Jurnal Matematika & Sains, Desember 2012, Vol. 17 Nomor 3

Time-Invariant Fuzzy Max-Plus Linear Systems

Marcellinus Andy Rudhito
Department of Mathematics Education
Universitas Sanata Dharma, Paingan Maguwoharjo Yogyakarta
e-mail: arudhito@yahoo.co.id

Received 12 July 2012, accepted for publication 19 December 2012

Abstract

In time-invariant max-plus linear systems (IMLS), the activity times are real numbers. In time-invariant fuzzy max-plus linear systems (IFMLSI), as there is uncertainty in the activity times, the activity times are modeled as fuzzy numbers. This article discusses a generalization of IMLSI to IFMLS, especially IFMLS with single input single output (SISO), and input-output analysis of IFMLS-SISO. We show that input-output analysis IFMLS-SISO associated with the latest times input problem can be solved by using a solution of a system of fuzzy number max-plus linear equations.

Keywords: Linear Systems, Max-Plus, Fuzzy Number, Time-Invariant, Input-Output.

Sistem Linier *Max-Plus* Samar yang Invarian terhadap Waktu

Abstrak

Dalam sistem linear max-plus waktu invarian (SLMI), waktu aktifitas berupa bilangan real. Dalam sistem linear max-plus kabur waktu invarian (SLMKI), ada ketidakpastian dalam waktu aktifitas, sehingga waktu aktifitas dimodelkan sebagai bilangan kabur. Artikel ini membahas suatu generalisasi SLMI menjadi SLMKI, khususnya SLMKI dengan satu input dan satu output (SISO), dan analisis input-output SLMKI-SISO. Dapat ditunjukkan bahwa analisis input-output SLMKI-SISO yang berkaitan dengan masalah waktu input paling lambat, dapat diselesaikan dengan menggunakan suatu penyelesaian sistem persamaan linear max-plus kabur.

Kata kunci: Sistem Linear, Max-Plus, Bilangan Kabur, Waktu-Invariant, Input-Output.

1. Introduction

In modeling and analyzing a network sometimes its activity time is not known, because, for example its phase design, data on time activity or distribution which are not fixed. The activity can be estimated based on the experience and opinions from experts and network operators. This network activity times are modeled using fuzzy number, and it is called fuzzy activity times (Chanas and Zielinski, 2001; Soltoni and Haji, 2007).

Max-plus algebra, namely the set of all real numbers **R** with operations max and plus, has been used to model and analyze network problems algebraically, such as problem of project scheduling and queuing system, with a deterministic time activity (Chanas and Zielinski, 2001; Heidergott *et al.,* 2005; Krivulin, 2001; Rudhito, 2003). Rudhito (2004), de Schutter (1996) have discussed a model of simple production system dynamics using max-plus algebra approach. In general, this model is a system of max-plus linear time-invariant.

Fuzzy number max-plus algebra is an extension of max-plus algebra, in which the elements are fuzzy numbers, (Rudhito *et al.,* 2008; 2011a). Morever Rudhito *et al.* (2011b) studied matrices over fuzzy number max-plus algebra and systems of fuzzy number linear equations.

In a manner analogous to Rudhito (2003) and de Schutter (1996), and taking into account the results on fuzzy numbers max-plus algebra, this paper will discuss the time-invariant fuzzy max-plus linear system using fuzzy number max-plus algebra.

2. Results and Discussion

First, we review some definitions and basic concepts of max-plus algebra and matrices over max-plus algebra. For further details see (Chanas & Zielinski, 2001; Rudhito, 2003). Let $\mathbf{R}_\varepsilon := \mathbf{R} \cup \{\varepsilon\}$ be the set of all real numbers and $\varepsilon := -\infty$. In $\mathbf{R}_\varepsilon$ we define two operations as follow:

for every $a, b \in \mathbf{R}_\varepsilon$,

$$a \oplus b := \max(a, b) \text{ and } a \otimes b := a + b.$$

We can show that ($\mathbf{R}_\varepsilon$, $\oplus$, $\otimes$) is a commutative semiring whose neutral element $\varepsilon = -\infty$ and unity element $e = 0$. Moreover, ($\mathbf{R}_\varepsilon$, $\oplus$, $\otimes$) is a semifield, that is ($\mathbf{R}_\varepsilon$, $\oplus$, $\otimes$) is a commutative semiring, where for every $a \in \mathbf{R}$ there exists $-a$ such that $a \otimes (-a) = 0$. We call ($\mathbf{R}_\varepsilon$, $\oplus$, $\otimes$) a *max-plus algebra*, and write it as $\mathbf{R}_{\max}$. The max-plus algebra $\mathbf{R}_{\max}$ has no zero divisors, that is for every

$x, y \in \mathbf{R}_\varepsilon$, if $x \otimes y = \varepsilon$, then $x = \varepsilon$ or $y = \varepsilon$. The relation "$\preceq_m$" as $x \preceq_m y$ iff $x \oplus y = y$. The max-plus algebra $\mathbf{R}_{\max}$ can be partially ordered by introducing a partial order relation. Operations $\oplus$ and $\otimes$ are *consistent* with respect to the order $\preceq_m$, that is, for every $a, b, c \in \mathbf{R}_{\max}$, if $a \preceq_m b$, then $a \oplus c \preceq_m b \oplus c$, and $a \otimes c \preceq_m b \otimes c$. We define $x^{\otimes 0} := 0$, $x^{\otimes k} := x \otimes x^{\otimes k-1}$ and $\varepsilon^{\otimes k} := \varepsilon$, for all $k = 1, 2, \ldots$.

The operations $\oplus$ and $\otimes$ in $\mathbf{R}_{\max}$ can be extended to matrix operations in $\mathbf{R}_{\max}^{m\times n}$, where $\mathbf{R}_{\max}^{m\times n} := \{A = (A_{ij}) \mid A_{ij} \in \mathbf{R}_{\max}$, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n\}$ is the set of all m-by-n matrices over max-plus algebra. Specifically, for $A, B \in \mathbf{R}_{\max}^{n\times n}$ and $\alpha \in \mathbf{R}_{\max}$ we define

$$(\alpha \otimes A)_{ij} = \alpha \otimes A_{ij},\ (A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$$

$$\text{and } (A \otimes B)_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}.$$

For any matrix $A \in \mathbf{R}_{\max}^{n\times n}$, one can define

$$A^{\otimes 0} = E_n \text{ where } (E)_{ij} := \begin{cases} 0 \text{ where } i = j \\ \varepsilon \text{ where } i \neq j \end{cases}$$

and

$$A^{\otimes k} = A \otimes A^{\otimes k-1} \text{ for } k = 1, 2, \ldots .$$

The relation "$\preceq_m$" is defined on $\mathbf{R}_{\max}^{m\times n}$ as $A \preceq_m B$ iff $A \oplus B = B$. It is a partial order in $\mathbf{R}_{\max}^{m\times n}$. In $(\mathbf{R}_{\max}^{n\times n}, \oplus, \otimes)$, operations $\oplus$ and $\otimes$ are consistent with respect to the order $\preceq_m$, that is for every $A, B, C \in \mathbf{R}_{\max}^{n\times n}$, if $A \preceq_m B$, then $A \oplus C \preceq_m B \oplus C$ and $A \otimes C \preceq_m B \otimes C$.

We also review some definitions and some basic concepts of interval max-plus algebra, matrices over interval max-plus algebra. Further details can be found in Rudhito *et al.* (2011a) and de Schutter (1996). The (closed) interval x in $\mathbf{R}_{\max}$ is a subset of $\mathbf{R}_{\max}$ of the form $\mathrm{x} = [\underline{\mathrm{x}}, \overline{\mathrm{x}}] = \{x \in \mathbf{R}_{\max} \mid \underline{\mathrm{x}} \preceq_m x \preceq_m \overline{\mathrm{x}}\}$. The interval x in $\mathbf{R}_{\max}$ is called *max-plus interval* or shortly *interval*. Define $\mathbf{I}(\mathbf{R})_\varepsilon := \{\mathrm{x} = [\underline{\mathrm{x}}, \overline{\mathrm{x}}] \mid \underline{\mathrm{x}}, \overline{\mathrm{x}} \in \mathbf{R}, \varepsilon \prec_m \underline{\mathrm{x}} \preceq_m \overline{\mathrm{x}}\} \cup \{\varepsilon\}$, where $\varepsilon := [\varepsilon, \varepsilon]$. In $\mathrm{I(R)}_\varepsilon$ we can define two operations $\overline{\oplus}$ and $\overline{\otimes}$ where

$$\mathrm{x}\ \overline{\oplus}\ \mathrm{y} = [\underline{\mathrm{x}} \oplus \underline{\mathrm{y}}, \overline{\mathrm{x}} \oplus \overline{\mathrm{y}}]$$

and

$$\mathrm{x}\ \overline{\otimes}\ \mathrm{y} = [\underline{\mathrm{x}} \otimes \underline{\mathrm{y}}, \overline{\mathrm{x}} \otimes \overline{\mathrm{y}}]$$

for every $\mathrm{x}, \mathrm{y} \in \mathbf{I}(\mathbf{R})_\varepsilon$. The algebraic structure $(\mathbf{I}(\mathbf{R})_\varepsilon, \overline{\oplus}, \overline{\otimes})$ is a commutative semiring with neutral element $\varepsilon = [\varepsilon, \varepsilon]$ and unity element $0 = [0, 0]$. This commutative semiring $(\mathbf{I}(\mathbf{R})_\varepsilon, \overline{\oplus}, \overline{\otimes})$ is called the *interval max-plus algebra* and is written as $\mathbf{I}(\mathbf{R})_{\max}$. Relation "$\preceq_{\mathrm{Im}}$" defined on $\mathbf{I}(\mathbf{R})_{\max}$ as $\mathrm{x} \preceq_{\mathrm{Im}} \mathrm{y} \Leftrightarrow \mathrm{x}\ \overline{\oplus}\ \mathrm{y} = \mathrm{y}$, and it is a partial order on $\mathbf{I}(\mathbf{R})_{\max}$. Notice that $\mathrm{x}\ \overline{\oplus}\ \mathrm{y} = \mathrm{y} \Leftrightarrow \underline{\mathrm{x}} \preceq_m \underline{\mathrm{y}}$ and $\overline{\mathrm{x}} \preceq_m \overline{\mathrm{y}}$.

Define $\mathbf{I}(\mathbf{R})_{\max}^{m\times n} := \{\mathrm{A} = (\mathrm{A}_{ij}) \mid \mathrm{A}_{ij} \in \mathbf{I}(\mathbf{R})_{\max}$, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n\}$. The elements of $\mathbf{I}(\mathbf{R})_{\max}^{m\times n}$ are called *matrices over interval max-plus algebra* or shortly *interval matrices*. The operations $\overline{\oplus}$ and $\overline{\otimes}$ in $\mathbf{I}(\mathbf{R})_{\max}$ can be extended to matrix operations on $\mathbf{I}(\mathbf{R})_{\max}^{m\times n}$. Specifically, for A, $\mathrm{B} \in \mathbf{I}(\mathbf{R})_{\max}^{n\times n}$ and $\alpha \in \mathbf{I}(\mathbf{R})_{\max}$ we define $(\alpha\ \overline{\otimes}\ \mathrm{A})_{ij} = \alpha\ \overline{\otimes}\ \mathrm{A}_{ij}$, $(\mathrm{A}\ \overline{\oplus}\ \mathrm{B})_{ij} = \mathrm{A}_{ij}\ \overline{\oplus}\ \mathrm{B}_{ij}$ and $(\mathrm{A}\ \overline{\otimes}\ \mathrm{B})_{ij} = \overline{\bigoplus_{k=1}^{n}} \mathrm{A}_{ik}\ \overline{\otimes}\ \mathrm{B}_{kj}$.

Relation "$\preceq_{\mathrm{Im}}$" defined on $\mathbf{I}(\mathbf{R})_{\max}^{m\times n}$ with $\mathrm{A} \preceq_{\mathrm{Im}} \mathrm{B} \Leftrightarrow \mathrm{A}\ \overline{\oplus}\ \mathrm{B} = \mathrm{B}$ is a partial order. Notice that $\mathrm{A}\ \overline{\oplus}\ \mathrm{B} = \mathrm{B} \Leftrightarrow \mathrm{A}_{ij}\ \overline{\oplus}\ \mathrm{B}_{ij} = \mathrm{B}_{ij} \Leftrightarrow \mathrm{A}_{ij} \preceq_{\mathrm{Im}} \mathrm{B}_{ij} \Leftrightarrow \underline{\mathrm{A}_{ij}} \preceq_m \underline{\mathrm{B}_{ij}}$ and $\overline{\mathrm{A}_{ij}} \preceq_m \overline{\mathrm{B}_{ij}}$ for every i and j.

In further discussion, we assume that the reader knows about some basic concepts in fuzzy set and fuzzy number. For further details see Lee (2005) and Zimmermann (1991). Firstly, we will review a theorem in the fuzzy set that we will use later on.

Theorem 2.1 (Decomposition Theorem, Zimmermann, 1991) If A^α is *a α-cut* of fuzzy set $\widetilde{A}$ in X and $\widetilde{A}^\alpha$ is a fuzzy set in X with membership function $\mu_{\widetilde{A}^\alpha}(x) = \alpha\chi_{A^\alpha}(x)$, where χ_{A^α} is the charateristic function of set A^α, then $\widetilde{A} = \bigcup_{\alpha\in[0,1]} \widetilde{A}^\alpha$.

Proof: (Zimmermann, 1991).

Definition 2.2 (Rudhito *et al.,* 2011b). Let $\mathbf{F}(\mathbf{R})_{\widetilde{\varepsilon}} := \mathbf{F}(\mathbf{R}) \cup \{\widetilde{\varepsilon}\}$, where $\mathbf{F}(\mathbf{R})$ is the set of all fuzzy numbers and $\widetilde{\varepsilon} := \{-\infty\}$, with the *$\alpha$-cut* of $\widetilde{\varepsilon}$ is $\varepsilon^\alpha = [-\infty, -\infty]$ for every $\alpha \in [0, 1]$. We define two operations $\widetilde{\oplus}$ and $\widetilde{\otimes}$ as follows : for every $\widetilde{a}, \widetilde{b} \in \mathbf{F}(\mathbf{R})_{\widetilde{\varepsilon}}$ with $a^\alpha = [\underline{a}^\alpha, \overline{a}^\alpha] \in \mathbf{I}(\mathbf{R})_{max}$ and $b^\alpha = [\underline{b}^\alpha, \overline{b}^\alpha] \in \mathbf{I}(\mathbf{R})_{max}$,

i) Maximum of $\widetilde{a}$ and $\widetilde{b}$, written $\widetilde{a}\ \widetilde{\oplus}\ \widetilde{b}$, is a fuzzy number whose *α-cut* is the interval $[\underline{a}^\alpha \oplus \underline{b}^\alpha, \overline{a}^\alpha \oplus \overline{b}^\alpha]$ for every $\alpha \in [0, 1]$

ii) Addition of $\tilde{a}$ and $\tilde{b}$, written $\tilde{a} \mathbin{\tilde{\otimes}} \tilde{b}$, is a fuzzy number whose α-cut is interval $[\underline{a}^{\alpha} \otimes \underline{b}^{\alpha}, \overline{a}^{\alpha} \otimes \overline{b}^{\alpha}]$ for every $\alpha \in [0, 1]$.

We can show that α-cut in this definition satisfies the conditions of α-cut of a fuzzy number. The commutative semiring $\mathbf{F(R)}_{\max} := ((\mathbf{F(R)}_{\tilde{\varepsilon}}, \tilde{\oplus}, \tilde{\otimes})$ is called fuzzy number max-plus algebra, and is written as $\mathbf{F(R)}_{\max}$. Relation "$\preceq_{\mathrm{Fm}}$" defined on $\mathbf{F(R)}_{\max}$ as $\tilde{x} \preceq_{\mathrm{Fm}} \tilde{y} \Leftrightarrow \tilde{x} \mathbin{\tilde{\oplus}} \tilde{y} = \tilde{y}$ is a partial order in $\mathbf{F(R)}_{\max}$. Notice that $\tilde{x} \mathbin{\tilde{\oplus}} \tilde{y} = \tilde{y} \Leftrightarrow x^{\alpha} \mathbin{\overline{\oplus}} y^{\alpha} = y^{\alpha} \Leftrightarrow x^{\alpha} \preceq_{\mathrm{Im}} y^{\alpha} \Leftrightarrow \underline{x}^{\alpha} \preceq_{\mathrm{m}} \underline{y}^{\alpha}$ and $\overline{x}^{\alpha} \preceq_{\mathrm{m}} \overline{y}^{\alpha}$ for every $\alpha \in [0, 1]$.

Definition 2.3 (Rudhito *et al.,* 2011a) *Define* $\mathbf{F(R)}_{\max}^{m\times n} := \{ \tilde{A} = (\tilde{A}_{ij}) \mid \tilde{A}_{ij} \in \mathrm{F(R)}_{\max}$, *for* $i = 1, 2, ..., m$ *and* $j = 1, 2, ..., n$ $\}$. The elements of $\mathbf{F(R)}_{\max}^{m\times n}$ are called matrices over fuzzy number max-plus algebra.

These matrices are also called *fuzzy number matrix*. The operations $\tilde{\oplus}$ and $\tilde{\otimes}$ in $\mathbf{F(R)}_{\max}$ can be extended to the operations of fuzzy number matrices in $(\mathbf{F(R)}_{\max}^{m\times n}$. Specifically, for the matrices $\tilde{A}$, $\tilde{B} \in \mathbf{F(R)}_{\max}^{n\times n}$, we define

$$(\tilde{A} \mathbin{\tilde{\oplus}} \tilde{B})_{ij} = \tilde{A}_{ij} \mathbin{\tilde{\oplus}} \tilde{B}_{ij}$$

and

$$(\tilde{A} \mathbin{\tilde{\otimes}} \tilde{B})_{ij} = \tilde{\bigoplus_{k=1}^{n}} \tilde{A}_{ik} \otimes \tilde{B}_{kj}.$$

For every $\tilde{A} \in \mathbf{F(R)}_{\max}^{m\times n}$ and for some number $\alpha \in [0, 1]$ define a *α-cut matrix of* $\tilde{A}$ as the interval matrix $A^{\alpha} = (A_{ij}^{\alpha})$, with A_{ij}^{α} is the α-cut of $\tilde{A}_{ij}$ for every i and j. Matrices $\underline{A}^{\alpha} = (\underline{A}_{ij}^{\alpha}) \in \mathbf{R}_{\max}^{m\times n}$ *and* $\overline{A}^{\alpha} = (\overline{A}_{ij}^{\alpha}) \in \mathbf{R}_{\max}^{m\times n}$ are called ***lower bound*** and ***upper bound of matrix*** A^{α}*, respectively*. We conclude that the matrices $\tilde{A}$, $\tilde{B} \in \mathbf{F(R)}_{\max}^{m\times n}$ are *equal* iff $A^{\alpha} = B^{\alpha}$, that is $A_{ij}^{\alpha} = B_{ij}^{\alpha}$ for every $\alpha \in [0, 1]$ and for every i and j. For every fuzzy number matrix $\tilde{A}$, $A^{\alpha} = [\underline{A}^{\alpha}, \overline{A}^{\alpha}]$. Let $\tilde{\lambda} \in \mathbf{F(R)}_{\max}$, $\tilde{A}$, $\tilde{B} \in \mathbf{F(R)}_{\max}^{m\times n}$. We can show that $(\lambda \otimes A)^{\alpha} = [\underline{\lambda}^{\alpha} \otimes \underline{A}^{\alpha}, \overline{\lambda}^{\alpha} \otimes \overline{A}^{\alpha}]$ and $(A \oplus B)^{\alpha} = [\underline{A}^{\alpha} \oplus \underline{B}^{\alpha}, \overline{A}^{\alpha} \oplus \overline{B}^{\alpha}]$ for every $\alpha \in [0, 1]$. Let $\tilde{A} \in \mathbf{F(R)}_{\max}^{m\times p}$, $\tilde{B} \in \mathbf{F(R)}_{\max}^{p\times n}$. We can show that $(A \otimes B)^{\alpha} = [\underline{A}^{\alpha} \otimes \underline{B}^{\alpha}, \overline{A}^{\alpha} \otimes \overline{B}^{\alpha}]$ for every $\alpha \in [0, 1]$. Relation "$\preceq_{\mathrm{Fm}}$" defined on $\mathbf{F(R)}_{\max}^{m\times n}$ with $\tilde{A} \preceq_{\mathrm{Fm}} \tilde{B} \Leftrightarrow \tilde{A} \mathbin{\tilde{\oplus}} \tilde{B} = \tilde{B}$ is a partial order. Notice that $\tilde{A} \mathbin{\tilde{\oplus}} \tilde{B} = \tilde{B} \Leftrightarrow \tilde{A}_{ij} \mathbin{\tilde{\oplus}} \tilde{B}_{ij} = \tilde{B}_{ij} \Leftrightarrow A^{\alpha}_{ij} \mathbin{\overline{\oplus}} B^{\alpha}_{ij} = B^{\alpha}_{ij} \Leftrightarrow A^{\alpha}_{ij} \preceq_{\mathrm{Im}} B^{\alpha}_{ij} \Leftrightarrow \underline{A}^{\alpha}_{ij} \preceq_{\mathrm{m}} \underline{B}^{\alpha}_{ij}$ and $\overline{A}^{\alpha}_{ij} \preceq_{\mathrm{m}} \overline{B}^{\alpha}_{ij}$ for every $\alpha \in [0, 1]$ and for every i and j. Define $\mathbf{F(R)}_{\max}^{n} := \{ \tilde{\mathbf{x}} = [\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n]^{\mathrm{T}} \mid \tilde{x}_i \in \mathbf{F(R)}_{\max}, i = 1, 2, ..., n\}$. The elements in $\mathbf{F(R)}_{\max}^{n}$ are called fuzzy number vectors over $\mathbf{F(R)}_{\max}$ or shortly fuzzy number vectors.

Definition 2.4 (Rudhito *et al.,* 2011b) Given $\tilde{A} \in \mathbf{F(R)}_{\max}^{n\times n}$ and $\tilde{\boldsymbol{b}} \in \mathbf{F(R)}_{\max}^{n}$. A fuzzy number vector $\tilde{\boldsymbol{x}}^{*} \in \mathbf{F(R)}_{\max}^{n}$ is called fuzzy number solution of system $\tilde{A} \mathbin{\tilde{\otimes}} \tilde{\mathbf{x}} = \tilde{\boldsymbol{b}}$ if $\tilde{\boldsymbol{x}}^{*}$ satisfies the system. A fuzzy number vector $\tilde{\boldsymbol{x}}' \in \mathbf{F(R)}_{\max}^{n}$ is called fuzzy number subsolution of system $\tilde{A} \mathbin{\tilde{\otimes}} \tilde{\mathbf{x}} = \tilde{\boldsymbol{b}}$ if $\tilde{A} \mathbin{\tilde{\otimes}} \tilde{\boldsymbol{x}}' \preceq_{\mathrm{Fm}} \tilde{\boldsymbol{b}}$.

Definition 2.5 (Rudhito *et al.,* 2011b) Given $\tilde{A} \in \mathbf{F(R)}_{\max}^{n\times n}$ and $\tilde{\boldsymbol{b}} \in \mathbf{F(R)}_{\max}^{n}$. A fuzzy number vector $\hat{\tilde{\boldsymbol{x}}} \in \mathbf{F(R)}_{\max}^{n}$ is called greatest fuzzy number subsolution of system $\tilde{A} \mathbin{\tilde{\otimes}} \tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}}$ if $\tilde{\boldsymbol{x}}' \preceq_{\mathrm{Fm}} \hat{\tilde{\boldsymbol{x}}}$ for every fuzzy number subsolution $\tilde{\boldsymbol{x}}'$ of the system $\tilde{A} \mathbin{\tilde{\otimes}} \tilde{\mathbf{x}} = \tilde{\mathbf{b}}$.

Definition 2.6 (Rudhito *et al.,* 2011) Given $\tilde{A} \in \mathbf{F(R)}_{\max}^{n\times n}$ with the entries of each column are not all equal to $\tilde{\varepsilon}$ and $\tilde{\boldsymbol{b}} \in \mathbf{F(R)}_{\max}^{n}$. Defined a fuzzy number vector $\hat{\tilde{\boldsymbol{x}}}$ whose components are $\hat{\tilde{x}}_i$, that is a fuzzy number with the α-cut is $\hat{x}_i^{\alpha} = [\underline{\hat{x}}_i^{\alpha}, \overline{\hat{x}}_i^{\alpha}]$. The bounds of $\hat{x}_i^{\alpha}$ are defined recursively as follows. Let

$\underline{\hat{x}}_i^{\alpha} = \min\{-((\underline{A}^{\alpha})^{\mathrm{T}} \otimes (-\underline{\boldsymbol{b}}^{\alpha}))_i, -((\overline{A}^{\alpha})^{\mathrm{T}} \otimes -\overline{\boldsymbol{b}}^{\alpha}))_i\}$ and $\overline{\hat{x}}_i^{\alpha} = -((\overline{A}^{\alpha})^{\mathrm{T}} \otimes -\overline{\boldsymbol{b}}^{\alpha}))_i$, follows

$$\underline{\hat{x}}_i^{\alpha} = \begin{cases} \min\limits_{\beta\in[0,1]} \{\overline{\hat{x}}_i^{\beta}\}, & \text{if } \min\limits_{\beta\in[0,1]} \{\overline{\hat{x}}_i^{\beta}\} \preceq_{\mathrm{m}} \underline{\hat{x}}_i^{\alpha} \\ \begin{cases} \underline{\hat{x}}_i^{\beta}, \text{if } \underline{\hat{x}}_i^{\beta} \preceq_{\mathrm{m}} \underline{\hat{x}}_i^{\alpha} \\ \underline{\hat{x}}_i^{\alpha}, \text{if } \underline{\hat{x}}_i^{\beta} \succ_{\mathrm{m}} \underline{\hat{x}}_i^{\alpha} \end{cases}, \forall \alpha, \beta \in [0,1], \alpha > \beta, & \text{if } \underline{\hat{x}}_i^{\alpha} \prec_{\mathrm{m}} \min\limits_{\beta\in[0,1]} \{\overline{\hat{x}}_i^{\beta}\} \end{cases},$$

$$\overline{\hat{x}}_i^{\alpha} = \begin{cases} \overline{\hat{x}}_i^{\beta}, \text{if } \overline{\hat{x}}_i^{\beta} \preceq_{\mathrm{m}} \overline{\hat{x}}_i^{\alpha} \\ \overline{\hat{x}}_i^{\alpha}, \text{if } \overline{\hat{x}}_i^{\beta} \succ_{\mathrm{m}} \overline{\hat{x}}_i^{\alpha} \end{cases}, \forall \alpha, \beta \in [0,1] \text{ with } \alpha < \beta.$$

We can show that the α-cut family of the components of the fuzzy number vector $\hat{\tilde{\mathbf{x}}}$ as in Definition 2.6 is really an α-cut family of a fuzzy number. The following theorem gives an existence of the greatest fuzzy number subsolution of the system $\tilde{A} \tilde{\otimes} \tilde{\mathbf{x}} = \tilde{\boldsymbol{b}}$.

Theorem 2.7 (Rudhito *et al.,* 2011b) *Given* $\tilde{A} \in \mathbf{F(R)}_{\max}^{n\times n}$ with the entries of each column are not all equal to $\tilde{\varepsilon}$ *and* $\tilde{\boldsymbol{b}} \in \mathbf{F(R)}_{\max}^{n}$. Fuzzy number vector $\hat{\tilde{\mathbf{x}}}$ which components are defined in Definition 2.7 is the greatest fuzzy number subsolution of system $\tilde{A} \tilde{\otimes} \tilde{\mathbf{x}} = \tilde{\boldsymbol{b}}$.

Proof : See Rudhito *et al.* (2011b).

Next we will discuss time invariant fuzzy max-plus linear systems (IFMLS) with single input single output (SISO).

Definition 2.8 (IFMLS-SISO $(\tilde{A},\tilde{B},\tilde{C},\hat{x}_0)$). A time-invariant fuzzy max-plus linear system with single input single output is a discrete even system which can be written as follows

$$\tilde{\mathbf{x}}(k+1) = \tilde{A} \tilde{\otimes} \tilde{\mathbf{x}}(k) \tilde{\oplus} \tilde{B} \tilde{\otimes} \tilde{u}(k+1)$$

$$\tilde{y}(k) = \tilde{C} \tilde{\otimes} \tilde{\mathbf{x}}(k)$$

for $k = 1, 2, 3, \ldots$, with initial condition $\tilde{\mathbf{x}}(0) = \tilde{\mathbf{x}}_0$, $\tilde{A} \in \mathbf{F(R)}_{\max}^{n\times n}$, $\tilde{B} \in \mathbf{F(R)}_{\max}^{n}$ and $\tilde{C} \in \mathbf{F(R)}_{\max}^{1\times n}$. Fuzzy number vector $\tilde{\mathbf{x}}$ (k) $\in \mathbf{F(R)}_{\max}^{n}$ represents the state, $\tilde{u}$ (k) $\in \mathbf{F(R)}_{\max}$ is the input fuzzy sclar and $\tilde{y}$ (k) $\in \mathbf{F(R)}_{\max}$ is the output fuzzy scalar of system for the *k*-th time.

If the initial condition and the input sequences are given for these systems, we can determine recursively the output sequence of the system. The general form of the input-output for the system is given in the following theorem.

Theorem 2.9 (Input-Output IFMLS-SISO ($\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{\mathbf{x}}_0$)) *Given a positive integer p.* If $\tilde{\boldsymbol{u}}$ = $[\tilde{u}(1), \tilde{u}(2), \ldots, \tilde{u}(p)]^{\mathrm{T}}$ *and* $\tilde{\mathbf{y}} = [\tilde{y}(1), \tilde{y}(2), \ldots, \tilde{y}(p)]^{\mathrm{T}}$ *are fuzzy vector of input sequences and the fuzzy vector of output sequences, respectively, for* FMLSI-SISO($\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{\mathbf{x}}_0$), *then*

$$\tilde{\mathbf{y}} = \tilde{K} \tilde{\otimes} \tilde{\mathbf{x}}_0 \tilde{\oplus} \tilde{H} \tilde{\otimes} \tilde{\mathbf{u}}$$

with

$$\tilde{K} = \begin{bmatrix} \tilde{C} \tilde{\otimes} \tilde{A} \\ \tilde{C} \tilde{\otimes} \tilde{A}^{\tilde{\otimes}2} \\ \vdots \\ \tilde{C} \tilde{\otimes} \tilde{A}^{\tilde{\otimes}p} \end{bmatrix} \text{ and }$$

$$\tilde{H} = \begin{bmatrix} \tilde{C} \tilde{\otimes} \tilde{B} & \tilde{\varepsilon} & \cdots & \tilde{\varepsilon} \\ \tilde{C} \tilde{\otimes} \tilde{A} \tilde{\otimes} \tilde{B} & \tilde{C} \tilde{\otimes} \tilde{B} & \cdots & \tilde{\varepsilon} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{C} \tilde{\otimes} \tilde{A}^{\tilde{\otimes}p-1} \tilde{\otimes} \tilde{B} & \tilde{C} \tilde{\otimes} \tilde{A}^{\tilde{\otimes}p-2} \tilde{\otimes} \tilde{B} & \cdots & \tilde{C} \tilde{\otimes} \tilde{B} \end{bmatrix}$$

Proof: Using mathematical induction we can show that

$\tilde{\mathbf{x}}(k) = (\tilde{A}^{\tilde{\otimes}k} \tilde{\otimes} \tilde{\mathbf{x}}_0) \tilde{\oplus} (\tilde{\bigoplus}_{i=1}^{k} (\tilde{A}^{\tilde{\otimes}k-i} \tilde{\otimes} \tilde{B} \tilde{\otimes} \tilde{u}(i))$ for $k = 1, 2, 3, \ldots$.

Hence $\tilde{y}(k) = (\tilde{C} \tilde{\otimes} \tilde{A}^{\tilde{\otimes}k} \tilde{\otimes} \tilde{\mathbf{x}}_0) \tilde{\oplus} (\tilde{\bigoplus}_{i=1}^{k} \tilde{C} \tilde{\otimes} \tilde{A}^{\tilde{\otimes}k-i} \tilde{\otimes} \tilde{B} \tilde{\otimes} \tilde{u}(i))$, and then

$$\begin{bmatrix} \tilde{y}(1) \\ \tilde{y}(2) \\ \vdots \\ \tilde{y}(p) \end{bmatrix} = \begin{bmatrix} \tilde{C} \tilde{\otimes} \tilde{A} \\ \tilde{C} \tilde{\otimes} \tilde{A}^{\tilde{\otimes}2} \\ \vdots \\ \tilde{C} \tilde{\otimes} \tilde{A}^{\tilde{\otimes}p} \end{bmatrix} \tilde{\otimes} \tilde{\mathbf{x}}_0 \tilde{\oplus}$$

$$\begin{bmatrix} \tilde{C} \tilde{\otimes} \tilde{B} & \tilde{\varepsilon} & \cdots & \tilde{\varepsilon} \\ \tilde{C} \tilde{\otimes} \tilde{A} \tilde{\otimes} \tilde{B} & \tilde{C} \tilde{\otimes} \tilde{B} & \cdots & \tilde{\varepsilon} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{C} \tilde{\otimes} \tilde{A}^{\tilde{\otimes}p-1} \tilde{\otimes} \tilde{B} & \tilde{C} \tilde{\otimes} \tilde{A}^{\tilde{\otimes}p-2} \tilde{\otimes} \tilde{B} & \cdots & \tilde{C} \tilde{\otimes} \tilde{B} \end{bmatrix} \tilde{\otimes} \begin{bmatrix} \tilde{u}(1) \\ \tilde{u}(2) \\ \vdots \\ \tilde{u}(p) \end{bmatrix}. \blacksquare$$

Let p be a positive integer, p, $\tilde{\boldsymbol{u}} = [\tilde{u}(1), \tilde{u}(2), \ldots, \tilde{u}(\mathrm{p})]^{\mathrm{T}}$ and $\tilde{\mathbf{y}} = [\tilde{y}(1), \tilde{y}(2), \ldots, \tilde{y}(\mathrm{p})]^{\mathrm{T}}$ are fuzzy vector of input sequences and fuzzy vector of output sequences, respectively. We will determine the vector of the latest times input (the greatest fuzzy input vector $\tilde{\mathbf{u}}$) such that $\tilde{K} \tilde{\otimes} \tilde{\mathbf{x}}_0 \tilde{\oplus} \tilde{H} \tilde{\otimes} \tilde{\mathbf{u}} \preceq_{\mathrm{Fm}} \tilde{\mathbf{y}}$, with $\tilde{K}$ and $\tilde{H}$ are given by Theorem 2.9.

Theorem 2.10 Given a IFMLS-SISO ($\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{\mathbf{x}}_0$) with $\tilde{C} \tilde{\otimes} \tilde{B} \neq \tilde{\mathcal{E}}$, where $(\tilde{\mathcal{E}})_{\mathrm{ij}} := \tilde{\varepsilon}$ for every i and j. If $\tilde{K} \tilde{\otimes} \tilde{\mathbf{x}}_0 \preceq_{\mathrm{Fm}} \tilde{\mathbf{y}}$, then solution of the latest times input problem for the system is given by $\hat{\tilde{\mathbf{u}}}$ whose components are $\tilde{u}_i$, *i.e.* it is a fuzzy number with the α-cut is $\hat{u}_i^{\alpha} = [\underline{\hat{u}_i^{\alpha}}, \underline{H^{\alpha}}]$. The bounds of $\hat{u}_i^{\alpha}$ are defined recursively as follows. Let

$$\underline{\hat{u}_i^{\alpha}} = \min\{-((\underline{H^{\alpha}})^{\mathrm{T}} \otimes (-\underline{\mathbf{y}^{\alpha}}))_i, -((\overline{H^{\alpha}})^{\mathrm{T}} \otimes -\overline{\mathbf{y}^{\alpha}}))_i\} \text{ and}$$

$$\overline{\hat{u}_i^{\alpha}} = -((\overline{H^{\alpha}})^{\mathrm{T}} \otimes -\overline{\mathbf{y}^{\alpha}}))_i ,$$

$$\underline{\widehat{u}_i^{\alpha}} =$$

$$\begin{cases} \min\limits_{\beta\in[0,1]} \{\overline{\hat{u}_i^{\beta}}\}, & \text{if } \min\limits_{\beta\in[0,1]} \{\overline{\hat{u}_i^{\beta}}\} \preceq_{\mathrm{m}} \underline{\hat{u}_i^{\alpha}} \\ \begin{cases} \underline{\hat{u}_i^{\beta}}, \text{if } \hat{u}_i^{\beta} \preceq_{\mathrm{m}} \hat{u}_i^{\alpha} \\ \underline{\hat{u}_i^{\alpha}}, \text{if } \hat{u}_i^{\beta} \succ_{\mathrm{m}} \hat{u}_i^{\alpha} \end{cases}, \forall \alpha, \beta \in [0,1]\ \alpha > \beta, & \text{if } \underline{\hat{u}_i^{\alpha}} \prec_{\mathrm{m}} \min\limits_{\beta\in[0,1]} \{\overline{\hat{u}_i^{\beta}}\} \end{cases},$$

$$\overline{\widehat{u}_i^{\alpha}} = \begin{cases} \overline{\hat{u}_i^{\beta}}, \text{if } \overline{\hat{u}_i^{\beta}} \preceq_{\mathrm{m}} \overline{\hat{u}_i^{\alpha}} \\ \overline{\hat{u}_i^{\alpha}}, \text{if } \overline{\hat{u}_i^{\beta}} \succ_{\mathrm{m}} \overline{\hat{u}_i^{\alpha}} \end{cases}, \forall \alpha, \beta \in [0,1] \text{ with } \alpha < \beta \cdot$$

Proof: Since $\widetilde{K} \mathbin{\widetilde{\otimes}} \widetilde{\mathbf{x}}_0 \preceq_{\mathrm{Fm}} \widetilde{\mathbf{y}}$, then $\widetilde{K} \mathbin{\widetilde{\otimes}} \widetilde{\mathbf{x}}_0 \mathbin{\widetilde{\oplus}} \widetilde{H} \mathbin{\widetilde{\otimes}} \widetilde{\mathbf{u}} = \widetilde{\mathbf{y}} \Leftrightarrow \widetilde{H} \mathbin{\widetilde{\otimes}} \widetilde{\mathbf{u}} = \widetilde{\mathbf{y}}$. Hence, the latest times input problem for FMLSI-SISO ($\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{\mathbf{x}}_0$) become the problem of determining the greatest fuzzy input vector $\widetilde{\boldsymbol{u}}$ such that $\widetilde{H} \mathbin{\widetilde{\otimes}} \widetilde{\mathbf{u}} \preceq_{\mathrm{Fm}} \widetilde{\mathbf{y}}$, that is to find the greatest fuzzy number subsolution of system $\widetilde{H} \mathbin{\widetilde{\otimes}} \widetilde{\mathbf{u}} = \widetilde{\mathbf{y}}$. Since $\widetilde{C} \mathbin{\widetilde{\otimes}} \widetilde{B} \neq \widetilde{\varepsilon}$, then the entries of each column $\widetilde{H}$ are not all equal to $\widetilde{\varepsilon}$. According to the Theorem 2.7, the greatest fuzzy number subsolution of system $\widetilde{H} \mathbin{\widetilde{\otimes}} \widetilde{\mathbf{u}} = \widetilde{\boldsymbol{y}}$ is given by $\widehat{\widetilde{\mathbf{u}}}$ whose components are $\widehat{\widetilde{u}}_i$ as given by Theorem 2.10 above. ■

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