GENERAL NEUTRINO MASS MATRIX PATTERNS
AND ITS UNDERLYING FAMILY SYMMETRIES

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Abstract
Based on current experimental results, such as neutrino oscillations, and the neutrinoless double beta decays (i.e. data from Super-Kamiokande, KamLAND, SNO, etc.), the neutrino mixing matrix can be adequately determined. Though there are still certain parameters that have large possibility limits, but based on the current experimental results, it is possible to construct a general form of neutrino mass matrix.
Starting from this general form of the neutrino mass matrix, we put certain conditions in the context of the seesaw mechanism model to determine the possible pattern of the neutrino mass matrix that has texture zero. From the obtained neutrino mass matrix pattern, there are three class of pattern, where two of the class are known to be realized in the literature by the underlying family symmetries of the D_4 and A_4 groups, the dihedral and tetrahedral symmetry groups..

Keywords: neutrino mass matrix, seesaw mechanism, family symmetry.

Abstrak
Mengacu pada hasil-hasil eksperimen terkini, seperti osilasi neutrino dan peluruhan beta ganda tanpa neutrino (misalnya data dari Super-Kamiokande, KamLAND, SNO, dsb.), matriks campuran neutrino telah dapat ditentukan. Walaupun masih ada beberapa parameter tertentu yang mempunyai nila-nilai batas yang lebar, tetapi berdasarkan hasil eksperimen terkini, dimungkinkan untuk menyusun sebuah bentuk umum matrik massa neutrino.
Berawal dari bentuk umum matrik massa neutrino ini, dengan menerapkan beberapa syarat tertentu dalam konteks mekanisme seesaw ditentukan kemungkinan pola-pola matrik massa neutrino yang memiliki tekstur nol. Dari pola matrik massa neutrino yang diperoleh, terdapat tiga kelas pola, yang mana dua dari kelas tersebut diketahui terealisasi dalam literatur-literatur dengan simetri keluarga penopangnya adalah grup D_4 dan A_4, yaitu grup simetri dihedral dan tetrahedral.

Kata kunci: matriks massa neutrino, mekanisme seesaw, simetri keluarga.

1. Introduction

According to the Standard Model (SM) of Particle Physics, based on gauge symmetry $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ neutrinos are massless, neutral, and only left-handed neutrinos participate in weak and electromagnetic interactions via charge-current and neutral-current interactions. Even though the SM is very successful in both describing wide range of phenomena and its predictions on the related physical quantities, there is no fundamental reason to put neutrino to be massless like photon to be massless due to $U(1)_{EM}$ gauge invariance.

For more than two decades the solar neutrino flux measured on Earth has been much less than that predicted by the solar model[1)]. From recent experimental results, there is evidence that neutrinos are massive due to the experimental facts that the solar and atmospheric neutrinos undergo oscillations during its propagation[2,3,4)]. Theoretically, neutrino oscillations can be explained by using quantum mechanics and occurs if there is non-zero Δm_{ij}^2, where $\Delta m_{ij}^2 = m_i^2 - m_j^2$ ($i, j = 1, 2, 3$, m_i and m_j are the neutrino mass in the mass eigenstates basis) and the neutrino flavor eigenstates are different from neutrino mass eigenstates. The flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$ can be expressed as a linear combination of neutrino mass eigenstates (ν_1, ν_2, ν_3) as follows:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = V \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \qquad (1)$$

where V is a neutrino mixing matrix like the *Cabibbo-Kobayashi-Maskawa* (CKM) matrix in the quark sector.

From the theoretical side, several models have been proposed to explain the existence of the neutrino mass and its underlying symmetries. Among the proposed models, the one based on seesaw mechanism is the most popular because it can explain the smallness of the neutrino mass and also account for the large mixing angles simultaneously[5].

Recently, one of the interesting subjects is the determination of the lepton family symmetries underlying the neutrino mass patterns in the seesaw mechanism. The general neutrino mass matrix M_ν in the seesaw mechanism is given by[6]:

$$M_\nu = M_D M_N^{-1} M_D^T, \qquad (2)$$

where M_D and M_N are the Dirac mass and Majorana mass matrices respectively. Based only on phenomenological consideration, Ma proposed the *All-purpose neutrino mass matrix* [7] and then he also derived the same neutrino mass matrix by using the cubic symmetry group S_4 (contain A_4 and S_3 groups) as the family symmetry[8]. Several models based on the tetrahedral group A_4 has been proposed also by Zee[9].

In this paper, we derive the general neutrino mass matrix patterns from the neutrino mixing matrix V, suitable with the current experimental limits (Section 2). In Section 3, using the general neutrino mass matrix patterns obtained in section 2, we determine the possible structure of the M_N^{-1} matrix from the general neutrino mass matrix pattern obtained in section 2, but with additional requirement that is M_N has one or more texture zero. We mention also the corresponding underlying family symmetry for the neutrino mass matrix. Finally, in Section 4 we give the conclusion.

2. General Neutrino Mass Matrix Patterns

Following standard convention, let us denote the neutrino current eigenstates coupled to the charge leptons by the W bosons as ν_α ($\alpha = e, \mu, \tau$), and the neutrino mass eigenstates as ν_i ($i = 1, 2, 3$) then the mixing matrix V takes the form presented in Eq. (1). As we have stated explicitly above, we use the seesaw mechanism as the responsible mechanism for generating the neutrinos mass. The Majorana-mass Lagrangian term is given by:

$$L = -\nu_\alpha M_{\alpha\beta} C \nu_\beta + h.c. \qquad (3)$$

where C denotes the charge conjugation matrix. Thus, the neutrino mass matrix M is symmetric. For the sake of simplicity we will assume *CP* conservation so that M is real. Within this simplification, the neutrino mass matrix M is diagonalized by the orthogonal transformation:

$$V^T M V = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}. \qquad (4)$$

From recent experimental results, the explicit form of the mixing matrix moduli V has been known to be[10]:

$$|V| = \begin{pmatrix} 0.79 - 0.88 & 0.47 - 0.61 & < 0.20 \\ 0.19 - 0.52 & 0.42 - 0.73 & 0.58 - 0.82 \\ 0.20 - 0.53 & 0.44 - 0.74 & 0.56 - 0.81 \end{pmatrix}. \qquad (5)$$

According to the requirement that the mixing matrix V must be orthogonal, together with putting V in nice simple looking numbers, the following form of V are used to be proposed,

$$V = \begin{pmatrix} -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{pmatrix}, \qquad (6)$$

we will use this form of V as our mixing matrix. To determine the neutrino mass matrix M_ν, we employ the following relation:

$$M_\nu = V \left(V^T M V \right) V^T. \qquad (7)$$

Substitutions of Eq. (4) and Eq. (6) into Eq. (7) lead to:

$$M_v = \begin{pmatrix} -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \tag{8}$$

$$= \begin{pmatrix} \dfrac{2m_1 + m_2}{3} & \dfrac{m_2 - m_1}{3} & \dfrac{m_2 - m_1}{3} \\ \dfrac{m_2 - m_1}{3} & \dfrac{m_1 + 2m_2 + 3m_3}{6} & \dfrac{m_1 + 2m_2 - 3m_3}{6} \\ \dfrac{m_2 - m_1}{3} & \dfrac{m_1 + 2m_2 - 3m_3}{6} & \dfrac{m_1 + 2m_2 + 3m_3}{6} \end{pmatrix}.$$

Inspecting Eq. (8), one can write the general neutrino mass matrix patterns as follows:

$$M_v = \begin{pmatrix} a & b & b \\ b & c & d \\ b & d & c \end{pmatrix}. \tag{9}$$

From Eq. (9) we can see that the general neutrino mass matrix obtained is symmetric as required.

3. Possible Family Symmetries of the General Neutrino Mass Matrix

Having formulated the general neutrino mass matrix, we proceed to determine the kinds of the possible underlying family symmetries of the general neutrino mass matrix appearing in Eq. (9). In the seesaw mechanism scheme, the neutrino mass matrix is given by Eq. (2). If the general neutrino mass matrix in Eq. (9) is taken as the neutrino mass matrix generated by seesaw mechanism, then we can write M_v as follows:

$$M_v = \begin{pmatrix} a & b & b \\ b & c & d \\ b & d & c \end{pmatrix} = M_D M_N^{-1} M_D^T. \tag{10}$$

The kinds of the family symmetries can be determined from the structure of the Majorana mass matrix M_N as well as the Dirac mass

matrix M_D. We assume M_N has one or more of its element to be zero (texture zero). The textures zeros of the mass matrix indicate the existence of additional symmetries beyond the SM[11]. If M_N matrix has one or more of its element to be zero (textures zeros), then this implies M_N^{-1} matrix has one or more 2×2 sub-matrices with zero determinants.

If we take M_D to be diagonal:

$$M_D = \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{pmatrix},$$

with the condition $u_2 = u_3 = u$ (CP is conserved in neutrino oscillations)[12], then the property of M_N^{-1} matrix is preserved in M_v [13]. According to the above requirements, we have the M_N^{-1} matrix pattern as:

$$M_N^{-1} = \begin{pmatrix} A & B & B \\ B & C & D \\ B & D & C \end{pmatrix}. \tag{11}$$

Now we put the texture zero requirements on the M_N The possible patterns of M_N matrix from Eq. (11), is:

$$M_N = \frac{1}{A(C^2 - D^2) + 2B^2(D - C)} \begin{pmatrix} C^2 - D^2 & B(D-C) & B(D-C) \\ B(D-C) & AC - B^2 & -AD + B^2 \\ B(D-C) & -AD + B^2 & AC - B^2 \end{pmatrix} \tag{12}$$

The M_N matrix in Eq. (12) will have textures zero if one or more of the following relations are satisfied:

i. $C = -D$,

ii. $AD - B^2 = 0$,

iii. $AC - B^2 = 0$,

iv. $B = 0$.

$$\tag{13}$$

Once we can determine the form of M_N matrix, then the corresponding M_N^{-1} and M_v matrices can be obtained.

Case (i) $C = -D$, we obtain:

$$M_N = \begin{pmatrix} 0 & \dfrac{1}{2B} & \dfrac{1}{2B} \\ \dfrac{1}{2B} & \dfrac{-AD-B^2}{4B^2D} & \dfrac{-AD+B^2}{4B^2D} \\ \dfrac{1}{2B} & \dfrac{-AD+B^2}{4B^2D} & \dfrac{-AD-B^2}{4B^2D} \end{pmatrix} \qquad (14)$$

The M_N matrix in Eq. (14) has the corresponding M_N^{-1} and M_v matrices as follows:

$$M_N^{-1} = \begin{pmatrix} A & B & B \\ B & -D & D \\ B & D & -D \end{pmatrix}, \qquad (15)$$

and

$$M_v = \begin{pmatrix} P & Q & Q \\ Q & -R & S \\ Q & S & -R \end{pmatrix} \qquad (16)$$

where $P = A/u_1^2$, $Q = B/u_1 u$, and $R = S = D/u^2$.

Case (ii) $AD = B^2$, we obtain:

$$M_N = \begin{pmatrix} \dfrac{C+D}{A(C-D)} & \dfrac{-B}{A(C-D)} & \dfrac{-B}{A(C-D)} \\ \dfrac{-B}{A(C-D)} & \dfrac{1}{C-D} & 0 \\ \dfrac{-B}{A(C-D)} & 0 & \dfrac{1}{C-D} \end{pmatrix} \qquad (17)$$

with

$$M_N^{-1} = \begin{pmatrix} A & B & B \\ B & C & D \\ B & D & C \end{pmatrix} \qquad (18)$$

and

$$M_v = \begin{pmatrix} P & Q & Q \\ Q & R & S \\ Q & S & R \end{pmatrix} \qquad (19)$$

where $P = A/u_1^2$, $Q = B/u_1 u$, $R = C/u^2$, and $S = D/u^2$.

Case (iii) $AC = B^2$, we obtain:

$$M_N = \begin{pmatrix} \dfrac{C+D}{A(C-D)} & \dfrac{-B}{A(C-D)} & \dfrac{-B}{A(C-D)} \\ \dfrac{-B}{A(C-D)} & 0 & \dfrac{-1}{C-D} \\ \dfrac{-B}{A(C-D)} & \dfrac{-1}{C-D} & 0 \end{pmatrix} \qquad (20)$$

with

$$M_N^{-1} = \begin{pmatrix} A & B & B \\ B & C & D \\ B & D & C \end{pmatrix} \qquad (21)$$

and

$$M_v = \begin{pmatrix} P & Q & Q \\ Q & R & S \\ Q & S & R \end{pmatrix} \qquad (22)$$

where $P = A/u_1^2$, $Q = B/u_1 u$, $R = C/u^2$, and $S = D/u^2$.

Case (iv) $B = 0$, we obtain:

$$M_N = \begin{pmatrix} 1/A & 0 & 0 \\ 0 & \dfrac{C}{C^2-D^2} & \dfrac{-D}{C^2-D^2} \\ 0 & \dfrac{-D}{C^2-D^2} & \dfrac{C}{C^2-D^2} \end{pmatrix} \qquad (23)$$

with

$$M_N^{-1} = \begin{pmatrix} A & 0 & 0 \\ 0 & C & D \\ 0 & D & C \end{pmatrix} \qquad (24)$$

and

$$M_v = \begin{pmatrix} P & 0 & 0 \\ 0 & R & S \\ 0 & S & R \end{pmatrix} \qquad (25)$$

where $P = A/u_1^2$, $R = C/u^2$, and $S = D/u^2$.

Case (v) $C = -D$, and $AD = B^2$, we obtain:

$$M_N = \begin{pmatrix} 0 & 1/2B & 1/2B \\ 1/2B & -1/2D & 0 \\ 1/2B & 0 & -1/2D \end{pmatrix} \qquad (26)$$

with

$$M_N^{-1} = \begin{pmatrix} A & B & B \\ B & -D & D \\ B & D & -D \end{pmatrix} \qquad (27)$$

and

$$M_v = \begin{pmatrix} P & Q & Q \\ Q & -R & R \\ Q & R & -R \end{pmatrix} \qquad (28)$$

where $P = A/u_1^2$, $Q = B/u^2$, and $R = D/u^2$.

Case (vi) $C = -D$, and $AC = B^2$, we obtain:

$$M_N = \begin{pmatrix} 0 & 1/2B & 1/2B \\ 1/2B & 0 & -1/2C \\ 1/2B & -1/2C & 0 \end{pmatrix} \qquad (29)$$

with

$$M_N^{-1} = \begin{pmatrix} A & B & B \\ B & C & -C \\ B & -C & C \end{pmatrix} \qquad (30)$$

and

$$M_v = \begin{pmatrix} P & Q & Q \\ Q & R & -R \\ Q & -R & R \end{pmatrix} \qquad (31)$$

where $P = A/u_1^2$, $Q = B/u^2$, and $R = D/u^2$.

Case (vii) $B = 0$, and $AD = B^2$, we obtain:

$$M_N = \begin{pmatrix} 1/A & 0 & 0 \\ 0 & 1/C & 0 \\ 0 & 0 & 1/C \end{pmatrix} \tag{32}$$

with

$$M_N^{-1} = \begin{pmatrix} A & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{pmatrix} \tag{33}$$

and

$$M_\nu = \begin{pmatrix} P & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{pmatrix} \tag{34}$$

where $P = A/u_1^2$ and $R = C/u^2$.

Case (viii) $B = 0$, and $AC = B^2$, we obtain:

$$M_N = \begin{pmatrix} 1/A & 0 & 0 \\ 0 & 0 & 1/D \\ 0 & 1/D & 0 \end{pmatrix} \tag{35}$$

with

$$M_N^{-1} = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & D \\ 0 & D & 0 \end{pmatrix} \tag{36}$$

and

$$M_\nu = \begin{pmatrix} P & 0 & 0 \\ 0 & 0 & S \\ 0 & S & 0 \end{pmatrix} \tag{37}$$

where $P = A/u_1^2$, and $S = D/u^2$.

From the eight cases considered above, we can see that the structure of the M_N^{-1} and M_ν matrices could be classified into three classes due to their textures whether it contains textures zeros or not, they are:

1. Both M_N^{-1} and M_ν matrices have no textures zeros, it appears in the cases: $C = -D$, $AD = B^2$, $AC = B^2$, $C = -D$ combined with $AD = B^2$, and $C = -D$ combined with $AC = B^2$.

2. Both M_N^{-1} and M_ν matrices have four zeros as can be seen in Eqs. (24) and (25) for the case $B = 0$.

3. Both M_N^{-1} and M_ν matrices have six zeros as can be seen in Eqs. (33) and (34) for the case $B = 0$ combined with $AD = B^2$, and in

Eqs. (36) and (37) for the case $B = 0$ combined with $AC = B^2$.

Specifying the underlying family symmetry required specifying the particle contents of a model and the corresponding representation for each particle. There are numerous papers which start from certain family symmetry group in the quark and lepton sectors or only in the lepton sectors that tried to derive the mixing matrix and the neutrino mass matrix from the family symmetry groups. We can mention some of them, for example E. Ma[14] gives a models based on A_4 where he puts the lepton family doublet (ν_i, l_i), and the singlet l_i^c ($i = 1, 2, 3$) to be $\underline{3}$ and additional three heavy neutral fermion singlet N_i to be $\underline{1}$, $\underline{1}'$, $\underline{1}''$, this model gives M_N in the form of Eq. (35) correspond to the case (viii). In other paper, he used the same group A_4 but assigning (ν_i, l_i) to be $\underline{3}$ and l_i^c to be $\underline{1}'$, $\underline{1}''$, together with three Higgs doublet to be $\underline{3}$ and one Higgs singlet to be $\underline{1}$, he obtain a similar pattern.

The dihedral group D_4 has been used also as the underlying family symmetry, in one of the model by Ma[14]. In this model (ν_i, l_i) and ν_i^c are assigned to the $\underline{1}^{++} + \underline{2}$ representation of D_4 group. The Higgs sector has 3 doublets: $\Phi_{1,2,3} \sim \underline{1}^{++}, \underline{1}^{++}, \underline{1}^{+-}$ and 2 singlets: $\chi_{1,2} \sim \underline{2}$. Adding an extra Z_2 symmetry such that the $l_1^c, \nu_{1,2,3}^c, \Phi_1$ are odd, while all other fields are even, the charge lepton mass matrix is diagonal with 3 independent eigenvalues and the neutrino mass matrix was similar to case (vii).

Those two groups above are just some of the underlying family symmetry group that can realize the structure of the M_ν matrices for both class 1 and 3. Meanwhile, the underlying family symmetry to be responsible for M_N^{-1} and M_ν matrices structure in class 2 have not been considered in any paper that we know of. This may be a subject for further research.

4. Conclusion

By choosing a nice looking form of the mixing matrix elements V, fulfilling orthogonality condition, together with the condition that the Dirac mass matrix is diagonal with degenerate $u_2 = u_3 = u$, we obtain the general neutrino mass matrix pattern M_ν .

Assuming that the resulting general neutrino mass matrix to be a mass matrix generated by a seesaw mechanism combined with the requirements that the Majorana mass matrix has one or more textures zeros, we find that there are eight possible cases for the M_N matrix. Based on the structure of the M_N^{-1} matrices whether it contains textures zeros or not, the eight possible patterns of the M_N matrix lead to the three classes of the M_ν matrix.

Two of the M_ν classes (class 1 and 3) are known to have the A_4 and D_4 groups to be their possible underlying family symmetries. While the underlying family symmetry that can realize the M_ν matrix in class 2 were not known in any literature.

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