a group structure into a graph. The vertices of a graph are all elements of a group and changes to an element due to operations on the group are represented by directed edges. Furthermore, [2], [3] began to associate graphs with a broader structure, namely rings. Investigation of the ring structure is carried out through the colored representation of the graph. The representation of an algebraic structure on a graph opens up opportunities for visual investigation of the properties of a particular structure. An essential part in the process of representing a particular algebraic structure to a graph is how to define the connection between the vertices of the graph. Different ways of defining adjacent vertices can lead to different variations of properties as well.

One of the interesting things in the ring, which is about zero divisor. A non-zero element a is said to be a zero divisor if it can be found a non-zero element b such that $ab = 0$. From this structure, [4] proposed the origin of the zero divisor graph. The vertices of the graph are all zero divisors. Two vertices are adjacent if and only if the product of the two elements is zero. Many interesting properties result from this concept, one of which is about the combinatorics of a finite ring [5], [6], [7].

The concept is similar to zero divisor in the ring is the Annihilator. Badawi has started a study on annihilator graphs [8]. In its development, annihilating graphs are generalized into annihilating Ideal graphs. In [9], it is stated that Annihilator is an ideal I, namely $Ann(I) = \{r \in R | rl = 0 \; \forall \, l \in L\}$. If $Ann(I)$ is not a trivial set, then I is called an ideal annilator. In 2011, [10] started to represent a structure consisting of annihilator ideals into a graph. The graph that is formed is named Annihilating Ideal Graph. In line with the development of zero divisor graphs, [11] is continuing the study of Exact Annihilating Ideal graphs. The development of ideal annihilating and exact annihilating properties of graphs is studied separately. The general relationship between these two graphs began to be investigated by [12].

An integer modulo n, $\mathbb{Z}_n$ is a ring that has very interesting properties. This $\mathbb{Z}_n$ structure is widely used in graphs, for example in coloring Antimagic graphs [13] and Domination ratio [14]. The factorization theorem on integers is a motivation for developing graph studies involving a ring of integers modulo n. One of the graph studies carried out was a study on non-coprime for $\mathbb{Z}_n$ [15]. In this research, we combine the properties of (Exact) Annihilating Ideal Graph of arbitrary ring with factorization of ring integer modulo n. These properties will be used to represent integer factors in a graph.

2. RESEARCH METHODS

This research is a literature research that examines the properties of annihilating ideal and exact annihilating ideal graphs on integer rings modulo n, $\mathbb{Z}_n$. The properties studied are the relationship between the factorization of integer n and the vertex of an ideal annihilating graph, the adjacency of vertices, and the relationship between integer decomposition and graph decomposition. The definition of (Exact) Annihilating Ideal based on [10], [11] is as follows.

Definition 1. [10]
An Ideal I of commutative ring R with identity is a Annihilating Ideal if there exist non zero ideal J of R such that $\mathrm{IJ} = 0$. The set of all Annihilating Ideal of ring R is denoted by $\mathbb{A}(\mathrm{R})$.

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Definition 2. [11]
An ideal I of commutative ring R with identity is Exact Annihilating Ideal if there exist non zero ideal J of R such that $\mathrm{Ann}(\mathrm{I}) = \mathrm{J}$ and $\mathrm{Ann}(\mathrm{J}) = \mathrm{I}$. The set of all Exact Annihilating Ideal of ring R denoted by $\mathbb{EA}(\mathrm{R})$.

Based on the two definitions above, then (Exact) Annihilating Ideal Graph is defined as follows.

Definition 3. [10]
Annihilating Ideal graph of ring R denoted by $\mathbb{AG}(\mathrm{R})$ is a graph with vertices $\mathbb{A}(\mathrm{R})^* = \mathbb{A}(\mathrm{R}) \backslash \{(0)\}$ and $(\mathrm{I}, \mathrm{J}) \in \mathrm{E}(\mathbb{AG}(\mathrm{R}))$ if and only if $\mathrm{IJ} = (0)$.

Definition 4. [11]
Exact Annihilating Ideal graph of ring R denoted by $\mathbb{EAG}(\mathrm{R})$ is a graph with vertices $\mathbb{EA}(\mathrm{R})^* = \mathbb{EA}(\mathrm{R}) \backslash \{(0)\}$ and $(\mathrm{I}, \mathrm{J}) \in \mathrm{E}(\mathbb{EAG}(\mathrm{R}))$ if and only if $\mathrm{Ann}(\mathrm{I}) = \mathrm{J}$ and $\mathrm{Ann}(\mathrm{J}) = \mathrm{I}$.

Definition of (Exact) Annihilating Ideal Graph, this article further describes the properties of the graph with rings $\mathbb{Z}_n$. Comparison of the properties of annihilating ideal and exact annihilating ideal graph of ring $\mathbb{Z}_n$ is also presented in this article.

3. RESULTS AND DISCUSSION

Integers are partitioned into prime numbers and composite numbers. Integer factorization affects the cardinality of the set of all vertices of an ideal annihilating graph. Conversely, can also be observed from the ideal annihilating graph, the characteristics of these integers can be determined. The following theorem shows the relationship between integer factorization and vertex cardinality of an ideal annihilating graph.

Theorem 1.

Suppose $\mathbb{Z}_n$ ring of integer modulo n where n not prime.

1. If $n = p^2$, *where p is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.
2. If $n \neq p^2$, *where p is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$

Proof.

(1) Suppose $n = p^2$ then there exists uniquely non zero proper ideal in di $\mathbb{Z}_n$, $\langle \bar{p} \rangle = \{\overline{pz} | \bar{z} \in \mathbb{Z}_n\}$. If $n = p^2$ its means $\overline{p^2} = \bar{n} = \bar{0}$ such that $\langle \bar{p} \rangle \langle \bar{p} \rangle = \langle \bar{0} \rangle$. Ideal $\langle \bar{p} \rangle$ is an annihilating ideal of $\mathbb{Z}_n$ by Definition 1. Since ideal $\langle \bar{p} \rangle$ is the only one of proper non zero ideal in $\mathbb{Z}_n$, hence $\mathbb{A}(\mathbb{Z}_n)^* = \{\langle \bar{p} \rangle\}$ or $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.

(2) Suppose n is nonprime, that is $n = ab$ for some $a, b \in \mathbb{Z}$ where $1 < a < n$, $1 < b < n$, and $a \neq b$. The product of two ideal, $\langle a \rangle \langle b \rangle = \{(za)(yb) | a, b \in \mathbb{Z}, y, z \in \mathbb{Z}_n\}$. Since $n = ab$, $zy(ab) = zy(n)$ then $\langle a \rangle \langle b \rangle = \{zyn | z, y \in \mathbb{Z}_n\} = \langle \bar{0} \rangle$. Clearly, $\langle a \rangle \neq \langle \bar{0} \rangle$ and $\langle b \rangle \neq \langle \bar{0} \rangle$. Ideals $\langle a \rangle$ and $\langle b \rangle$ are annihilating ideal by Definition 1. Hence, $\langle a \rangle, \langle b \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$. That is prove that for any nonprime n, $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$. ■

Theorem 2.

Suppose $\mathbb{Z}_n$ ring of integer modulo n. The number of vertices of annihilating ideal graph $\mathbb{AG}(\mathbb{Z}_n)$ is $\varphi(n) - 2$, where $\varphi(n)$ is the number of positive factors of n.

Proof.

Suppose $n = (p_1)^{\alpha_1}(p_2)^{\alpha_2} \dots (p_n)^{\alpha_n}$ is prime factorization of n. If $x|n$ then $x = (p_1)^{\beta_1}(p_2)^{\beta_2} \dots (p_n)^{\beta_n}$ where $\beta_i \leq \alpha_i$ for all i. If $x|n$, also means that there exists integer y such that $xy = n$. Suppose $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \dots (p_n)^{\gamma_n}$ then $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \dots (p_n)^{\gamma_n}$, where $\alpha_i = \beta_i + \gamma_i$ for $1 \leq i \leq n$.

We construct principal ideal $\langle x \rangle = \{\overline{xz} | z \in \mathbb{Z}_n\}$ and $\langle y \rangle = \{\overline{yt} | t \in \mathbb{Z}_n\}$ of $\mathbb{Z}_n$. The product of these ideal $\langle x \rangle \langle y \rangle = \{(\overline{xz})(\overline{yt})\} = \{(\overline{xy})(\overline{zt})\}$. As $xy = n$ implies $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. For all $\langle x \rangle$, where x is a positive factor of n, there exists ideal $\langle y \rangle$ such that $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. The number of Ideal $\langle x \rangle$ that satisfied the condition is the number of positive factor of n, $\varphi(n)$. Suppose the set

$$\mathbb{I}(\mathbb{Z}_n) = \{\langle x \rangle \text{ ideal } \mathbb{Z}_n | \exists y \in \mathbb{Z} \text{ such that } xy = n\}$$

Based on the process above, we have $|\mathbb{I}(\mathbb{Z}_n)| = \varphi(n)$. All of elements $\mathbb{I}(\mathbb{Z}_n)$ is the elements of $\mathbb{A}(\mathbb{Z}_n)^*$ except $\langle 1 \rangle$ and $\langle n \rangle$. Hence $|\mathbb{A}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. ■

Theorem 3.

Suppose $\mathbb{Z}_n$ ring of integer modulo n. If $\langle \bar{a} \rangle$ is a vertex of graph $\mathbb{AG}(\mathbb{Z}_n)$ then a is a factor of n.

Proof.

Assume a isn't factor of n. We have $n = ax + y$, where x and y is integer and $0 < y < a$.
The product of ideal $\langle \bar{a} \rangle$ and $\langle \bar{x} \rangle$ is

$$\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\bar{a}r)(\bar{x}n)\} = \{\bar{a}(r\bar{x}n)\} = \{\bar{a}(\bar{x}rn)\} = \{(\bar{a}\bar{x})rn\} = \{(\overline{ax})rn\}$$

We have element $\bar{n} = \bar{y}$ because $n = ax + y$. Then $\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\overline{ax})rn\} = \{(\overline{axr})n\} = \langle \bar{n} \rangle = \langle \bar{y} \rangle$. In means $\langle \bar{a} \rangle$ isn't a ideal annihilator of $\mathbb{Z}_n$. Hence $\langle \bar{a} \rangle \notin \mathbb{A}(\mathbb{Z}_n)^*$. By the contraposition, we have if $\langle \bar{a} \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$ then a is a factor of n. ■

The converse of Theorem 3 is not true. For all $n \in \mathbb{Z}$, we have $1|n$, but clearly $\langle \bar{1} \rangle$ is not an ideal annihilator of $\mathbb{Z}_n$. It means $\langle \bar{1} \rangle$ is not a vertex ini $\mathbb{AG}(\mathbb{Z}_n)$.

Theorem 4.

Suppose $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are ideal in $\mathbb{Z}_n$. Vertex $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ if and only if $n|pq$.

Proof.
Suppose $\langle p\rangle = \{pa | a \in \mathbb{Z}_n\}$ and $\langle q\rangle = \{qb | r \in \mathbb{Z}_n\}$. The product $\langle p\rangle\langle q\rangle = \langle\overline{pq}\rangle$. If $n|pq$ then $\langle p\rangle\langle q\rangle = \langle\bar{0}\rangle = \{\bar{0}\}$. Hence $\langle\bar{p}\rangle$ and $\langle\bar{q}\rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ by Definition 3.
If $\langle\bar{p}\rangle$ and $\langle\bar{q}\rangle$ are adjacent then $\langle p\rangle\langle q\rangle = \{\bar{0}\}$. It means $(pq)(ab) = nk$ for some integer $a, b,$ and k. The equation $(pq)(ab) = nk$ implies $n|(pq)(ab)$, especially must be $n|pq$. ∎

We will continue to discuss the relation some part of annihilating ideal and exact annihilating ideal graph of any commutative ring R.

Lemma 5.
For any commutative ring R, $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$
Proof.
Take any ideal $I \in \mathbb{EA}(R)^*$. It means there exist ideal J of R such that $Ann(I) = J$ and $Ann(J) = I$. Based on definition of annihilator, the product of ideal $IJ = 0$. Hence $I \in \mathbb{A}(R)^*$.
Now, take any ideal $I \in \mathbb{A}(R)^*$. It means there exist nonzero ideal J such that $IJ = 0$. Ideal I is annihilator ideal then $Ann(I) \neq 0$. Suppose $J = Ann(I)$ then J is nonzero ideal of R. We have $Ann(J) = Ann\big(Ann(I)\big) = I$. We conclude $Ann(I) = J$ and $Ann(J) = I$. Hence $I \in \mathbb{EA}(R)^*$. ∎

Lemma 6.
For any commutative ring R, $\mathbb{EAG}(R)$ is a subgraph of $\mathbb{AG}(R)$.
Proof.
Lemma 5 show us that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. We will prove that for all $(I, J) \in E\big(\mathbb{EAG}(R)\big)$ then $(I, J) \in E(\mathbb{AG}(R))$. Adjacency of ideal I and J on $\mathbb{EAG}(R)$ means that $I = Ann(J)$ and $J = Ann(I)$. Based on properties of annihilator of ideal, we have $IJ = 0$. Based on definition of adjacency on $\mathbb{AG}(R)$, we have $(I, J) \in E(\mathbb{AG}(R))$. ∎
The converse of Theorem 4 not valid for exact annihilating ideal graph. Figure 1 below show that vertex $\langle\bar{6}\rangle$ and $\langle\overline{12}\rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_{24})$ but not adjacent in $\mathbb{EAG}(\mathbb{Z}_{24})$ although $24|12 \times 6$.

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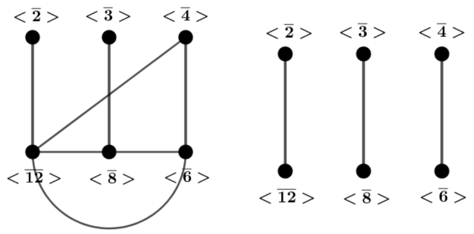

(a) **(b)**

Figure 1. **Representation of Annihilating Ideal and Exact Annihilating Ideal Graph (a)$\mathbb{AG}(\mathbb{Z}_{24})$, (b)$\mathbb{EAG}(\mathbb{Z}_{24})$**

Based on the situation, we will construct the criteria of adjacency in exact annihilating ideal graph.

Theorem 7.
Suppose commutative ring $\mathbb{Z}_n$ with identity $\bar{1}$. Ideals $\langle\bar{p}\rangle$ and $\langle\bar{q}\rangle$ are adjacent vertex of $\mathbb{EAG}(\mathbb{Z}_n)$ if and only if $n = pq$.
Proof.
($\leftarrow$). Assume $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent. We will proof $n \neq pq$. We have $\langle\bar{p}\rangle = \{\overline{pa} | \bar{a}, \bar{p} \in \mathbb{Z}_n\}$ and $\langle\bar{q}\rangle = \left\{\overline{qb} \middle| \bar{b}, \bar{q} \in \mathbb{Z}_n\right\}$ are not adjacent. It means $Ann(\langle\bar{p}\rangle) \neq \langle\bar{q}\rangle$ and $Ann(\langle\bar{q}\rangle) \neq \langle\bar{p}\rangle$ such that $\langle\bar{p}\rangle\langle\bar{q}\rangle \neq \{\bar{0}\}$. We use commutative and associative property of $\mathbb{Z}_n$ to get form $\langle\bar{p}\rangle\langle\bar{q}\rangle = \left\{(\overline{pa})\left(\overline{qb}\right)\right\} = \left\{(\overline{pq})\left(\overline{ab}\right)\right\} \neq \{\bar{0}\}$. It imply $pq \nmid n$. Hence $pq \neq n$.
($\rightarrow$). Assume $n \neq pq$. We will proof vertex $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$. If $n \neq pq$ then $n = pq + a$ with a is non-zero integer. We construct two principal ideal generated by p and q on $\mathbb{Z}_n$. Now, we have the product of these ideal

$$\langle\bar{p}\rangle\langle\bar{q}\rangle = \{(\overline{pr})(\overline{qt})\} = \{(\overline{n-a})(\overline{rt})\} = \langle\overline{-a}\rangle$$

We have $(\langle\bar{p}\rangle, \langle\bar{q}\rangle) \notin E\big(\mathbb{AG}(\mathbb{Z}_n)\big)$. Based on Lemma 6, vertex $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$. ∎

Theorem 8.
Suppose commutative ring $\mathbb{Z}_n$ with identity $\bar{1}$. If $n = r^2$ then $\langle \bar{r} \rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$.
Proof.
Suppose $n = r^2$ and principal ideal $\langle \bar{r} \rangle$ of ring $\mathbb{Z}_n$. We have $Ann(\langle \bar{r} \rangle) = \langle \bar{r} \rangle$. Its means $\langle \bar{r} \rangle$ is a vertex of $\mathbb{EAG}(\mathbb{Z}_n)$. Assume there is a vertex $\langle \bar{a} \rangle$ (not equal to $\langle \bar{r} \rangle$) of $\mathbb{EAG}(\mathbb{Z}_n)$ such that $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$ adjacent. The product of the ideals is $\langle \bar{a} \rangle \langle \bar{r} \rangle \neq \langle \bar{r} \rangle \langle \bar{r} \rangle = \langle \bar{0} \rangle$. Vertex $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$ adjacent on $\mathbb{EAG}(\mathbb{Z}_n)$ means that $\langle \bar{r} \rangle = Ann(\langle \bar{a} \rangle)$ and $\langle \bar{a} \rangle = Ann(\langle \bar{r} \rangle)$. Furthermore $\langle \bar{r} \rangle \langle \bar{a} \rangle = \langle \bar{0} \rangle$. Its contradiction with the product ideals $\langle \bar{r} \rangle$ and $\langle \bar{a} \rangle$. Hence there is no vertex adjacent with $\langle \bar{r} \rangle$ on $\mathbb{EAG}(\mathbb{Z}_n)$. ■

In [11] showed that $diam(\mathbb{EAG}(R)) < 1$ and $g(\mathbb{EAG}(R)) \leq 4$ for any commutative ring R. In this paper, we will show more specific result about diameter, girth, cycle existence of $\mathbb{EAG}(R)$.

Theorem 9.
Suppose commutative ring R. If $\mathbb{EAG}(R)$ is connected graph then $diam(\mathbb{EAG}(R)) = 1$.
Proof.
Suppose I and J are two different vertex of $\mathbb{EAG}(R)$. Assume $d(I,J) = 2 > 1$, means that exist a vertex A of $\mathbb{EAG}(R)$ such that $I - A - J$ is a path in $\mathbb{EAG}(R)$. Based on Definition 4 we have $I = Ann(A), A = Ann(I), A = Ann(J)$, and $J = Ann(A)$. It imply $I = Ann(A) = Ann(Ann(J))$. Based on Lemma 2.1 on [3], we get $Ann(Ann(J)) = J$. Two last equation imply $I = J$. We have a contradiction with ideal I and J must be different. So, $d(I,J) = 1$ for all ideal I and J. It proved that $diam(\mathbb{EAG}(R)) = 1$. ■

Collorary 10.
Suppose commutative ring R. If $\mathbb{EAG}(R)$ contain a cycle then $g(\mathbb{EAG}(R)) \leq 3$.

Proof.
If graph G contain a cycle then $g(G) \leq 2diam(G) + 1$. Theorem 9 has shown that $diam(\mathbb{EAG}(R)) = 1$. Finally, we have $g(\mathbb{EAG}(R)) \leq 2diam(\mathbb{EAG}(R)) + 1 = 3$. ■

Theorem 3.9 in [11] showed that $\mathbb{EAG}(\mathbb{Z}_{p^n})$ where p is prime can be represented as union of some complete graph. Figure 1 below show that $\mathbb{EAG}(\mathbb{Z}_{24})$ can be represented as union of K_2 graph, although $24 \neq p^n$ for any prime p. Based on this fact, we construct a theorem to generalize properties of representation of $\mathbb{EAG}(\mathbb{Z}_n)$.

Theorem 11.
The number of complete subgraph of Exact annihilating ideal graph of ring $\mathbb{Z}_n$ is $\left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$.
Proof.
Lemma 5 showed that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. Based on Theorem 2, we have $|\mathbb{EA}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. Theorem 9 showed that $diam(\mathbb{EAG}(R)) = 1$ for any commutative ring R. We conclude that the maximum number of edges $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)}{2} - 1$. Its means the maximum complete subgraph of $\mathbb{EAG}(\mathbb{Z}_n)$ is also $\frac{\varphi(n)}{2} - 1$.
Case 1: $n = r^2$ for some integer r
Based on Theorem 8, $\langle \bar{r} \rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$. We have $\varphi(n) - 3$ other vertices of $\mathbb{EAG}(\mathbb{Z}_n)$. Obviously there is no positive integer a such that $n = a^2$. In another word, we just found exactly one isolated vertex on $\mathbb{EAG}(\mathbb{Z}_n)$. We can partition $\mathbb{EAG}(\mathbb{Z}_n)$ to be $\frac{\varphi(n)-3}{2}$ graph K_2. Isolated vertex can be represented as K_1. The total of number complete graph that contain in $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)-3}{2} + 1 = \frac{\varphi(n)+1}{2} - 1 = \left\lceil \frac{\varphi(n)}{2} \right\rceil - 1 = \left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$.
Case 2: $n \neq r^2$ for any integer r
If $n \neq r^2$ for any integer r then $n = ab$ where $a \neq b$. Ideal $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ are vertices in $\mathbb{EAG}(\mathbb{Z}_n)$. Based on theorem 7, $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ adjacent in $\mathbb{EAG}(\mathbb{Z}_n)$. This condition means there is no isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$.

Graph $\mathbb{EAG}(\mathbb{Z}_n)$ is fully partition into complete graph K_2. Total number of K_2 is $\frac{\varphi(n)-2}{2} = \frac{\varphi(n)}{2} - 1 = \left\lceil\frac{\varphi(n)}{2}\right\rceil - 1 = \left\lceil\frac{\varphi(n)}{2} - 1\right\rceil$. ■

4. CONCLUSIONS

Factorization on $\mathbb{Z}_n$ characterizes the (Exact) Annihilating Ideal Graph, especially in 1) the number of vertices in an annihilating ideal graph, 2) adjacency of the vertices, and 3) decomposition of exact annihilating ideal graph.

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ANNIHILATING IDEAL AND EXACT ANNIHILATING IDEAL GRAPH OF RING $\mathbb{Z}_n$

Anindito Wisnu Susanto [1]**, Dewa Putu Wiadnyana Putra**[2*]

[1,2]*Department of Mathematics Education, Faculty of Teacher Training and Education, Sanata Dharma University*
Jl. Affandi, Mrican, Caturtunggal, Depok, Sleman, Yogyakarta, 55281, Indonesia

Corresponding author's e-mail: [2*] **dewa@usd.ac.id**

ABSTRACT

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The existence of annihilator in the ring motivates the emergence of studies on Annihilating Ideal and Exact Annihilating Ideal Graphs. The purpose of this research is to describe the characteristics of an (exact) annihilating ideal of ring $\mathbb{Z}_n$. The method used in this research is literature study. The results of this study discuss finiteness, adjacency, connectedness, vertices, and types of $\mathbb{AG}(\mathbb{Z}_n)$ and $\mathbb{EAG}(\mathbb{Z}_n)$. Furthermore, the number of vertices of an Annihilating Ideal Graph is determined by the factorization of n. The adjacency of two vertices is determined by the divisibleness of n. The results also show that $\mathbb{EAG}(\mathbb{Z}_n)$ is a subgraph of $\mathbb{AG}(\mathbb{Z}_n)$. $\mathbb{EAG}(\mathbb{Z}_n)$ can be represented as a union of several complete graphs.

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1. INTRODUCTION

The use of graphs in representing algebraic structures has been carried out since at least 1878 in [1]. This representation starts from representing a group structure into a graph. The vertices of a graph are all elements of a group and changes to an element due to operations on the group are represented by directed edges. Furthermore, [2], [3] began to associate graphs with a broader structure, namely rings. Investigation of the ring structure is carried out through the colored representation of the graph. The representation of an algebraic structure on a graph opens up opportunities for visual investigation of the properties of a particular structure. An essential part in the process of representing a particular algebraic structure to a graph is how to define the connection between the vertices of the graph. Different ways of defining adjacent vertices can lead to different variations of properties as well.

One of the interesting things in the ring, which is about zero divisor. A non-zero element a is said to be a zero divisor if it can be found a non-zero element b such that $ab = 0$. From this structure, [4] proposed the origin of the zero divisor graph. The vertices of the graph are all zero divisors. Two vertices are adjacent if and only if the product of the two elements is zero. Many interesting properties result from this concept, one of which is about the combinatorics of a finite ring [5], [6], [7].

The concept is similar to zero divisor in the ring is the Annihilator. Badawi has started a study on annihilator graphs [8]. In its development, annihilating graphs are generalized into annihilating Ideal graphs. In [9], it is stated that Annihilator is an ideal I, namely $Ann(I) = \{r \in R | rl = 0 \; \forall \, l \in L\}$. If $Ann(I)$ is not a trivial set, then I is called an ideal annilator. In 2011, [10] started to represent a structure consisting of annihilator ideals into a graph. The graph that is formed is named Annihilating Ideal Graph. In line with the development of zero divisor graphs, [11] is continuing the study of Exact Annihilating Ideal graphs. The development of ideal annihilating and exact annihilating properties of graphs is studied separately. The general relationship between these two graphs began to be investigated by [12].

An integer modulo n, $\mathbb{Z}_n$ is a ring that has very interesting properties. This $\mathbb{Z}_n$ structure is widely used in graphs, for example in coloring Antimagic graphs [13] and Domination ratio [14]. The factorization theorem on integers is a motivation for developing graph studies involving a ring of integers modulo n. One of the graph studies carried out was a study on non-coprime for $\mathbb{Z}_n$ [15]. In this research, we combine the properties of (Exact) Annihilating Ideal Graph of arbitrary ring with factorization of ring integer modulo n. These properties will be used to represent integer factors in a graph.

2. RESEARCH METHODS

This research is a literature research that examines the properties of annihilating ideal and exact annihilating ideal graphs on integer rings modulo n, $\mathbb{Z}_n$. The properties studied are the relationship between the factorization of integer n and the vertex of an ideal annihilating graph, the adjacency of vertices, and the relationship between integer decomposition and graph decomposition. The definition of (Exact) Annihilating Ideal based on [10], [11] is as follows.

Definition 1. [10]
An Ideal I of commutative ring R with identity is a Annihilating Ideal if there exist non zero ideal J of R such that $IJ = 0$. The set of all Annihilating Ideal of ring R is denoted by $\mathbb{A}(R)$.

Definition 2. [11]
An ideal I of commutative ring R with identity is Exact Annihilating Ideal if there exist non zero ideal J of R such that $Ann(I) = J$ and $Ann(J) = I$. The set of all Exact Annihilating Ideal of ring R denoted by $\mathbb{EA}(R)$.

Based on the two definitions above, then (Exact) Annihilating Ideal Graph is defined as follows.

Definition 3. [10]
Annihilating Ideal graph of ring R denoted by $\mathbb{A}G(R)$ is a graph with vertices $\mathbb{A}(R)^* = \mathbb{A}(R)\backslash\{(0)\}$ and $(I,J) \in E(\mathbb{A}G(R))$ if and only if $IJ = (0)$.

Definition 4. [11]
Exact Annihilating Ideal graph of ring R denoted by $\mathbb{EA}\mathbb{G}(R)$ is a graph with vertices $\mathbb{EA}(R)^* = \mathbb{EA}(R)\backslash\{(0)\}$ and $(I,J) \in E(\mathbb{EA}G(R))$ if and only if $Ann(I) = J$ and $Ann(J) = I$.

Definition of (Exact) Annihilating Ideal Graph, this article further describes the properties of the graph with rings $\mathbb{Z}_n$. Comparison of the properties of annihilating ideal and exact annihilating ideal graph of ring $\mathbb{Z}_n$ is also presented in this article.

3. RESULTS AND DISCUSSION

Integers are partitioned into prime numbers and composite numbers. Integer factorization affects the cardinality of the set of all vertices of an ideal annihilating graph. Conversely, can also be observed from the ideal annihilating graph, the characteristics of these integers can be determined. The following theorem shows the relationship between integer factorization and vertex cardinality of an ideal annihilating graph.

Theorem 1.
Suppose $\mathbb{Z}_n$ ring of integer modulo n where n not prime.

1. If $n = p^2$, *where p is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.
2. If $n \neq p^2$, *where p is prime then* $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$

Proof.

(1) Suppose $n = p^2$ then there exists uniquely non zero proper ideal in di $\mathbb{Z}_n$, $\langle \bar{p} \rangle = \{\overline{pz} | \bar{z} \in \mathbb{Z}_n\}$. If $n = p^2$ its means $\overline{p^2} = \bar{n} = \bar{0}$ such that $\langle \bar{p} \rangle \langle \bar{p} \rangle = \langle \bar{0} \rangle$. Ideal $\langle \bar{p} \rangle$ is an annihilating ideal of $\mathbb{Z}_n$ by **Definition 1.** Since ideal $\langle \bar{p} \rangle$ is the only one of proper non zero ideal in $\mathbb{Z}_n$, hence $\mathbb{A}(\mathbb{Z}_n)^* = \{\langle \bar{p} \rangle\}$ or $|\mathbb{A}(\mathbb{Z}_n)^*| = 1$.

(2) Suppose n is nonprime, that is $n = ab$ for some $a, b \in \mathbb{Z}$ where $1 < a < n$, $1 < b < n$, and $a \neq b$. The product of two ideal, $\langle a \rangle \langle b \rangle = \{(za)(yb) | a, b \in \mathbb{Z}, y, z \in \mathbb{Z}_n\}$. Since $n = ab$, $zy(ab) = zy(n)$ then $\langle a \rangle \langle b \rangle = \{zyn | z, y \in \mathbb{Z}_n\} = \langle \bar{0} \rangle$. Clearly, $\langle a \rangle \neq \langle \bar{0} \rangle$ and $\langle b \rangle \neq \langle \bar{0} \rangle$. Ideals $\langle a \rangle$ and $\langle b \rangle$ are annihilating ideal by **Definition 1.** Hence, $\langle a \rangle, \langle b \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$. That is prove that for any nonprime n, $|\mathbb{A}(\mathbb{Z}_n)^*| \geq 2$. ■

Theorem 2.
Suppose $\mathbb{Z}_n$ ring of integer modulo n. The number of vertices of annihilating ideal graph $\mathbb{AG}(\mathbb{Z}_n)$ is $\varphi(n) - 2$, where $\varphi(n)$ is the number of positive factors of n.

Proof.
Suppose $n = (p_1)^{\alpha_1}(p_2)^{\alpha_2} \dots (p_n)^{\alpha_n}$ is prime factorization of n. If $x | n$ then $x = (p_1)^{\beta_1}(p_2)^{\beta_2} \dots (p_n)^{\beta_n}$ where $\beta_i \leq \alpha_i$ for all i. If $x | n$, also means that there exists integer y such that $xy = n$. Suppose $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \dots (p_n)^{\gamma_n}$ then $y = (p_1)^{\gamma_1}(p_2)^{\gamma_2} \dots (p_n)^{\gamma_n}$, where $\alpha_i = \beta_i + \gamma_i$ for $1 \leq i \leq n$.

We construct principal ideal $\langle x \rangle = \{\overline{xz} | z \in \mathbb{Z}_n\}$ and $\langle y \rangle = \{\overline{yt} | t \in \mathbb{Z}_n\}$ of $\mathbb{Z}_n$. The product of these ideal $\langle x \rangle \langle y \rangle = \{(\overline{xz})(\overline{yt})\} = \{(\overline{xy})(\overline{zt})\}$. As $xy = n$ implies $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. For all $\langle x \rangle$, where x is a positive factor of n, there exists ideal $\langle y \rangle$ such that $\langle x \rangle \langle y \rangle = \{\bar{0}\}$. The number of Ideal $\langle x \rangle$ that satisfied the condition is the number of positive factor of n, $\varphi(n)$. Suppose the set

$$\mathbb{I}(\mathbb{Z}_n) = \{\langle x \rangle \text{ ideal } \mathbb{Z}_n | \exists y \in \mathbb{Z} \text{ such that } xy = n\}$$

Based on the process above, we have $|\mathbb{I}(\mathbb{Z}_n)| = \varphi(n)$. All of elements $\mathbb{I}(\mathbb{Z}_n)$ is the elements of $\mathbb{A}(\mathbb{Z}_n)^*$ except $\langle 1 \rangle$ and $\langle n \rangle$. Hence $|\mathbb{A}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. ■

Theorem 3.
Suppose $\mathbb{Z}_n$ ring of integer modulo n. If $\langle \bar{a} \rangle$ is a vertex of graph $\mathbb{AG}(\mathbb{Z}_n)$ then a is a factor of n.

Proof.
Assume a isn't factor of n. We have $n = ax + y$, where x and y is integer and $0 < y < a$. The product of ideal $\langle \bar{a} \rangle$ and $\langle \bar{x} \rangle$ is

$$\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\bar{a}r)(\bar{x}n)\} = \{\bar{a}(r\bar{x}n)\} = \{\bar{a}(\bar{x}rn)\} = \{(\bar{a}\bar{x})rn\} = \{(\overline{ax})rn\}$$

We have element $\bar{n} = \bar{y}$ because $n = ax + y$. Then $\langle \bar{a} \rangle \langle \bar{x} \rangle = \{(\overline{ax})rn\} = \{(\overline{axr})n\} = \langle \bar{n} \rangle = \langle \bar{y} \rangle$. In means $\langle \bar{a} \rangle$ isn't a ideal annihilator of $\mathbb{Z}_n$. Hence $\langle \bar{a} \rangle \notin \mathbb{A}(\mathbb{Z}_n)^*$. By the contraposition, we have if $\langle \bar{a} \rangle \in \mathbb{A}(\mathbb{Z}_n)^*$ then a is a factor of n. ■

The converse of Theorem 3 is not true. For all $n \in \mathbb{Z}$, we have $1 | n$, but clearly $\langle \bar{1} \rangle$ is not an ideal annihilator of $\mathbb{Z}_n$. It means $\langle \bar{1} \rangle$ is not a vertex ini $\mathbb{AG}(\mathbb{Z}_n)$.

Theorem 4.
Suppose $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are ideal in $\mathbb{Z}_n$. Vertex $\langle \bar{p} \rangle$ and $\langle \bar{q} \rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ if and only if $n | pq$.

Proof.
Suppose $\langle p\rangle = \{pa|a \in \mathbb{Z}_n\}$ and $\langle q\rangle = \{qb|r \in \mathbb{Z}_n\}$. The product $\langle p\rangle\langle q\rangle = \langle\overline{pq}\rangle$. If $n|pq$ then $\langle p\rangle\langle q\rangle = \langle\bar{0}\rangle = \{\bar{0}\}$. Hence $\langle\bar{p}\rangle$ and $\langle\bar{q}\rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_n)$ by Definition 3.
If $\langle\bar{p}\rangle$ and $\langle\bar{q}\rangle$ are adjacent then $\langle p\rangle\langle q\rangle = \{\bar{0}\}$. It means $(pq)(ab) = nk$ for some integer $a, b,$ and k. The equation $(pq)(ab) = nk$ implies $n|(pq)(ab)$, especially must be $n|pq$. ■

We will continue to discuss the relation some part of annihilating ideal and exact annihilating ideal graph of any commutative ring R.

Lemma 5.
For any commutative ring R, $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$
Proof.
Take any ideal $I \in \mathbb{EA}(R)^*$. It means there exist ideal J of R such that $Ann(I) = J$ and $Ann(J) = I$. Based on definition of annihilator, the product of ideal $IJ = 0$. Hence $I \in \mathbb{A}(R)^*$.
Now, take any ideal $I \in \mathbb{A}(R)^*$. It means there exist nonzero ideal J such that $IJ = 0$. Ideal I is annihilator ideal then $Ann(I) \neq 0$. Suppose $J = Ann(I)$ then J is nonzero ideal of R. We have $Ann(J) = Ann\big(Ann(I)\big) = I$. We conclude $Ann(I) = J$ and $Ann(J) = I$. Hence $I \in \mathbb{EA}(R)^*$. ■

Lemma 6.
For any commutative ring R, $\mathbb{EAG}(R)$ is a subgraph of $\mathbb{AG}(R)$.
Proof.
Lemma 5 show us that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. We will prove that for all $(I,J) \in E\big(\mathbb{EAG}(R)\big)$ then $(I,J) \in E(\mathbb{AG}(R))$. Adjacency of ideal I and J on $\mathbb{EAG}(R)$ means that $I = Ann(J)$ and $J = Ann(I)$. Based on properties of annihilator of ideal, we have $IJ = 0$. Based on definition of adjacency on $\mathbb{AG}(R)$, we have $(I,J) \in E(\mathbb{AG}(R))$. ■

The converse of Theorem 4 not valid for exact annihilating ideal graph. The counter example of converse Theorem 4 is in Example 1 below.
Example 1.
In ring $\mathbb{Z}_{24}$, vertex $\langle\bar{6}\rangle$ and $\langle\overline{12}\rangle$ are adjacent in $\mathbb{AG}(\mathbb{Z}_{24})$ but not adjacent in $\mathbb{EAG}(\mathbb{Z}_{24})$ although $24|12 \times 6$. Figure 1 below show the representation both graph of ring $\mathbb{Z}_{24}$.

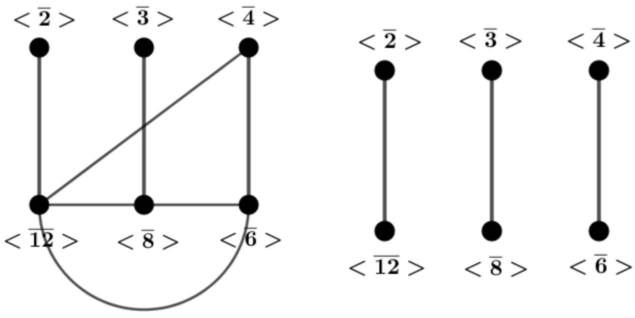

Figure 1. Representation of Annihilating Ideal and Exact Annihilating Ideal Graph (a)$\mathbb{AG}(\mathbb{Z}_{24})$, (b)$\mathbb{EAG}(\mathbb{Z}_{24})$

Based on the situation, we will construct the criteria of adjacency in exact annihilating ideal graph.
Theorem 7.
Suppose commutative ring $\mathbb{Z}_n$ with identity $\bar{1}$. Ideals $\langle\bar{p}\rangle$ and $\langle\bar{q}\rangle$ are adjacent vertex of $\mathbb{EAG}(\mathbb{Z}_n)$ if and only if $n = pq$.
Proof.
$(\leftarrow)$. Assume $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent. We will proof $n \neq pq$. We have $\langle\bar{p}\rangle = \{\overline{pa}|\bar{a}, \bar{p} \in \mathbb{Z}_n\}$ and $\langle\bar{q}\rangle = \{\overline{qb}|\bar{b}, \bar{q} \in \mathbb{Z}_n\}$ are not adjacent. It means $Ann(\langle\bar{p}\rangle) \neq \langle\bar{q}\rangle$ and $Ann(\langle\bar{q}\rangle) \neq \langle\bar{p}\rangle$ such that $\langle\bar{p}\rangle\langle\bar{q}\rangle \neq \{\bar{0}\}$. We use commutative and associative property of $\mathbb{Z}_n$ to get form $\langle\bar{p}\rangle\langle\bar{q}\rangle = \{(\overline{pa})(\overline{qb})\} = \{(\overline{pq})(\overline{ab})\} \neq \{\bar{0}\}$. It imply $pq \nmid n$. Hence $pq \neq n$.

($\rightarrow$). Assume $n \neq pq$. We will proof vertex $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$. If $n \neq pq$ then $n = pq + a$ with a is non-zero integer. We construct two principal ideal generated by p and q on $\mathbb{Z}_n$. Now, we have the product of these ideal

$$\langle\bar{p}\rangle\langle\bar{q}\rangle = \{(\overline{pr})(\overline{qt})\} = \{(\overline{n-a})(\overline{rt})\} = \langle\overline{-a}\rangle$$

We have $(\langle\bar{p}\rangle, \langle\bar{q}\rangle) \notin E\big(\mathbb{AG}(\mathbb{Z}_n)\big)$. Based on Lemma 6, vertex $\langle\bar{p}\rangle$ *and* $\langle\bar{q}\rangle$ are not adjacent in graph $\mathbb{EAG}(\mathbb{Z}_n)$.■

Theorem 8.

Suppose commutative ring $\mathbb{Z}_n$ *with identity* $\bar{1}$. *If* $n = r^2$ *then* $\langle\bar{r}\rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$.

Proof.

Suppose $n = r^2$ and principal ideal $\langle\bar{r}\rangle$ of ring $\mathbb{Z}_n$. We have $Ann(\langle\bar{r}\rangle) = \langle\bar{r}\rangle$. Its means $\langle\bar{r}\rangle$ is a vertex of $\mathbb{EAG}(\mathbb{Z}_n)$. Assume there is a vertex $\langle\bar{a}\rangle$ (not equal to $\langle\bar{r}\rangle$) of $\mathbb{EAG}(\mathbb{Z}_n)$ such that $\langle\bar{r}\rangle$ and $\langle\bar{a}\rangle$ adjacent. The product of the ideals is $\langle\bar{a}\rangle\langle\bar{r}\rangle \neq \langle\bar{r}\rangle\langle\bar{r}\rangle = \langle\bar{0}\rangle$. Vertex $\langle\bar{r}\rangle$ and $\langle\bar{a}\rangle$ adjacent on $\mathbb{EAG}(\mathbb{Z}_n)$ means that $\langle\bar{r}\rangle = Ann(\langle\bar{a}\rangle)$ and $\langle\bar{a}\rangle = Ann(\langle\bar{r}\rangle)$. Furthermore $\langle\bar{r}\rangle\langle\bar{a}\rangle = \langle\bar{0}\rangle$. Its contradiction with the product ideals $\langle\bar{r}\rangle$ and $\langle\bar{a}\rangle$. Hence there is no vertex adjacent with $\langle\bar{r}\rangle$ on $\mathbb{EAG}(\mathbb{Z}_n)$. ■

In [11] showed that $diam\big(\mathbb{EAG}(R)\big) < 1$ and $g\big(\mathbb{EAG}(R)\big) \leq 4$ for any commutative ring R. In this paper, we will show more specific result about diameter, girth, cycle existence of $\mathbb{EAG}(R)$.

Theorem 9.

Suppose commutative ring R. *If* $\mathbb{EAG}(R)$ *is connected graph then* $diam\big(\mathbb{EAG}(R)\big) = 1$.

Proof.

Suppose I and J are two different vertex of $\mathbb{EAG}(R)$. Assume $d(I, J) = 2 > 1$, means that exist a vertex A of $\mathbb{EAG}(R)$ such that $I - A - J$ is a path in $\mathbb{EAG}(R)$. Based on Definition 4 we have $I = Ann(A), A = Ann(I), A = Ann(J),$ and $J = Ann(A)$. It imply $I = Ann(A) = Ann\big(Ann(J)\big)$. Based on Lemma 2.1 on [3], we get $Ann\big(Ann(J)\big) = J$. Two last equation imply $I = J$. We have a contradiction with ideal I and J must be different. So, $d(I, J) = 1$ for all ideal I and J. It proved that $diam\big(\mathbb{EAG}(R)\big) = 1$. ■

Collorary 10.

Suppose commutative ring R. *If* $\mathbb{EAG}(R)$ *contain a cycle then* $g\big(\mathbb{EAG}(R)\big) \leq 3$.

Proof.

If graph G contain a cycle then $g(G) \leq 2diam(G) + 1$. **Theorem 9** has shown that $diam\big(\mathbb{EAG}(R)\big) = 1$. Finally, we have $g\big(\mathbb{EAG}(R)\big) \leq 2diam\big(\mathbb{EAG}(R)\big) + 1 = 3$. ■

Theorem 3.9 in [11] showed that $\mathbb{EAG}\big(\mathbb{Z}_{p^n}\big)$ where p is prime can be represented as union of some complete graph. Figure 1 below show that $\mathbb{EAG}(\mathbb{Z}_{24})$ can be represented as union of K_2 graph, although $24 \neq p^n$ for any prime p. Based on this fact, we construct a theorem to generalize properties of representation of $\mathbb{EAG}(\mathbb{Z}_n)$.

Theorem 11.

The number of complete subgraph of Exact annihilating ideal graph of ring $\mathbb{Z}_{\boldsymbol{n}}$ *is* $\left\lceil\frac{\varphi(n)}{2} - 1\right\rceil$.

Proof.

Lemma 5 showed that $\mathbb{EA}(R)^* = \mathbb{A}(R)^*$. Based on **Theorem 2**, we have $|\mathbb{EA}(\mathbb{Z}_n)^*| = \varphi(n) - 2$. **Theorem 9** showed that $diam\big(\mathbb{EAG}(R)\big) = 1$ for any commutative ring R. We conclude that the maximum number of edges $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)}{2} - 1$. Its means the maximum complete subgraph of $\mathbb{EAG}(\mathbb{Z}_n)$ is also $\frac{\varphi(n)}{2} - 1$.

Case 1: $n = r^2$ for some integer r

Based on Theorem 8, $\langle\bar{r}\rangle$ is a isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$. We have $\varphi(n) - 3$ other vertices of $\mathbb{EAG}(\mathbb{Z}_n)$. Obviously there is no positive integer a such that $n = a^2$. In another word, we just found exactly one isolated vertex on $\mathbb{EAG}(\mathbb{Z}_n)$. We can partition $\mathbb{EAG}(\mathbb{Z}_n)$ to be $\frac{\varphi(n)-3}{2}$ graph K_2. Isolated vertex can be represented as K_1. The total of number complete graph that contain in $\mathbb{EAG}(\mathbb{Z}_n)$ is $\frac{\varphi(n)-3}{2} + 1 = \frac{\varphi(n)+1}{2} - 1 = \left\lceil\frac{\varphi(n)}{2}\right\rceil - 1 = \left\lceil\frac{\varphi(n)}{2} - 1\right\rceil$.

Case 2: $n \neq r^2$ for any integer r

If $n \neq r^2$ for any integer r then $n = ab$ where $a \neq b$. Ideal $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ are vertices in $\mathbb{EAG}(\mathbb{Z}_n)$. Based on theorem 7, $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ adjacent in $\mathbb{EAG}(\mathbb{Z}_n)$. This condition means there is no isolated vertex in $\mathbb{EAG}(\mathbb{Z}_n)$. Graph $\mathbb{EAG}(\mathbb{Z}_n)$ is fully partition into complete graph K_2. Total number of K_2 is $\frac{\varphi(n)-2}{2} = \frac{\varphi(n)}{2} - 1 = \left\lceil \frac{\varphi(n)}{2} \right\rceil - 1 = \left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$. ■

4. CONCLUSIONS

Factorization on $\mathbb{Z}_n$ characterizes the (Exact) Annihilating Ideal Graph, especially in 1) the number of vertices in an annihilating ideal graph, 2) adjacency of the vertices, and 3) decomposition of exact annihilating ideal graph. The number of vertices of annihilating ideal is equal to the number vertices of exact annihilating graph of ring $\mathbb{Z}_n$, that is $\varphi(n) - 2$, where $\varphi(n)$ is the number of positive factors of n. In $\mathbb{AG}(\mathbb{Z}_n)$, two vertices $\langle p \rangle$ and $\langle q \rangle$ are adjacent if and only if n divides the product of p and q. But, in $\mathbb{EAG}(\mathbb{Z}_n)$ these two vertices are adjacent if and only if n must equal to the product of p and q. $\mathbb{EAG}(\mathbb{Z}_n)$ is decomposed into $\left\lceil \frac{\varphi(n)}{2} - 1 \right\rceil$ complete subgraph.

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