Vol. 03 No. 01 Januari 2011

ISSN: 2087-5126

BULLETIN OF MATHEMATICS

Published by
Indonesian Mathematical Society Aceh-Sumut

Bulletin of Mathematics
(Indonesian Mathematical Society Aceh - North Sumatera)
c/o Department of Mathematics, University of Sumatera Utara
Jalan Bioteknologi 1, Kampus USU, Medan 20155, Indonesia
E-mail:bulletinmathematics@yahoo.com **Website:** http://indoms-nadsumut.org & http://bull-math.org

Bulletin of Mathematics
Volume 03, Number 01, January 2011

Bulletin of Mathematics
Vol. 03, No. 01 (2011), pp. 79–85.

AN ALTERNATIVE DERIVATION OF THE SHALLOW WATER EQUATIONS

Sudi Mungkasi

Abstract. *The one dimensional shallow water equations consist of the equation of mass conservation and that of momentum conservation. In this paper, those equations are derived. We utilize a constant approximation of integration in the derivation. We call our derivation an alternative derivation since it is different from the classical derivation. The classical derivation of the shallow water equations involves a velocity-potential function, while our derivation does not.*

1. INTRODUCTION

The shallow water equations, or SWE for short, model shallow water flows as the name suggest. These equations are according to the transport of mass and momentum, and based on conservation laws. These equations form a nonlinear hyperbolic system, and often admit discontinuous solution even when the initial condition is smooth. They have been widely used for many applications, for example: dam-breaks, tsunamis, river flows, and flood inundation. The SWE also known as the Saint-Venant system, for the one dimensional case, can be written as two simultaneous equations

$$h_t + (hu)_x = 0\,, \tag{1}$$

Received 21-01-2011, Accepted 31-01-2011.
2010 Mathematics Subject Classification: 76B07, 76B15.
Key words and Phrases: shallow water equations, conservation of mass, conservation of momentum.

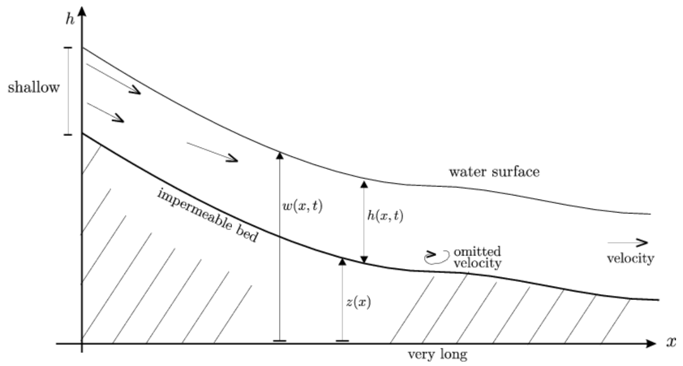

Figure 1: Shallow water flow in one-dimension

$$(hu)_t + (hu^2 + \frac{1}{2}gh^2)_x = -ghz_x \, . \qquad (2)$$

Here, we have that: x represents the distance variable along the water flow, t represents the time variable, $z(x)$ is the fixed water bed, $h(x,t)$ is the height of the water at point x and at time t, and $u(x,t)$ denotes the velocity of the water flow at point x and at time t. In addition, g is a constant denoting the acceleration due to the gravity. The shallow water flow is illustrated in Figure 1, where the water stage w is defined by $w(x,t) = z(x) + h(x,t)$.

In this paper, we present an alternative derivation of the one dimensional SWE (1), (2) using a constant approximation of integration. We call our derivation as "an alternative derivation", because our derivation is different from the classical derivation, which usually uses a velocity-potential function (see Stoker [3] for the classical derivation). Note that LeVeque [2] has derived some conservation laws, but also in a different way. We start from an arbitrary control volume, then derive the shallow water equations based on the considered control volume. Since the control volume is arbitrary, the shallow water equations hold for an arbitrary domain.

This paper is organized as follows. Section 2 recalls a constant approximation for integral form, and also presents the derivation of the SWE consisting of equation (1) of mass conservation and equation (2) of momentum conservation. We provide some concluding remarks in Section 3.

2. DERIVATION OF THE SWE

In this section, we derive equation (1) of mass conservation and equation (2) of momentum conservation. These two equations form the SWE simultaneously. In the derivations, we apply a constant approximation for integral

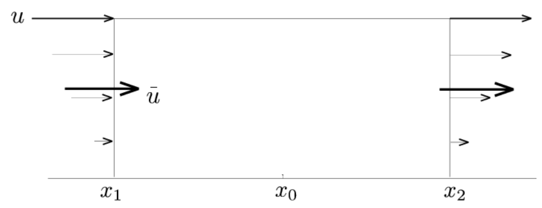

Figure 2: The inflow and outflow of the control volume

form. The properties of the approximations are given in the following theorem, where the proof has been provided by Laney [1, pp.176–177].

Theorem 2.1 *Suppose that we are given an arbitrary interval* $[x_1, x_2]$ *. Let* $\Delta x = x_2 - x_1$ *. If* $x_0 \in [x_1, x_2]$ *but* x_0 *is not the centroid of the interval, then*

$$\int_{x_1}^{x_2} f(x)\ dx = f(x_0)\Delta x + O(\Delta x^2)\,. \tag{3}$$

If x_0 *is the centroid of the interval* $[x_1, x_2]$ *, then*

$$\int_{x_1}^{x_2} f(x)\ dx = f(x_0)\Delta x + O(\Delta x^3)\,. \tag{4}$$

2.1. Conservation of mass

Conservation of mass means that the mass is neither created nor destroyed. Several assumptions, involved in the derivation of the equation of mass conservation, follows. First, the water flow is assumed to be laminar that is the turbulent velocity is neglected. Second, the water is incompressible so that the density, ρ, of the water at each point is constant. In addition, because the conservation of mass is applied and it is assumed that the water bed is impermeable, the mass in any control volume can change only due to the flow acrossing the end points of the control volume.

The total mass m of water in an arbitrary control volume $[x_1, x_2]$ is given by

$$m = \int_{x_1}^{x_2} \rho h(x,t)\ dx \tag{5}$$

The total mass given by (5) holds because the mass density over the depth at an arbitrary point (x, t) is $\rho h(x, t)$ which can be calculated by integration of ρ from $z(x)$ to $w(x, t)$, where we know that $w(x, t) = z(x) + h(x, t)$.

The rate of flow of water past any point (x, t) over the depth is called the *mass flux* which is given by

$$\text{mass flux } = \rho h(x,t)u(x,t)\,. \tag{6}$$

Using (6), we obtain that the rate of change of the total mass is

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho h(x,t)\; dx = \rho h(x_1,t)u(x_1,t) - \rho h(x_2,t)u(x_2,t)\,. \tag{7}$$

For smooth solutions, (7) is equivalent to

$$\begin{aligned}\int_{x_1}^{x_2} h(x,t+\Delta t)dx \quad &= \quad \int_{x_1}^{x_2} h(x,t)dx + \int_{t}^{t+\Delta t} h(x_1,s)u(x_1,s)ds \\ &\quad - \int_{t}^{t+\Delta t} h(x_2,s)u(x_2,s)ds\,. \end{aligned} \tag{8}$$

Equation (8) means that the mass at time step $t + \Delta t$ is equal to the mass at time t added by the total amount of mass moving into the control volume during Δt-period, as illustrated in Figure 2.

Let Δx and Δt be small quantities. Using a constant approximation (4) of integration, we can rewrite equation (8) as

$$\begin{aligned} h(x,t+\Delta t)\Delta x \quad &= \quad h(x,t)\Delta x + h(x - \frac{\Delta x}{2}, t)u(x - \frac{\Delta x}{2}, t)\Delta t \\ &\quad -h(x + \frac{\Delta x}{2}, t)u(x + \frac{\Delta x}{2}, t)\Delta t \\ &\quad +O(\Delta t^3) + O(\Delta x^3) \end{aligned} \tag{9}$$

which can be rewritten as

$$\frac{h(x,t+\Delta t) - h(x,t)}{\Delta t} = -\frac{(hu)|_{(x+\frac{\Delta x}{2},t)} - (hu)|_{(x-\frac{\Delta x}{2},t)}}{\Delta x}\,, \tag{10}$$

in which $O(\Delta t^3)$ and $O(\Delta x^3)$ terms are neglected. As Δx and Δt approach zero, equation (10) becomes

$$h_t + (uh)_x = 0 \tag{11}$$

Equation (11) is then called the equation of mass conservation.

2.2. Conservation of momentum

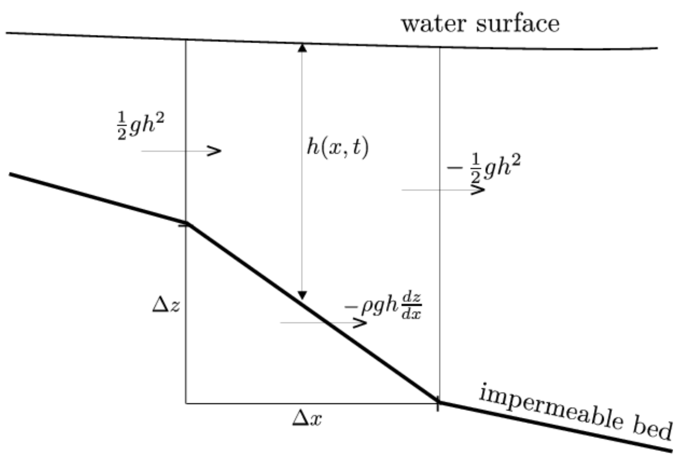

Figure 3: Pressure in a slope area

In this subsection, the conservation of momentum equation using Newton's second law of motion is presented. The law can be mathematically written as

$$F = \frac{dp}{dt}\,, \tag{12}$$

where the inertial force F is defined as the rate of change of the momentum p with respect to time t.

The total momentum of water movement in any control volume from x_1 to x_2 at time t is

$$p(t) = \int_{x_1}^{x_2} \rho h(x,t)u(x,t)\; dx\,. \tag{13}$$

The forces at points x_1 and x_2 over the depth at time t are

$$F_1(t) = \frac{1}{2}\rho g h^2(x_1,t) \tag{14}$$

and

$$F_2(t) = -\frac{1}{2}\rho g h^2(x_2,t) \tag{15}$$

respectively, where $g > 0$ is constant denoting the acceleration due to the gravity as we have stated in the introduction section. Furthermore, the force over Δz as shown in Figure 3 is

$$\Delta F_3 = -\rho g h(x,t)\Delta z \tag{16}$$

or can be written as

$$\Delta F_3 = -\rho g h(x,t)\frac{\Delta z}{\Delta x}\Delta x \tag{17}$$

and therefore the force over the bottom of the control volume is

$$F_3 = \int_{x_1}^{x_2} -\rho g h(x,t) z_x dx \tag{18}$$

Hence, the total force over the control volume denoted by F is the sum of F_1 , F_2 , and F_3 , that is,

$$F = \frac{1}{2}\rho g h^2(x_1,t) - \frac{1}{2}\rho g h^2(x_2,t) - \int_{x_1}^{x_2} \rho g h(x,t)\frac{dz}{dx}dx\,. \tag{19}$$

Using Leibniz's rule for differentiation under the integral sign, we obtain that the first derivative of p with respect to t is

$$\begin{aligned} \frac{dp}{dt} &= \frac{d}{dt}\int_{x_1}^{x_2} \rho h(x,t)u(x,t)\ dx \\ &= \int_{x_1}^{x_2} \frac{\partial}{\partial t}\rho h(x,t)u(x,t)\ dx \\ &\quad +\rho h(x_2,t)u^2(x_2,t) - \rho h(x_1,t)u^2(x_1,t)\,. \end{aligned} \tag{20}$$

According to Newton's second law (12) of motion, we have that the result in equation (20) is equal to that in equation (19). Hence, for Δt-period it follows that

$$\begin{aligned} &\int_t^{t+\Delta t}\int_{x_1}^{x_2} (\rho h u)_t\ dx\ dt + \int_t^{t+\Delta t} \rho h(x_2,t)u^2(x_2,t)\ dt \\ &-\int_t^{t+\Delta t} \rho h(x_1,t)u^2(x_1,t)\ dt = \int_t^{t+\Delta t} \frac{1}{2}\rho g h^2(x_1,t)\ dt \\ &-\int_t^{t+\Delta t} \frac{1}{2}\rho g h^2(x_2,t)\ dt - \int_t^{t+\Delta t}\int_{x_1}^{x_2} \rho g h(x,t) z_x\ dx\ dt \end{aligned} \tag{21}$$

Using a constant approximation (4) of integration, we can rewrite equation (21) as

$$(hu)_t + (hu^2 + \frac{1}{2}gh^2)_x = -ghz_x\,, \tag{22}$$

in which $O(\Delta t^3)$ and $O(\Delta x^3)$ terms are neglected. This last equation is called the equation of momentum conservation.

3. CONCLUSION

We have derived the one dimensional shallow water equations, which consist of the equation of mass conservation and the equation of momentum conservation, using a constant approximation of integration. Our presentation suggests that higher dimensional shallow water equations can also be alternatively derived using a similar technique.

ACKNOWLEDGEMENT

The author thanks Associate Professor Stephen Roberts at The Australian National University (ANU) for his advice. This work was done when the author was undertaking a masters program at the ANU.

References

[1] C. B. Laney. 1998. *Computational Gasdynamics*, Cambridge University Press, Cambridge.

[2] R. J. LeVeque. 2002. *Finite-Volume Methods for Hyperbolic Problems*, Cambridge University Press, Cambridge.

[3] J. J. Stoker. 1957. *Water Waves: The Mathematical Theory with Application*, Interscience Publishers, New York.

Sudi Mungkasi: Department of Mathematics, Mathematical Sciences Institute, The Australian National University, Canberra, Australia; Department of Mathematics, Faculty of Science and Technology, Sanata Dharma University, Yogyakarta, Indonesia.
E-mail: sudi.mungkasi@anu.edu.au; sudi@usd.ac.id