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Yogyakarta - Indonesia, 12th - 15th July 2011

# MATHEMATICS AND ITS APPLICATIONS IN THE DEVELOPMENT OF SCIENCES AND TECHNOLOGY

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# PROCEEDINGS OF THE 6TH SOUTHEAST ASIAN MATHEMATICAL SOCIETY GADJAH MADA UNIVERSITY INTERNATIONAL CONFERENCE ON MATHEMATICS AND ITS APPLICATIONS 2011

**Yogyakarta, Indonesia, 12th – 15th July 2011**

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**FACULTY OF MATHEMATICS AND NATURAL SCIENCES**
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# PROCEEDINGS OF THE $6^{TH}$ SOUTHEAST ASIAN MATHEMATICAL SOCIETY-GADJAH MADA UNIVERSITY INTERNATIONAL CONFERENCE ON MATHEMATICS AND ITS APPLICATIONS 2011

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# PREFACE

It is an honor and great pleasure for the Department of Mathematics – Universitas Gadjah Mada, Yogyakarta – INDONESIA, to be entrusted by the Southeast Asian Mathematical Society (SEAMS) to organize an international conference every four years. Appreciation goes to those who have developed and established this tradition of the successful series of conferences. The SEAMS - Gadjah Mada University (SEAMS-GMU) 2011 International Conference on Mathematics and Its Applications took place in the Faculty of Mathematics and Natural Sciences of Universitas Gadjah Mada on July 12th – 15th, 2011. The conference was the follow up of the successful series of events which have been held in 1989, 1995, 1999, 2003 and 2007.

The conference has achieved its main purposes of promoting the exchange of ideas and presentation of recent development, particularly in the areas of pure, applied, and computational mathematics which are represented in Southeast Asian Countries. The conference has also provided a forum of researchers, developers, and practitioners to exchange ideas and to discuss future direction of research. Moreover, it has enhanced collaboration between researchers from countries in the region and those from outside.

More than 250 participants from over the world attended the conference. They come from USA, Austria, The Netherlands, Australia, Russia, South Africa, Taiwan, Iran, Singapore, The Philippines, Thailand, Malaysia, India, Pakistan, Mongolia, Saudi Arabia, Nigeria, Mexico and Indonesia. During the four days conference, there were 16 plenary lectures and 217 contributed short communication papers. The plenary lectures were delivered by Halina France-Jackson (South Africa), Jawad Y. Abuihlail (Saudi Arabia), Andreas Rauber (Austria), Svetlana Borovkova (The Netherlands), Murk J. Bottema (Australia), Ang Keng Cheng (Singapore), Peter Filzmoser (Austria), Sergey Kryzhevich (Russia), Intan Muchtadi-Alamsyah (Indonesia), Reza Pulungan (Indonesia), Salmah (Indonesia), Yudi Soeharyadi (Indonesia), Subanar (Indonesia) Supama (Indonesia), Asep K. Supriatna (Indonesia) and Indah Emilia Wijayanti (Indonesia). Most of the contributed papers were delivered by mathematicians from Asia.

We would like to sincerely thank all plenary and invited speakers who warmly accepted our invitation to come to the Conference and the paper contributors for their overwhelming response to our call for short presentations. Moreover, we are very grateful for the financial assistance and support that we received from Universitas Gadjah Mada, the Faculty of Mathematics and Natural Sciences, the Department of Mathematics, the Southeast Asian Mathematical Society, and UNESCO.

We would like also to extend our appreciation and deepest gratitude to all invited speakers, all participants, and referees for the wonderful cooperation, the great coordination, and the fascinating efforts. Appreciation and special thanks are addressed to our colleagues and staffs who help in editing process. Finally, we acknowledge and express our thanks to all friends, colleagues, and staffs of the Department of Mathematics UGM for their help and support in the preparation during the conference.

The Editors
October, 2012

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# APPLICATION OF FUZZY NUMBER MAX-PLUS ALGEBRA TO CLOSED SERIAL QUEUING NETWORK WITH FUZZY ACTIVITY TIME

M. ANDY RUDHITO, SRI WAHYUNI, ARI SUPARWANTO,
F. SUSILO

**Abstract.** The activity times in a queuing network are seldom precisely known, and then could be represented into the fuzzy number, that is called fuzzy activity times. This paper aims to determine the dynamical model of a closed serial queuing network with fuzzy activity time and its periodic properties using max-plus algebra approach. The finding shows that the dynamics of the network can be modeled as a recursive system of fuzzy number max-plus linear equations. The periodic properties of the network can be obtained from the fuzzy number max-plus eigenvalue and eigenvector of matrix in the system. In the network, for a given level of risk, it can be determined the earliest of early departure time of a customer, so that the customer's departure interval time will be in the smallest interval where the lower bound and upper bound are periodic.
*Keywords and Phrases:* max-plus algebra, queuing network, fuzzy activity times, periodic.

## 1. INTRODUCTION

We will discuss the closed serial queuing network of n single-server, with a infinite buffer capacity and *n* customers (Krivulin [4]). The network works with the principle of First-In First-Out (FIFO). In the system, the customers have to pass through the queues consecutively so as to receive service at each server. One cycle of network services is the process of entry of customers into the buffer of 1st server to leave the nth server. After completion of service to the nth server, customers return to the first queue for a new cycle of network services. Suppose at the initial time of observation, all the servers do not give service, in which the buffer of ith server contains one customer, for each i = 1, 2, ..., *n*. It is assumed that the transition of customers from a queue to the next one requires no time.

Figure 1 (Krivulin [5]) gives the initial state of the closed serial queuing network, where customers are expressed by "•".

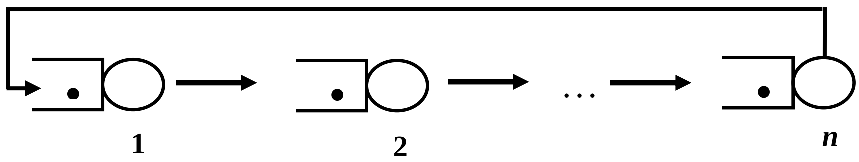


**Figure 1 Closed Serial Queuing Network**

The closed serial queuing network can be found in the assembly plant systems, such as assembling cars and electronic goods. Customers in this system are palettes while the server is a machine assembler. Palette is a kind of desk or place where the components or semi-finished goods are placed and moved to visit machines assemblers. At first, 1st pallete enters to the buffer of 1st engine and then enters to the 1st machine and the 2nd pallete enters to the buffer of 1st engine. In the 1st engine, components are placed and prepared for assembly in the next machine. Next, 1st palette enters buffer of 2nd machine and 2nd pallette enter 1st machine. And so forth for n palettes are available, so that it reaches the state as in Figure 1 above, where the initial state observation is reached. After assembly is completed in the nth machine, the assembly of goods will leave the network, while the palette will go back to the buffer of 1st engine, to begin a new cycle of network services, and so on.

Max-plus algebra (Baccelli, et al. [1]; Heidergott, B. B, et. al. [3]), namely the set of all real numbers **R** with the operations max and plus, has been used to model a closed serial queuing network algebraically, with a deterministic time activity (Krivulin [4]; Krivulin [5]). In the problem of modeling and analysis of a network sometimes its activity times is not known, for instance due to its design phase, data on time activity or distribution are not fixed. This activity can be estimated based on the experience and opinions from experts and network operators. This network activity times are modeled using fuzzy number, that is called fuzzy activity times. Scheduling problems involving fuzzy number can be seen in Chanas and Zielinski [2], and Soltoni and Haji [9]. As for the issue network model involving fuzzy number can be seen in Lüthi and Haring [6].

In this paper we determine the dynamical model of a closed serial queuing network with fuzzy activity time and its periodic properties using max-plus algebra approach. This approach will use some concepts such as: fuzzy number max-plus algebra, fuzzy number max-plus eigenvalue and eigenvector (Rudhito [8]). We will discuss a closed serial queuing network as discussed in Krivulin [4] and Krivulin [5], where crisp activity time will be replaced with fuzzy activity time, where can be modeled by fuzzy number. The dynamical model of the network can be obtained analogous with crisp activity time case. The periodic properties of the network can be obtained from the fuzzy number max-plus eigenvalue and eigenvector of matrix in the system. We will use some concepts and result on max-plus algebra, interval max-plus algebra and fuzzy number max-plus algebra.

## 2. MAX-PLUS ALGEBRA

In this section we will review some concepts and results of max-plus algebra, matrix over max-plus algebra and max-plus eigenvalue. Further details can be found in Baccelli, et al. [1].

Let $\mathbf{R}_\varepsilon := \mathbf{R} \cup \{\varepsilon\}$ with $\mathbf{R}$ is the set of all real numbers and $\varepsilon := -\infty$. Define two operations $\oplus$ and $\otimes$ such that

$$a \oplus b := \max(a, b) \text{ and } a \otimes b := a + b$$

for every $a, b \in \mathbf{R}_\varepsilon$.

We can show that $(\mathbf{R}_\varepsilon, \oplus, \otimes)$ is a commutative idempotent semiring with neutral element $\varepsilon = -\infty$ and unity element $e = 0$. Moreover, $(\mathbf{R}_\varepsilon, \oplus, \otimes)$ is a semifield, that is $(\mathbf{R}_\varepsilon, \oplus, \otimes)$ is a commutative semiring, where for every $a \in \mathbf{R}$ there exists $-a$ such that $a \otimes (-a) = 0$. Then, $(\mathbf{R}_\varepsilon, \oplus, \otimes)$ is called *max-plus algebra*, and is written as $\mathbf{R}_{\max}$. The relation "$\preceq_m$" defined on $\mathbf{R}_{\max}$ as $x \preceq_m y$ iff $x \oplus y = y$. In $\mathbf{R}_{\max}$, operations $\oplus$ and $\otimes$ are *consistent* with respect to the order $\preceq_m$, that is for every $a, b, c \in \mathbf{R}_{\max}$, if $a \preceq_m b$, then $a \oplus c \preceq_m b \oplus c$, and $a \otimes c \preceq_m b \otimes c$. Define $x^{\otimes 0} := 0$, $x^{\otimes k} := x \otimes x^{\otimes k-1}$ and $\varepsilon^{\otimes k} := \varepsilon$, for $k = 1, 2, \ldots$ .

Define $\mathbf{R}_{\max}^{m\times n} := \{A = (A_{ij}) \mid A_{ij} \in \mathbf{R}_{\max}, i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n\}$, that is set of all matrices over max-plus algebra. Specifically, for $A, B \in \mathbf{R}_{\max}^{n\times n}$ and $\alpha \in \mathbf{R}_{\max}$ we define

$$(\alpha \otimes A)_{ij} = \alpha \otimes A_{ij},\ (A \oplus B)_{ij} = A_{ij} \oplus B_{ij} \text{ and } (A \otimes B)_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}.$$

We define matrix $E \in \mathbf{R}_{\max}^{n\times n}$, $(E)_{ij} := \begin{cases} 0 & \text{if } i = j \\ \varepsilon & \text{if } i \neq j \end{cases}$ and matrix $\varepsilon \in \mathbf{R}_{\max}^{n\times n}$, $(\varepsilon)_{ij} := \varepsilon$ for every $i$ and $j$. For any matrix $A \in \mathbf{R}_{\max}^{n\times n}$, one can define $A^{\otimes 0} = E_n$ and $A^{\otimes k} = A \otimes A^{\otimes k-1}$ for $k = 1, 2, \ldots$ . The relation "$\preceq_m$" defined on $\mathbf{R}_{\max}^{m\times n}$ as $A \preceq_m B$ iff $A \oplus B = B$. In $(\mathbf{R}_{\max}^{n\times n}, \oplus, \otimes)$, operations $\oplus$ and $\otimes$ are consistent with respect to the order $\preceq_m$, that is for every $A, B, C \in \mathbf{R}_{\max}^{n\times n}$, if $A \preceq_m B$, then $A \oplus C \preceq_m B \oplus C$, and $A \otimes C \preceq_m B \otimes C$.

Define $\mathbf{R}_{\max}^{n} := \{\mathbf{x} = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \mid x_i \in \mathbf{R}_{\max}, i = 1, 2, \ldots, n\}$. Note that $\mathbf{R}_{\max}^{n}$ can be viewed as $\mathbf{R}_{\max}^{n\times 1}$. The elements of $\mathbf{R}_{\max}^{n}$ are called *vectors* over $\mathbf{R}_{\max}$ or shortly *vectors*. A vector $\mathbf{x} \in \mathbf{R}_{max}^{n}$ is said to be *not equal* to vector $\boldsymbol{\varepsilon}$, and is written as $\mathbf{x} \neq \boldsymbol{\varepsilon}$, if there exists $i \in \{1, 2, ..., n\}$ such that $x_i \neq \varepsilon$.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ with $\mathcal{V} = \{1, 2, \ldots, p\}$ is non empty finite set which is its elements is called *node* and $\mathcal{A}$ is a set of ordered pairs of nodes. A directed graph $\mathcal{G}$ is said to be *weighted* if every arch $(j, i) \in \mathcal{A}$ corresponds to a real number $A_{ij}$. The real number $A_{ij}$ is called the *weight* of arch $(j, i)$, and is written as $w(j, i)$. In pictorial representation of weighted directed graph, archs are labelled by its weight. Define a *precedence graph* of a matrix $A \in \mathbf{R}_{max}^{n\times n}$ as weighted directed graph $\mathcal{G}(A) = (\mathcal{V}, \mathcal{A})$ with $\mathcal{V} = \{1, 2, \ldots, n\}$, $\mathcal{A} = \{(j, i) \mid w(i, j) = A_{ij} \neq \varepsilon\}$.

Conversely, for every weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$, can be defined a matrix $A \in \mathbf{R}_{max}^{n\times n}$, which is called the *weighting matrix* of graph $\mathcal{G}$, where

$A_{ij} = \begin{cases} w(j,i) & \text{if } (j,i) \in \mathcal{A} \\ \varepsilon & \text{if } (j,i) \notin \mathcal{A}. \end{cases}$ The mean weight of a path is defined as the sum of the weights of the individual arcs of this path, divided by the length of this path. If such a path is a circuit one talks about the mean weight of the circuit, or simply the cycle mean. It follow that a formula for *maximum mean cycle mean* $\lambda_{\max}(A)$ in $\mathcal{G}(A)$ is $\lambda_{\max}(A) = \bigoplus_{k=1}^{n} (\frac{1}{k} \bigoplus_{i=1}^{n} (A^{\otimes k})_{ii})$..

The matrix $A \in \mathbf{R}_{\max}^{n\times n}$ is said to be *irreducible* if its precedence graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ is strongly connected, that is for every $i, j \in \mathcal{V}$, $i \neq j$, there is a path from $i$ to $j$. We can show that matrix $A \in \mathbf{R}_{\max}^{n\times n}$ is irreducible if and only if $(A \oplus A^{\otimes 2} \oplus ... \oplus A^{\otimes n-1})_{ij} \neq \varepsilon$ for every $i, j$ where $i \neq j$ (Schutter, 1996).

Given $A \in \mathbf{R}_{\max}^{n\times n}$. Scalar $\lambda \in \mathbf{R}_{\max}$ is called the *max-plus eigenvalue of matrix A* if there exists a vector $\boldsymbol{v} \in \mathbf{R}_{\max}^{n}$ with $\boldsymbol{v} \neq \boldsymbol{\varepsilon}_{n\times 1}$ such that $A \otimes \boldsymbol{v} = \lambda \otimes \boldsymbol{v}$. Vector $\boldsymbol{v}$ is called *max-plus eigenvector of matrix A associated with* $\lambda$. We can show that $\lambda_{\max}(A)$ is a max-plus eigenvalue of matrix $A$. For matrix $B = -\lambda_{\max}(A) \otimes A$, if $B_{ii}^{+} = 0$, then $i$-th column of matrix $B^{*}$ is an eigenvector corresponding with eigenvalue $\lambda_{\max}(A)$. The eigenvector is called *fundamental max-plus eigenvector* associated with eigenvalues $\lambda_{\max}(A)$ (Bacelli, et al., 2001). A linear combination of fundamental max-plus eigenvector of matrix $A$ is also an eigenvector assosiated with $\lambda_{\max}(A)$. We can show that if matrix $A \in \mathbf{R}_{\max}^{n\times n}$ is irreducible, then $\lambda_{\max}(A)$ is the unique max-plus egenvalue of $A$ and the max-plus eigenvector associated with $\lambda_{\max}(A)$ is $\boldsymbol{v}$, where $v_i \neq \varepsilon$ for every $i \in \{1, 2, ..., n\}$ (Bacelli, et al., 2001).

## 3. INTERVAL MAX-PLUS ALGEBRA

In this section we will review some concepts and results of interval max-plus algebra, matrix over interval max-plus algebra and interval max-plus eigenvalue. Further details can be found in Rudhito, et al. [7] and Rudhito [8].

The (closed) max-plus interval x in $\mathbf{R}_{\max}$ is a subset of $\mathbf{R}_{\max}$ of the form

$$\mathrm{x} = [\underline{x}, \overline{x}] = \{x \in \mathbf{R}_{\max} \mid \underline{x} \preceq_m x \preceq_m \overline{x}\},$$

which is shortly called *interval*. The interval $\mathrm{x} \subseteq \mathrm{y}$ if and only if $\underline{y} \preceq_m \underline{x} \preceq_m \overline{x} \preceq_m \overline{y}$. Especially x = y if and only if $\underline{x} = \underline{y}$ and $\overline{x} = \overline{y}$. The number $x \in \mathbf{R}_{\max}$ can be represented as interval $[x, x]$. Define $\mathbf{I}(\mathbf{R})_\varepsilon := \{\mathrm{x} = [\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \mathbf{R}, \varepsilon \prec_m \underline{x} \preceq_m \overline{x}\} \cup \{\varepsilon\}$, where $\varepsilon := [\varepsilon, \varepsilon]$. Define $\mathrm{x} \overline{\oplus} \mathrm{y} = [\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}]$ and $\mathrm{x} \overline{\otimes} \mathrm{y} = [\underline{x} \otimes \underline{y}, \overline{x} \otimes \overline{y}]$ for every x, y $\in \mathbf{I}(\mathbf{R})_\varepsilon$. We can show that $(\mathbf{I}(\mathbf{R})_\varepsilon, \overline{\oplus}, \overline{\otimes})$ is a commutative idempotent semiring with neutral element $\varepsilon = [\varepsilon, \varepsilon]$ and unity element $0 = [0, 0]$. This commutative idempotent semiring

$(\mathbf{I}(\mathbf{R})_\varepsilon, \overline{\oplus}, \overline{\otimes})$ is called *interval max-plus algebra*, and is written as $\mathbf{I}(\mathbf{R})_{\max}$. Relation "$\preceq_{\mathrm{Im}}$"defined on $\mathbf{I}(\mathbf{R})_{\max}$ as $x \preceq_{\mathrm{Im}} y \Leftrightarrow x \overline{\oplus} y = y$ is a partial order on $\mathbf{I}(\mathbf{R})_{\max}$. Notice that $x \overline{\oplus} y = y \Leftrightarrow \underline{x} \preceq_{\mathrm{m}} \underline{y}$ and $\overline{x} \preceq_{\mathrm{m}} \overline{y}$.

Define $\mathbf{I}(\mathbf{R})_{\max}^{m\times n} := \{A = (A_{ij}) \mid A_{ij} \in \mathbf{I}(\mathbf{R})_{\max},\ i = 1, 2, ..., m,\ j = 1, 2, ..., n\}$. The elements of $\mathbf{I}(\mathbf{R})_{\max}^{m\times n}$ are called *matrices over interval max-plus algebra* or shortly *interval matrices*. The operations on interval matrices can be defined in the same way with the operations on matrices over max-plus algebra. For any matrix $A \in \mathbf{I}(\mathbf{R})_{\max}^{m\times n}$, Define the matrix $\underline{A} = (\underline{A_{ij}}) \in \mathbf{R}_{\max}^{m\times n}$ and $\overline{A} = (\overline{A_{ij}}) \in \mathbf{R}_{\max}^{m\times n}$, which are called *lower bound matrix* and *upper bound matrix of* A, respectively. Define a *matrix interval of* A, that is $[\underline{A}, \overline{A}] = \{A \in \mathbf{R}_{\max}^{m\times n} \mid \underline{A} \preceq_{\mathrm{m}} A \preceq_{\mathrm{m}} \overline{A}\}$ and $\mathbf{I}(\mathbf{R}_{\max}^{m\times n})_{\mathrm{b}} = \{[\underline{A}, \overline{A}] \mid A \in (\mathbf{R})_{\max}^{n\times n}\}$. The matrix interval $[\underline{A}, \overline{A}]$ and $[\underline{B}, \overline{B}] \in \mathbf{I}(\mathbf{R}_{\max}^{m\times n})_{\mathrm{b}}$ are *equal* if $\underline{A} = \underline{B}$ and $\overline{A} = \overline{B}$. We can show that for every matrix interval $A \in \mathbf{I}(\mathbf{R}_{\max}^{m\times n})$ we can determine matrix interval $[\underline{A}, \overline{A}] \in \mathbf{I}(\mathbf{R}_{\max}^{m\times n})_{\mathrm{b}}$ and conversely. The matrix interval $[\underline{A}, \overline{A}]$ is called *matrix interval associated with the interval matrix* A, and is written as $A \approx [\underline{A}, \overline{A}]$. Moreover, we have $\alpha \overline{\otimes} A \approx [\underline{\alpha} \otimes \underline{A}, \overline{\alpha} \otimes \overline{A}]$, $A \overline{\oplus} B \approx [\underline{A} \oplus \underline{B}, \overline{A} \oplus \overline{B}]$ and $A \overline{\otimes} B \approx [\underline{A} \otimes \underline{B}, \overline{A} \otimes \overline{B}]$.

Define $\mathbf{I}(\mathbf{R})_{\max}^{n} := \{\mathbf{x} = [x_1, x_2, ..., x_n]^{\mathrm{T}} \mid x_i \in \mathbf{I}(\mathbf{R})_{\max},\ i = 1, 2, ..., n\}$. Note that $\mathbf{I}(\mathbf{R})_{\max}^{n}$ can be viewed as $\mathbf{I}(\mathbf{R})_{\max}^{n\times 1}$. The elements of $\mathbf{I}(\mathbf{R})_{\max}^{n}$ are called *interval vectors over* $\mathbf{I}(\mathbf{R})_{\max}$ or shortly *interval vectors*. An interval vector $\mathbf{x} \in \mathbf{I}(\mathbf{R})_{\max}^{n}$ is said to be *not equal* to interval vector $\boldsymbol{\varepsilon}$, and is written as $\mathbf{x} \neq \boldsymbol{\varepsilon}$, if there exists $i \in \{1, 2, ..., n\}$ such that $x_i \neq \varepsilon$.

Interval matrix $A \in \mathbf{I}(\mathbf{R})_{\max}^{n\times n}$, where $A \approx [\underline{A}, \overline{A}]$, is said to be *irreducible* if every matrix $A \in [\underline{A}, \overline{A}]$ is irreducible. We can show that interval matrix $A \in \mathbf{I}(\mathbf{R})_{\max}^{n\times n}$, where $A \approx [\underline{A}, \overline{A}]$ is irreducible if and only if $\underline{A} \in \mathbf{R}_{\max}^{n\times n}$ is irreducible (Rudhito, et al. [7]).

## 4. FUZZY NUMBER MAX-PLUS ALGEBRA

In this section we will review some concepts and results of fuzzy number max-plus algebra, matrix over fuzzy number max-plus algebra and fuzzy number max-plus eigenvalue. Further details can be found in Rudhito [8].

Fuzzy set $\tilde{K}$ in universal set $X$ is represented as the set of ordered pairs $\tilde{K} = \{(x, \mu_{\tilde{K}}(x)) \mid x \in X\}$ where $\mu_{\tilde{K}}$ is a membership function of fuzzy set $\tilde{K}$, which is a mapping from universal set $X$ to closed interval [0, 1]. *Support* of a fuzzy set $\tilde{K}$ is *supp*($\tilde{K}$) = $\{x \in X \mid \mu_{\tilde{K}}(x) > 0\}$. *Height of a* fuzzy set $\tilde{K}$ is *height*($\tilde{K}$) = $\sup_{x \in X} \{\mu_{\tilde{K}}(x)\}$. A fuzzy set $\tilde{K}$ is said to be *normal* if *height*($\tilde{K}$) = 1. For a number $\alpha \in [0, 1]$, *α-cut* of a fuzzy set $\tilde{K}$ is cut$^{\alpha}$($\tilde{K}$) = $K^{\alpha}$ = $\{x \in X \mid \mu_{\tilde{K}}(x) \geq \alpha\}$. A fuzzy sets $\tilde{K}$ is said to be *convex* if $K^{\alpha}$ is convex, that is contains line segment between any two points in the $K^{\alpha}$, for every $\alpha \in [0, 1]$,

*Fuzzy number* $\tilde{a}$ is defined as a fuzzy set in universal set **R** which satisfies the following properties: *i* ) normal, that is $a^1 \neq \emptyset$, *ii* ) for every $\alpha \in (0, 1]$ $a^{\alpha}$ is closed in **R**, that is there exists $\underline{a}^{\alpha}$, $\overline{a}^{\alpha} \in \mathbf{R}$ with $\underline{a}^{\alpha} \leq \overline{a}^{\alpha}$ such that $a^{\alpha} = [\underline{a}^{\alpha}, \overline{a}^{\alpha}] = \{x \in \mathbf{R} \mid \underline{a}^{\alpha} \leq x \leq \overline{a}^{\alpha}\}$, *iii*) *supp*($\tilde{a}$) is bounded. For $\alpha = 0$, define $a^0$ = [inf(*supp*($\tilde{a}$)), sup(*supp*($\tilde{a}$))]. Since every closed interval in **R** is convex, $a^{\alpha}$ is convex for every $\alpha \in [0, 1]$, hence $\tilde{a}$ is convex.

Let $\mathbf{F}(\mathbf{R})_{\tilde{\varepsilon}} := \mathbf{F}(\mathbf{R}) \cup \{\tilde{\varepsilon}\}$, where **F**(**R**) is set of all fuzzy numbers and $\tilde{\varepsilon} := \{-\infty\}$ with the $\alpha$-cut of $\tilde{\varepsilon}$ is $\varepsilon^{\alpha} = [-\infty, -\infty]$. Define two operations $\tilde{\oplus}$ and $\tilde{\otimes}$ such that for every $\tilde{a}, \tilde{b} \in \mathbf{F}(\mathbf{R})_{\tilde{\varepsilon}}$, with $a^{\alpha} = [\underline{a}^{\alpha}, \overline{a}^{\alpha}] \in \mathbf{I}(\mathbf{R})_{\max}$ and $b^{\alpha} = [\underline{b}^{\alpha}, \overline{b}^{\alpha}] \in \mathbf{I}(\mathbf{R})_{\max}$,

*i)* Maximum of $\tilde{a}$ and $\tilde{b}$, written $\tilde{a} \,\tilde{\oplus}\, \tilde{b}$, is a fuzzy number whose $\alpha$-cut is interval $[\underline{a}^{\alpha} \oplus \underline{b}^{\alpha}, \overline{a}^{\alpha} \oplus \overline{b}^{\alpha}]$ for every $\alpha \in [0, 1]$

*ii)* Addition of $\tilde{a}$ and $\tilde{b}$, written $\tilde{a} \,\tilde{\otimes}\, \tilde{b}$, is a fuzzy number whose $\alpha$-cut is interval $[\underline{a}^{\alpha} \otimes \underline{b}^{\alpha}, \overline{a}^{\alpha} \otimes \overline{b}^{\alpha}]$ for every $\alpha \in [0, 1]$.

We can show that $(\mathbf{F}(\mathbf{R})_{\tilde{\varepsilon}}, \tilde{\oplus}, \tilde{\otimes})$ is a commutative idempotent semiring. The commutative idempotent semiring $\mathbf{F}(\mathbf{R})_{\max} := (\mathbf{F}(\mathbf{R})_{\tilde{\varepsilon}}, \tilde{\oplus}, \tilde{\otimes})$ is called *fuzzy number max-plus algebra*, and is written as $\mathbf{F}(\mathbf{R})_{\max}$ (Rudhito, et al. [8]).

Define $\mathbf{F}(\mathbf{R})_{\max}^{m \times n} := \{\tilde{A} = (\tilde{A}_{ij}) \mid \tilde{A}_{ij} \in \mathbf{F}(\mathbf{R})_{\max}, i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n\}$. The elements of $\mathbf{F}(\mathbf{R})_{\max}^{m \times n}$ are called *matrices over fuzzy number max-plus algebra* or shortly *fuzzy number matrices***.** The operations on fuzzy number matrices can be defined in the same way with the operations on matrices over max-plus algebra. Define matrix $\tilde{E} \in \mathbf{F}(\mathbf{R})_{\max}^{n \times n}$, with $(\tilde{E})_{ij} := \begin{cases} \tilde{0} & \text{if } i = j \\ \tilde{\varepsilon} & \text{if } i \neq j \end{cases}$, and matrix $\tilde{\mathcal{E}} \in \mathbf{F}(\mathbf{R})_{\max}^{n \times n}$, with $(\tilde{\mathcal{E}})_{ij} := \tilde{\varepsilon}$ for every $i$ and $j$.

For every $\tilde{A} \in \mathbf{F}(\mathbf{R})_{\max}^{m \times n}$ and $\alpha \in [0, 1]$, define *α-cut matrix of* $\tilde{A}$ as the interval matrix $A^{\alpha} = (A_{ij}^{\alpha})$, with $A_{ij}^{\alpha}$ is the α-cut of $\tilde{A}_{ij}$ for every $i$ and $j$. Define matrix $\underline{A}^{\alpha} = (\underline{A}_{ij}^{\alpha}) \in \mathbf{R}_{\max}^{m \times n}$ and $\overline{A}^{\alpha} = (\overline{A}_{ij}^{\alpha}) \in \mathbf{R}_{\max}^{m \times n}$ which are called *lower bound* and *upper bound of matrix* $A^{\alpha}$*, respectively*. We can conclude that the matrices $\tilde{A}$, $\tilde{B} \in \mathbf{F}(\mathbf{R})_{\max}^{m \times n}$ are *equal* iff $A^{\alpha} = B^{\alpha}$, that is $A_{ij}^{\alpha} = B_{ij}^{\alpha}$ for every $\alpha \in [0, 1]$ and for every $i$ and $j$. For every fuzzy number matrix $\tilde{A}$, $A^{\alpha} \approx [\underline{A}^{\alpha}, \overline{A}^{\alpha}]$. Let $\tilde{\lambda} \in \mathbf{F(R)}_{\max}$, $\tilde{A}$, $\tilde{B} \in \mathbf{F(R)}_{\max}^{m\times n}$. We can show that $\lambda \otimes A)^{\alpha} \approx [\underline{\lambda}^{\alpha} \otimes \underline{A}^{\alpha}, \overline{\lambda}^{\alpha} \otimes \overline{A}^{\alpha}]$ and $(A \oplus B)^{\alpha} \approx [\underline{A}^{\alpha} \oplus \underline{B}^{\alpha}, \overline{A}^{\alpha} \oplus \overline{B}^{\alpha}]$ for every $\alpha \in [0, 1]$. Let $\tilde{A} \in \mathbf{F(R)}_{\max}^{m\times p}$, $\tilde{A} \in \mathbf{F(R)}_{\max}^{p\times n}$. We can show that $(A \otimes B)^{\alpha} \approx [\underline{A}^{\alpha} \otimes \underline{B}^{\alpha}, \overline{A}^{\alpha} \otimes \overline{B}^{\alpha}]$ for every $\alpha \in [0, 1]$.

Define $\mathbf{F(R)}_{\max}^{n} := \{ \tilde{\mathbf{x}} = [\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n]^{\mathrm{T}} \mid \tilde{x}_i \in \mathbf{F(R)}_{\max}, i = 1, \ldots, n \}$. The elements in $\mathbf{F(R)}_{\max}^{n}$ are called *fuzzy number vectors over* $\mathbf{F(R)}_{\max}$ or shortly *fuzzy number vectors*. A fuzzy number vector $\tilde{\mathbf{x}} \in \mathbf{F(R)}_{\max}^{n}$ is said to be *not equal* to fuzzy number vector $\tilde{\boldsymbol{\varepsilon}}$, written $\tilde{\mathbf{x}} \neq \tilde{\boldsymbol{\varepsilon}}$, if there exists $i \in \{1, 2, ..., n\}$ such that $\tilde{x}_i \neq \tilde{\varepsilon}$.

Fuzzy number matrix $\tilde{A} \in \mathbf{F(R)}_{\max}^{n\times n}$ is said to be *irreducible* if $A^{\alpha} \in \mathbf{I(R)}_{\max}^{n\times n}$ is irreducible for every $\alpha \in [0, 1]$. We can show that $\tilde{A}$ is irreducible if and only if $\underline{A}^{0} \in \mathbf{R}_{\max}^{n\times n}$ is irreducible (Rudhito, et al. [7]).

Let $\tilde{A} \in \mathbf{F(R)}_{\max}^{n\times n}$. The fuzzy number scalar $\tilde{\lambda} \in \mathbf{F(R)}_{\max}$ is called *fuzzy number max-plus eigenvalue of matrix* $\tilde{A}$ if there exists a fuzzy number vector $\tilde{\mathbf{v}} \in \mathbf{F(R)}_{\max}^{n}$ with $\tilde{\mathbf{v}} \neq \tilde{\boldsymbol{\varepsilon}}_{n\times 1}$ such that $\tilde{A} \,\tilde{\otimes}\, \tilde{\mathbf{v}} = \tilde{\lambda} \,\tilde{\otimes}\, \tilde{\mathbf{v}}$. The vector $\tilde{\mathbf{v}}$ is called *fuzzy number max-plus eigenvectors of matrix* $\tilde{A}$ *associaed with* $\tilde{\lambda}$. Given $\tilde{A} \in \mathrm{F(R)}_{\max}^{n\times n}$. We can show that the fuzzy number scalar $\tilde{\lambda}_{\max}(\tilde{A}) = \bigcup_{\alpha\in[0,1]} \tilde{\lambda}_{\max}^{\alpha}$, where $\tilde{\lambda}_{\max}^{\alpha}$ is a fuzzy set in R with membership function $\mu_{\tilde{\lambda}_{\max}^{\alpha}}(x) = \alpha\, \chi_{\tilde{\lambda}_{\max}^{\alpha}}(x)$, and $\chi_{\tilde{\lambda}_{\max}^{\alpha}}$ is the characteristic function of the set $[\lambda_{\max}(\underline{A}^{\alpha}), \lambda_{\max}(\overline{A}^{\alpha})]$, is a fuzzy number max-plus eigenvalues of matrix $\tilde{A}$. Based on fundamental max-plus eigenvector associated with eigenvalues $\lambda_{\max}(\underline{A}^{\alpha})$ and $\lambda_{\max}(\overline{A}^{\alpha})$, we can find *fundamental fuzzy number max-plus eigenvector* associated with $\tilde{\lambda}_{\max}^{\alpha}$ (Rudhito [8]). Moreover, if matrix $\tilde{A}$ is irreducible, then $\tilde{\lambda}_{\max}(\tilde{A})$ is the unique fuzzy number max-plus eigenvalue of matrix $\tilde{A}$ and the fuzzy number max-plus eigenvector associated with $\lambda_{\max}(\tilde{A})$ is $\tilde{\mathbf{v}}$, where $\tilde{v}_i \neq \tilde{\varepsilon}$ for every $i \in \{1, 2, ..., n\}$.

## 5. DYNAMICAL MODEL OF A CLOSED SERIAL QUEUING NETWORK WITH FUZZY ACTIVITIY TIME

We discuss the closed serial queuing network of $n$ single-server, with a infinite buffer capacity and $n$ customers, as in Figure 1.

Let $\tilde{a}_i(k)$ = fuzzy arrival time of $k$th customer at server $i$,

$\tilde{d}_i(k)$ = fuzzy departure time of $k$th customer at server $i$,

$\tilde{t}_i$ = fuzzy service time of $k$th customer at server $i$.

for $k$ = 1, 2, ... and $i$ = 1, 2, ..., $n$.

The dynamical of queuing at server $i$ can be written as

$$\tilde{d}_i(k) = \max(\tilde{t}_i + \tilde{a}_i(k),\ \tilde{t}_i + \tilde{d}_i(k-1)) \tag{1}$$

$$\tilde{a}_i(k) = \begin{cases} \tilde{d}_n(k-1) & \text{if } i = 1 \\ \tilde{d}_{i-1}(k-1) & \text{if } i = 2,\ldots,n \end{cases}. \tag{2}$$

Using fuzzy number max-plus algebra notation, equation (1) can be written as

$$\tilde{d}_i(k) = (\tilde{t}_i \,\tilde{\otimes}\, \tilde{a}_i(k)) \,\tilde{\oplus}\, (\tilde{t}_i \,\overline{\otimes}\, \tilde{d}_i(k-1)). \tag{3}$$

Let $\tilde{\boldsymbol{d}}(k)$ = $[\tilde{d}_1(k), \tilde{d}_2(k), \ldots, \tilde{d}_n(k)]^{\mathrm{T}}$, $\tilde{\boldsymbol{a}}(k) = [\tilde{a}_1(k), \tilde{a}_2(k), \ldots, \tilde{a}_n(k)]^{\mathrm{T}}$ and $\tilde{T}$ = $\begin{bmatrix} \tilde{t}_1 & & \tilde{\varepsilon} \\ & \ddots & \\ \tilde{\varepsilon} & & \tilde{t}_n \end{bmatrix}$, then equations (3) and (2) can be written as

$$\tilde{\boldsymbol{d}}(k) = (\tilde{T} \,\tilde{\otimes}\, \tilde{\boldsymbol{a}}(k)) \,\tilde{\oplus}\, (\tilde{T} \,\tilde{\otimes}\, \tilde{\boldsymbol{d}}(k-1). \tag{4}$$

$$\tilde{\boldsymbol{a}}(k) = \tilde{G} \,\tilde{\otimes}\, \tilde{\boldsymbol{d}}(k-1), \tag{5}$$

with matrix $\tilde{G}$ = $\begin{bmatrix} \tilde{\varepsilon} & \cdots & \tilde{\varepsilon} & \tilde{0} \\ \tilde{0} & \ddots & \tilde{\varepsilon} & \tilde{\varepsilon} \\ & \ddots & \ddots & \vdots \\ \tilde{\varepsilon} & & \tilde{0} & \tilde{\varepsilon} \end{bmatrix}$.

Substituting equation (5) to the equation (4), can be obtained the equation

$$\begin{aligned} \tilde{\boldsymbol{d}}(k) &= \tilde{T} \,\tilde{\otimes}\, \tilde{G} \,\tilde{\otimes}\, \tilde{\boldsymbol{d}}(k-1) \,\tilde{\oplus}\, \tilde{T} \,\tilde{\otimes}\, \tilde{\boldsymbol{d}}(k-1) \\ &= \tilde{T} \,\tilde{\otimes}(\tilde{G} \,\tilde{\oplus}\, \tilde{E}) \,\tilde{\otimes}\, \tilde{\boldsymbol{d}}(k-1) \\ &= \tilde{A} \,\tilde{\otimes}\, \tilde{\boldsymbol{d}}(k-1) \end{aligned} \tag{6}$$

with fuzzy number matrix $\tilde{A} = \tilde{T} \,\tilde{\otimes}\, (\tilde{G} \,\tilde{\oplus}\, \tilde{E}) = \begin{bmatrix} \tilde{t}_1 & \tilde{\varepsilon} & \cdots & \tilde{\varepsilon} & \tilde{t}_1 \\ \tilde{t}_2 & \tilde{t}_2 & \tilde{\varepsilon} & \cdots & \tilde{\varepsilon} \\ \tilde{\varepsilon} & \ddots & \ddots & & \vdots \\ \vdots & & \tilde{t}_{n-1} & \tilde{t}_{n-1} & \tilde{\varepsilon} \\ \tilde{\varepsilon} & \cdots & \tilde{\varepsilon} & \tilde{t}_n & \tilde{t}_n \end{bmatrix}$.

The equation (6) is dynamical model of the closed serial queuing network.

## 6. PERIODIC PROPERTIES OF A CLOSED SERIAL QUEUING NETWORK WITH FUZZY ACTIVITIY TIME

Dynamical model recursive equation of the closed serial queuing network (6) can be represented through the early departure time of customer $\tilde{\boldsymbol{d}}(0)$, with its $\alpha$-cut $\boldsymbol{d}^{\alpha}(0) \approx [\underline{\boldsymbol{d}}^{\alpha}(0), \overline{\boldsymbol{d}}^{\alpha}(0)]$ for every $\alpha \in [0, 1]$. For every $\alpha \in [0, 1]$ hold $\boldsymbol{d}^{\alpha}(k) = A^{\alpha} \overline{\otimes}\ \boldsymbol{d}^{\alpha}(k-1) \approx [\underline{A}^{\alpha} \otimes \underline{\boldsymbol{d}}^{\alpha}(k-1), \overline{A^{\alpha}} \otimes \overline{\boldsymbol{d}}^{\alpha}(k-1)] = [(\underline{A}^{\alpha})^{\otimes k} \otimes \underline{\boldsymbol{d}}^{\alpha}(0), (\overline{A^{\alpha}})^{\otimes k} \otimes \overline{\boldsymbol{d}}^{\alpha}(0)] \approx (\mathrm{A}^{\alpha})^{\overline{\otimes} k}\ \overline{\otimes}$ $\mathbf{d}^{\alpha}(0)$. Thus, for every $\alpha \in$ [0, 1] hold $\boldsymbol{d}^{\alpha}(k) = (\mathrm{A}^{\alpha})^{\overline{\otimes} k}\ \overline{\otimes}\ \mathbf{d}^{\alpha}(0)$. Hence we have $\tilde{\boldsymbol{d}}(k) = \tilde{A}^{\tilde{\otimes} k}\ \tilde{\otimes}\ \tilde{\boldsymbol{d}}(0)$. Since the early departure time of customer can be determined exactly, it is a crisp time, that is a point fuzzy number $\tilde{\boldsymbol{d}}(0)$, with $\boldsymbol{d}^{\alpha}(0) \approx [\underline{\boldsymbol{d}}^{\alpha}(0), \overline{\boldsymbol{d}}^{\alpha}(0)]$ where $\underline{\boldsymbol{d}}^{\alpha}(0) = \overline{\boldsymbol{d}}^{\alpha}(0)$ for every $\alpha \in [0, 1]$.

Since precedence of matrix $\underline{A}^{0}$ in the model of the closed serial queuing network (Figure 1) is strongly connected, the matrix $\underline{A}^{0}$ is irredusible. Hence, matrix $\tilde{A}$ in the equation (6) is irredusibel. Thus, matrix $\tilde{A}$ has unique fuzzy number max-plus eigenvalue, that is $\tilde{\lambda}_{\max}(\tilde{A})$ where $\tilde{\boldsymbol{v}}$ is the fundamental fuzzy number max-plus eigenvector associated with $\tilde{\lambda}_{\max}(\tilde{A})$, where $\tilde{v}_i \neq \tilde{\varepsilon}$ for every $i \in \{1, 2, ..., n\}$.

We construct fuzzy number vector $\tilde{\boldsymbol{v}}^{*}$ where its $\alpha$-cut vector is $\boldsymbol{v}^{*\alpha} \approx [\underline{\boldsymbol{v}}^{*\alpha}, \overline{\boldsymbol{v}}^{*\alpha}]$, using the following steps. For every $\alpha \in [0, 1]$ dan $i = 1, 2, ..., n$, form

1. $\underline{\boldsymbol{v}'}^{\alpha} = \delta_1 \otimes \underline{\boldsymbol{v}}^{\alpha}$, $\overline{\boldsymbol{v}'^{\alpha}} = \delta_1 \otimes \overline{\boldsymbol{v}^{\alpha}}$, with $\delta_1 = -\min_i(\underline{v}^{0}_{i})$.
2. $\underline{\boldsymbol{v}''}^{\alpha} = \delta_2(\alpha) \otimes \underline{\boldsymbol{v}'}^{\alpha}$, $\overline{\boldsymbol{v}''^{\alpha}} = \delta_2(\alpha) \otimes \overline{\boldsymbol{v}'^{\alpha}}$, with $\delta_2(\alpha) = -\min_i(\underline{v'}^{\alpha}_{i} - \underline{v'}^{0}_{i})$.
3. $\overline{\boldsymbol{v}'''^{\alpha}} = \delta_3 \otimes \overline{\boldsymbol{v}''^{\alpha}}$, with $\delta_3 = -\min_i(\underline{v''}^{0}_{i} - \overline{v''^{0}}_{i})$.
4. $\underline{\boldsymbol{v}}^{*\alpha} = \underline{\boldsymbol{v}''}^{\alpha}$, $\overline{\boldsymbol{v}}^{*\alpha} = \delta_4(\alpha) \otimes \overline{\boldsymbol{v}'''^{\alpha}}$, with $\delta_4(\alpha) = -\min_i(\overline{v'''^{0}}_{i} - \overline{v'''^{\alpha}}_{i})$.

The fuzzy number vector $\tilde{\boldsymbol{v}}^{*}$ is also a fuzzy number max-plus eigenvalue associated with $\tilde{\lambda}_{\max}(\tilde{A})$. From construction above, the components of $\underline{\boldsymbol{v}}^{*0}$, that is $\underline{v}^{*0}_{i}$ are all non-negative and there exist $i \in \{1, 2, ..., n\}$ such that $\underline{v}^{*\alpha}_{i} = 0$ for every $\alpha \in [0, 1]$. Meanwhile, its $\alpha$-cut vector is the smalest interval, in the sense that $\min_i(\overline{v}^{*0}_{i} - \underline{v}^{*0}_{i}) = 0$ for $i = 1, 2, ..., n$, among all possible fuzzy number max-plus eigenvector, the modification of the fundamental fuzzy number max-plus eigenvector $\tilde{\boldsymbol{v}}$, where all the lower bounds of its components are non-negative and at least one zero.

Since the fuzzy number vector $\tilde{\boldsymbol{v}}^*$ is a fuzzy number max-plus eigenvector associated with $\tilde{\lambda}_{\max}(\tilde{A})$

$$\tilde{A} \mathbin{\tilde{\otimes}} \tilde{\boldsymbol{v}}^* = \tilde{\lambda}_{\max}(\tilde{A}) \mathbin{\tilde{\otimes}} \tilde{\boldsymbol{v}}^* \text{ or } A^{\alpha} \mathbin{\overline{\otimes}} \boldsymbol{v}^{*\alpha} = \lambda_{\max}(A^{\alpha}) \mathbin{\overline{\otimes}} \boldsymbol{v}^{*\alpha} \text{ or}$$

$$[\, \underline{A^{\alpha}} \otimes \underline{\boldsymbol{v}^{*\alpha}},\ \overline{A^{\alpha}} \otimes \overline{\boldsymbol{v}^{*\alpha}}\,] = [\lambda_{\max}(\underline{A^{\alpha}}) \otimes \underline{\boldsymbol{v}^{*\alpha}},\ \lambda_{\max}(\overline{A^{\alpha}}) \otimes \overline{\boldsymbol{v}^{*\alpha}}\,].$$

Hence $\underline{A^{\alpha}} \otimes \underline{\boldsymbol{v}^{*\alpha}} = \lambda_{\max}(\underline{A^{\alpha}}) \otimes \underline{\boldsymbol{v}^{*\alpha}}$ and $\overline{A^{\alpha}} \otimes \overline{\boldsymbol{v}^{*\alpha}} = \lambda_{\max}(\overline{A^{\alpha}}) \otimes \overline{\boldsymbol{v}^{*\alpha}}$ for every $\alpha \in [0, 1]$.

For some $\alpha \in [0, 1]$, we can take the early departure time of customer $\tilde{\boldsymbol{d}}(0) = \underline{\boldsymbol{v}^{*\alpha}}$, that is the earliest of early departure time of a customer, such that the lower bound of customer departure time intervals are periodic. This is because there exist $i \in \{1, 2, ..., n\}$ such that $\underline{v^{*\alpha}}_i = 0$ for every $\alpha \in [0, 1]$. Since the operation $\oplus$ and $\otimes$ on matrix are consistent with respect to the order "$\preceq_{\mathrm{m}}$", then

$$(\underline{A^{\alpha}})^{\otimes k} \otimes \underline{\boldsymbol{v}^{*\alpha}} \preceq_{\mathrm{m}} (\overline{A^{\alpha}})^{\otimes k} \otimes \underline{\boldsymbol{v}^{*\alpha}} \preceq_{\mathrm{m}} (\overline{A^{\alpha}})^{\otimes k} \otimes \overline{\boldsymbol{v}^{*\alpha}}.$$

This resulted

$$\begin{aligned}
\boldsymbol{d}^{\alpha}(k) &\approx [\,(\underline{A^{\alpha}})^{\otimes k} \otimes \underline{\boldsymbol{v}^{*\alpha}},\ (\overline{A^{\alpha}})^{\otimes k} \otimes \underline{\boldsymbol{v}^{*\alpha}}\,] \subseteq [\,(\underline{A^{\alpha}})^{\otimes k} \otimes \underline{\boldsymbol{v}^{*\alpha}},\ (\overline{A^{\alpha}})^{\otimes k} \otimes \overline{\boldsymbol{v}^{*\alpha}}\,] \\
&= [\,(\lambda_{\max}(\underline{A^{\alpha}}))^{\otimes k} \otimes \underline{\boldsymbol{v}^{*\alpha}},\ (\lambda_{\max}(\overline{A^{\alpha}}))^{\otimes k} \otimes \overline{\boldsymbol{v}^{*\alpha}}\,] \\
&= [\,(\lambda_{\max}(\underline{A^{\alpha}}))^{\otimes k},\ (\lambda_{\max}(\overline{A^{\alpha}}))^{\otimes k}\,] \mathbin{\overline{\otimes}} [\,\underline{\boldsymbol{v}^{*\alpha}}, \overline{\boldsymbol{v}^{*\alpha}}\,] \\
&= [\,\lambda_{\max}(\underline{A^{\alpha}}), \lambda_{\max}(\overline{A^{\alpha}})]^{\otimes k} \mathbin{\overline{\otimes}} [\,\underline{\boldsymbol{v}^{*\alpha}}, \overline{\boldsymbol{v}^{*\alpha}}\,] \text{ for every } k = 1, 2, 3, \ldots.
\end{aligned}$$

Thus for some $\alpha \in [0, 1]$, vector $\underline{\boldsymbol{v}^{*\alpha}}$ is the earliest of early departure time of a customer, so that the customer's departure interval time will be in the smallest interval where the lower bound and upper bound are periodic with the period $\lambda_{\max}(\underline{A^{\alpha}})$ and $\lambda_{\max}(\overline{A^{\alpha}})$, respectively.

In the same way as above, we can show that for some $\alpha \in [0, 1]$, if we take the early departure time $\tilde{\boldsymbol{d}}(0) = \boldsymbol{v}$, where $\underline{\boldsymbol{v}^{*\alpha}} \preceq_{\mathrm{m}} \boldsymbol{v} \preceq_{\mathrm{m}} \overline{\boldsymbol{v}^{\alpha}}$, then we have

$$\begin{aligned}
\boldsymbol{d}^{\alpha}(k) &\approx [\,(\underline{A^{\alpha}})^{\otimes k} \otimes \boldsymbol{v},\ (\overline{A^{\alpha}})^{\otimes k} \otimes \boldsymbol{v}] \subseteq [\,(\underline{A^{\alpha}})^{\otimes k} \otimes \underline{\boldsymbol{v}^{*\alpha}},\ (\overline{A^{\alpha}})^{\otimes k} \otimes \overline{\boldsymbol{v}^{*\alpha}}\,] \\
&= [\,\lambda_{\max}(\underline{A^{\alpha}}), \lambda_{\max}(\overline{A^{\alpha}})]^{\otimes k} \mathbin{\overline{\otimes}} [\,\underline{\boldsymbol{v}^{*\alpha}}, \overline{\boldsymbol{v}^{*\alpha}}\,].
\end{aligned}$$

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