Vol. 03 No. 02 Juli 2011 ISSN: 2087-5126

BULLETIN OF MATHEMATICS

Published by
Indonesian Mathematical Society Aceh-Sumut

Bulletin of Mathematics

(Indonesian Mathematical Society Aceh - North Sumatera)
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Jalan Bioteknologi 1, Kampus USU, Medan 20155, Indonesia

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Bulletin of Mathematics
Volume 03, Number 02, July 2011

Bulletin of Mathematics
Vol. 03, No. 02 (2011), pp. 103–110.

AN ANALYTICAL SOLUTION TO THE PROBLEM OF PLANAR OSCILLATION OF WATER IN A PARABOLIC CANAL

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Abstract. *The problem of planar oscillation of water in a parabolic canal in two dimension has been analytically solved by Thacker [Some exact solutions to the nonlinear shallow water equations, Journal of Fluid Mechanics, 107(1981):499–508]. However, for the one dimensional case, Thacker did not present the solution of the problem. Therefore, we derive here an analytical solution to the problem of planar oscillation of water in a parabolic canal for the one dimensional case.*

1. INTRODUCTION

The exact solutions to the oscillations of water in a parabolic canal considered by Thacker [6] are of two types, planar and paraboloid. Those analytical solutions have been widely applied by a number of authors, such as Balzano [1], Casulli [2], Murillo and García-Navarro [3], to test some numerical methods used to solve the shallow water equations.

Thacker [6] has solved the problem of planar oscillation of water in a parabolic canal in two dimension, but not derived the solution for the one dimensional case. In addition to the presentation of Thacker [6], Sampson et al [4] have derived analytical solutions for the one dimensional case based on the characteristic equation of an ordinary differential equation. Taking

Received 15-12-2010, Accepted 12-02-2011.
2010 Mathematics Subject Classification: 00A79.
Key words and Phrases: shallow water equations, planar oscillation, parabolic canal.

the presentations of Thacker [6], and Sampson et al [4] into account, in this paper we present an alternative derivation for the problem with some specific conditions or assumptions. Our derivation, which is for the one dimensional case, follows step by step from the derivation of Thacker [6] for two dimensional problems.

2. DERIVATION OF THE SOLUTION

This section presents an analytical solution to time-dependent motion of water in a parabolic canal without friction and without Coriolis force. The solution presented here is in conjunction to the work of Thacker [6]. An interesting feature of the solution is that no shock forms as the water flows up and down the sloping sides of the canal.

The motion of water in a shallow canal, as illustrated in Figure 1, is governed by the mathematical equations

$$\left.\begin{array}{c} \eta_t + \big[u(D+\eta)\big]_x = 0 \\ u_t + uu_x = -g\eta_x \end{array}\right\} \tag{1}$$

called the shallow water equations. Here, x is the one dimensional spatial variable; t is the time variable; $\eta(x,t)$ is the vertical measure from the horizontal reference passing the origin O_1 to the water surface; $D(x)$ is the vertical measure from the horizontal reference passing the origin O_1 to the bed topography; $u(x,t)$ is the water velocity; and g is a constant representing the acceleration due to the gravity. The derivation of (1) can be found in some text books, such as that written by Stoker [5]. According to (1), the instantaneous shoreline is determined by $D + \eta = 0$. The moving shoreline separates a region in which the total depth is positive from another region in which it is negative. It follows from the continuity equation, which is the first equation in (1), that the volume of water within the region for which the total depth is positive remains constant in time as the shoreline moves about.

We assume that there exists a solution for u of the form

$$u = u_0 + u_x x \tag{2}$$

where u_0 and u_x are functions only of time. Substitution (2) into (1) leads to an idea that the solution for η have the form

$$\eta = \eta_0 + \eta_x x + \frac{1}{2}\eta_{xx}x^2 \tag{3}$$

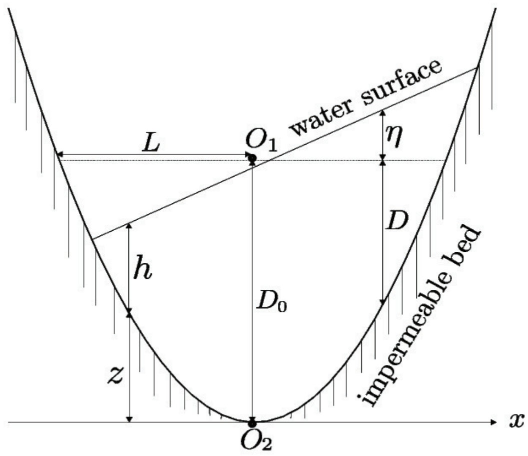

Figure 1: An illustration of a planar free surface in a parabolic canal.

where

$$\eta_x = -\frac{1}{g}\left[\frac{du_0}{dt} + u_0 u_x\right], \tag{4}$$

$$\eta_{xx} = -\frac{1}{g}\left[\frac{du_x}{dt} + u_x^2\right] \tag{5}$$

and η_0 is a function only of t.

In order to satisfy the continuity equation, D must be a polynomial similar to η. In particular, it can be assumed that the canal is a parabola of the form

$$D = D_0\left(1 - \frac{x^2}{L^2}\right). \tag{6}$$

Furthermore, the equilibrium shoreline is determined by the condition $D = 0$ which means

$$x = \pm L. \tag{7}$$

Substitution (2) and (3) into the continuity equation leads to

$$\left[\eta_0 + \eta_x x + \frac{1}{2}\eta_{xx}x^2\right]_t \cdot \left[(u_0 + u_x x)(D_0 - \frac{x^2}{L^2} + \eta_0 + \eta_x x + \frac{1}{2}\eta_{xx}x^2)\right]_x = 0 \tag{8}$$

which is equivalent to

$$\Big[\eta_0 + \eta_x x + \frac{1}{2}\eta_{xx}x^2\Big]_t \cdot \Big[\big[u_x(D_0+\eta_0) + u_0\eta_x\big] + \big[u_0(\eta_{xx} - 2\frac{D_0}{L^2}) + 2u_x\eta_x\big]x + \frac{3}{2}u_x(\eta_{xx} - 2\frac{D_0}{L^2})x^2\Big] = 0\,. \tag{9}$$

Since the last equation holds for all points, then the time-varying coefficients of the linearly independent terms must separately equal to zero. This means that

$$\frac{d\eta_0}{dt} + u_x(D_0+\eta_0) + u_0\eta_x = 0\,, \tag{10}$$

and

$$\frac{d\eta_x}{dt} + u_0\big(\eta_{xx} - 2\frac{D_0}{L^2}\big) + 2u_x\eta_x \;\; = \;\; 0\,, \tag{11}$$

and also

$$\frac{d\eta_{xx}}{dt} + 3u_x\big(\eta_{xx} - 2\frac{D_0}{L^2}\big) = 0\,. \tag{12}$$

These three equations can determine the corresponding three unknown functions (η_0 , η_x , and η_{xx}) of time.

Now, assume that $u_x = 0$, so that according to (5) it is clear that $\eta_{xx} = 0$. Then only two functions, u_0 and h_0 , must be determined. Equation (12) is identically satisfied, and equations (10) and (11) can be written respectively as

$$\frac{d\eta_0}{dt} + \eta_x u_0 = 0 \tag{13}$$

and

$$\frac{d\eta_x}{dt} - \frac{2D_0}{L^2}u_0 = 0\,. \tag{14}$$

Substitution (4) into (13) and (14) yields, respectively,

$$\frac{d\eta_0}{dt} - \frac{u_0}{g}\frac{du_0}{dt} = 0\,, \tag{15}$$

$$\frac{d^2u_0}{dt^2} + \frac{2gD_0}{L^2}u_0 = 0\,. \tag{16}$$

The general solution of (16) is

$$u_0 = c_1\sin(\omega t) + c_2\cos(\omega t) \tag{17}$$

where $\omega = \sqrt{2gD_0/L^2}$, and c_1 and c_2 are constants. Using the initial condition that $u_0 = 0$ for $t = 0$, we obtain that $c_2 = 0$. Therefore,

$$u_0 = c_1 \sin(\omega t)\,. \tag{18}$$

This implies that the horizontal displacement δ, which is the integral of the velocity, has the form of

$$\delta = -\frac{c_1}{\omega}\cos(\omega t) + c_3 \tag{19}$$

where c_3 is another constant. Applying the conditions $\delta(0) = A$ and $\delta(\frac{\pi}{2\omega}) = 0$, where the constant A determines the amplitude of the motion, we get $c_1 = -A\omega$ and $c_3 = 0$. Hence, the horizontal displacement is

$$\delta = A\cos(\omega t)\,; \tag{20}$$

the shorelines has the formula

$$x = \delta \pm L = A\cos(\omega t) \pm L\,; \tag{21}$$

and the velocity has the form

$$u = u_0 = -A\omega\sin(\omega t)\,. \tag{22}$$

To determine the form of the surface elevation η, the quantities η_x and η_0 need to be specified. Based on equation (14) and (22), we have

$$\frac{d\eta_x}{dt} = \frac{2D_0u_0}{L^2} = \frac{2D_0(-A\omega\sin(\omega t))}{L^2} = -\frac{2A\omega D_0}{L^2}\sin(\omega t) \tag{23}$$

and this yields

$$\eta_x = \frac{2AD_0}{L^2}\cos(\omega t) + c_4 \tag{24}$$

for some constant c_4. Based on (15) and (22), we have

$$\frac{d\eta_0}{dt} = \frac{u_0}{g}\frac{du_0}{dt} = \frac{A^2\omega^3}{2g}\sin(2\omega t) \tag{25}$$

and this yields

$$\eta_0 = -\frac{A^2\omega^2}{4g}\cos(2\omega t) + c_5 \tag{26}$$

for some constant c_5. As a result, it can be evaluated that

$$\begin{aligned}\eta &= \eta_0 + \eta_x x\\ &= \frac{A^2D_0}{2L^2}[1 - 2\cos^2(\omega t)] + c_5 + [\frac{2AD_0}{L^2}\cos(\omega t) + c_4]x\,.\end{aligned} \tag{27}$$

For $x = 0$ and $t = \frac{\pi}{2\omega}$, we have that $\eta = 0$; this implies $c_5 = -\frac{A^2 D_0}{2L^2}$. In addition, for $x = A\cos(\omega t) + L$ and $t = \frac{\pi}{2\omega}$, the water surface is horizontal which means $\eta = 0$; this implies $c_4 = 0$. Therefore, the closed form of the water surface is given by

$$\eta \quad = \quad \frac{2AD_0}{L^2}\cos(\omega t)\big[x - \frac{A}{2}\cos(\omega t)\big] . \tag{28}$$

Here the angular frequency of oscillation and the period are given by $\omega = \frac{\sqrt{2gD_0}}{L}$ and $T = \frac{2\pi}{\omega}$ respectively. It should be stessed that the solution (28), (22) obtained above is under the assumption that the origin point is O_1 , as given in Figure 1.

If the origin is O_2 , then a linear transformation is needed. Considering O_2 as the origin, we have that the canal profile is then given by

$$z = \frac{D_0}{L^2}x^2 , \tag{29}$$

the velocity and the shorelines are, still the same as before, given by

$$u = -A\omega\sin(\omega t) , \tag{30}$$

and

$$x = A\cos(\omega t) \pm L , \tag{31}$$

while the water stage w has the form

$$w \quad = \quad D_0 + \frac{2AD_0}{L^2}\cos(\omega t)\big[x - \frac{A}{2}\cos(\omega t)\big] . \tag{32}$$

Here, the water stage w is the vertical measure from the horizontal reference passing O_2 to the water surface, that is, $w(x,t) = z(x) + h(x,t)$ where $h(x,t) = D(x) + \eta(x,t)$, as illustrated in Figure 1. In short, the equations (32), (30) are the solution to the problem if the origin is O_2 , as illustrated in Figure 1.

3. CONCLUSION

We have derived an analytical solution to the problem of planar oscillation of water in a parabolic canal for the one dimensional case. This analytical solution can be applied to test the performance of numerical methods used to solve the one dimensional shallow water equations.

ACKNOWLEDGEMENT

The author thanks Associate Professor Stephen Roberts at The Australian National University (ANU) for his advice. This work was done when the author was undertaking a masters program at the ANU.

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