Students' Meaning-Making of Horizontal Function Translation with a Dynamic Visualization

By Yosep Dwi Kristanto[1,2], Teo Paoletti[3], Houssam El-Kasti[4], and Zsolt Lavicza[1]

[1]Department for STEM Didactics, Johannes Kepler University Linz, AUSTRIA
[2]Mathematics Education Study Program, Sanata Dharma University, Yogyakarta, INDONESIA
[3]School of Education, University of Delaware, USA
[4]Faculty of General Education, Qatar University, Doha, QATAR

yosepdwikristanto@usd.ac.id; teop@udel.edu; houssamkasti@gmail.com; zsolt.lavicza@jku.at

Received: 13 December 2025 Revised: 17 February 2026

DOI: 10.1564/tme_v33.2.07

Previous studies have shown that the conceptual complexity of horizontal function translation can pose challenges for students, motivating subsequent research to propose instructional approaches that support students in developing more productive ways of thinking about this transformation. This study investigates how students can use emergent graphical shape thinking (EGST) to construct meanings for horizontal function translation when supported by a dynamic visualization. A case study approach was employed to trace the learning progression of two students as they worked through a digital task incorporating this visualization. Their progression unfolded in three phases. First, they identified a relationship between the original function and its horizontal translation, though the direction of this relationship was the reverse of what was intended. Second, in an effort to refine the relationship between the functions, they established a correspondence relationship between the inputs of the original and translated functions. Finally, they used this correspondence relationship to generalize the functional relationship that characterizes a horizontal translation. How EGST and the dynamic visualization supported this meaning-making are subsequently discussed.

1. INTRODUCTION

Let $y = f(x)$ be the graph of a function. In many precalculus textbooks (e.g., Stewart et al., 2024; Sullivan, 2024), a horizontal function translation is defined as the transformation obtained by replacing the input variable x with $x - c$, where c is a constant. The graph of the resulting function $g(x) = f(x - c)$ represents a horizontal translation of the graph of f. When introducing function transformations many curricula describe such transformations as entailing a shifting or moving of a given graph to a new position; in the case of horizontal translations the graph moves left or right in a way that is often counterintuitive for students (i.e., the graph shifts right when c is positive and left when c is negative, the opposite of what many students anticipate; Yim & Lee, 2022). Beyond this intuitive challenge, horizontal function translation can also be conceptually complex. In the expression $g(x) = f(x - c)$, g is defined in terms of f, but the relationship between their inputs is reversed: the input of f must be expressed in terms of the input of g. This reversal can be potentially challenging for students when discovering the expression, as they may try to express the input of g in terms of the input of f instead of the other way around.

The example above illustrates the complexity of horizontal function translation and the challenges it poses for learners, consistent with prior research involving school students as well as preservice and in-service mathematics teachers (Yim & Lee, 2022; Zazkis et al., 2003). Researchers have attributed these challenges to the inherent complexity of horizontal translation (Baker et al., 2001), the counterintuitive graphical effect of its algebraic rule (Yim & Lee, 2022), students' unproductive meaning of functions (Baker et al., 2001; Lage & Gaisman, 2006), and various instructional obstacles (Yao, 2024; Zazkis et al., 2003). This body of work has led to a range of suggestions for teaching horizontal function translation, from rethinking how students conceptualize functions to drawing on multiple representations. In work focusing on multiple representations, many researchers have highlighted the affordances of technology for helping students make sense of function transformations (Anabousy et al., 2014; Borba & Confrey, 1996). With respect to reconceptualizing functions, Thompson and colleagues have argued for emphasizing a covariational meaning for functions (Oehrtman et al., 2008; Thompson & Carlson, 2017).

In the same vein, this study draws on the affordances of technology (Gibson, 2015) to promote a dynamic conceptualization of a function as a covariational relationship between two quantities that can be represented via a graph and an algebraic rule. To frame this work, we use emergent graphical shape thinking as a way of thinking that we aim to foster in students as they conceive function graphs and their transformations. We also draw on frameworks for dynamic visualizations to guide the design of tasks that support students' construction of meaning for horizontal function translation. In this study, we use meaning as the collection of actions, objects, or schemes an individual draws on when assimilating a situation (Thompson, 2015).

The paper is organized as follows. Section 2 outlines the theoretical framework, including emergent graphical shape thinking and learning from dynamic visualizations. Section 3 describes the task design and the research context. Section 4 presents an analysis of students' meanings for horizontal function translation, and Section 5 discusses the findings and their implications for research and instruction.

2. THEORETICAL FRAMEWORKS

2.1. Emergent graphical shape thinking

Emergent graphical shape thinking (EGST) involves conceptualizing a graph "in terms of what is made (a trace) and how it is made (covarying quantities)" (Moore & Thompson, 2015, p. 782). EGST emphasizes conceiving a graph as dynamically generated by the trace of a moving point determined by the covarying quantities, or viewing a completed graph as something that could be produced in this way (Paoletti et al., 2023). This stands in contrast to static graphical shape thinking, in which a graph is conceived as a manipulable shape, such as something that can be translated or rotated (Moore, 2021; Paoletti, Hardison, et al., 2022). Although static graphical shape thinking may be useful in school mathematics and is commonly used to introduce function transformations, such as by treating the transformed graph as a shift or dilation of the original graph, we argue that its value becomes limited when approaching function transformations that foreground quantities and their relationships, as is often the case in STEM contexts. For instance, such an approach does not sufficiently explain why a graph of global temperature anomaly over time should be shifted or dilated vertically or horizontally, nor what quantitative relationships are represented by these transformations. EGST, in contrast, offers a generative way of thinking about graphs that is productive across both mathematics and other STEM fields, as it foregrounds graphs as representations of relationships between quantities.

Paoletti et al. (2023) extend the idea of EGST by developing a framework that includes three interrelated main components: situational quantitative and covariational reasoning (M.S.), reasoning with graphical representations of covarying quantities (M.R.), and EGST itself (M.E.). According to this framework, for students to engage in M.E., they must engage in reasoning that bridges M.S. and M.R. (Paoletti et al., 2024). Put differently, constructing or interpreting a graph through EGST requires students to reason simultaneously about the covarying quantities situationally and graphically.

Moore and Thompson (2015) hypothesized that EGST has the potential to serve as a productive way of thinking about function transformations when instruction employs graphical representations of functions. We adopt this position and argue that EGST is particularly suited to supporting meaning-making for horizontal function translation because it frames graphs as dynamically generated traces of corresponding moving points defined by covarying quantities. By conceiving the original and translated graphs in this way, students may attend to invariant relationships across corresponding points (Borba & Confrey, 1996), for example, noticing that paired points share the same y-value while their x-values differ by a constant amount. Such coordination can support the inference that if the x-values of corresponding points differ by a constant c while their outputs remain equal, then $g(x) = f(x - c)$. In this sense, EGST may support students in developing an understanding of why the rule for horizontal function translation works (Yao, 2024), grounding symbolic expressions in relationships between quantities.

2.2. Learning from dynamic visualizations

Many studies have highlighted the potential of dynamic visualizations to support student learning, particularly for cognitively demanding tasks. Dynamic visualizations are those that can display change in space over time, either gradually or in a smooth, continuous manner (Ploetzner & Lowe, 2004). The effectiveness of such visualizations depends on at least two factors: their design and their content (Ehrhart & Lindner, 2023). Well-designed dynamic visualizations can help students more easily learn from the information presented (Schnotz & Rasch, 2008). They are also particularly well-suited for representing inherently dynamic content. Given these affordances, research in mathematics education frequently employs dynamic visualizations to help students make sense of mathematical ideas with dynamic characteristics, including function transformations.

Yao (2024) reports that many dynamic visualizations have been designed to support the teaching and learning of function transformations. Beyond being dynamic, these visualizations are also interactive, meaning that students have some control over how the changes are presented to them (Ploetzner & Lowe, 2004), providing opportunities to explore the transformations themselves. The value of interactivity is echoed in other studies, which suggest that interactive dynamic visualizations can support exploratory learning by enabling students to focus on specific variables of interest, generate hypotheses about relationships among those variables, and evaluate those hypotheses (Bodemer et al., 2004). Such opportunities for exploration are important for students as they learn about function transformations (Anabousy et al., 2014).

Dynamic visualizations, particularly those that present complex information, need to be designed in ways that help students identify relevant information. One way to achieve this is through cueing techniques. Such techniques not only guide students' attention toward essential information but also support them in relating and integrating different elements within the visualization (De Koning et al., 2009). This technique is implicitly suggested by Yao (2024) for designing dynamic visualizations whose goal is to help students discern the critical features of function transformations. To achieve this goal, students need cues that guide their attention to at least one pair of corresponding moving points on the graphs of the original and transformed functions (Yao, 2024). By attending to these points, students can observe how the quantities represented by each pair relate to one another, enabling them to identify the underlying transformation rule and, in turn, to understand why that rule works. In addition, providing a pair of corresponding moving points on the graphs can further support students in conceiving the graphs as representations of covarying quantities, aligning with the EGST framework described earlier.

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3. METHODS

This study aims to support students in using EGST to develop meanings for horizontal function translation through a digital task designed in alignment with the theoretical frameworks outlined earlier, namely EGST and research on learning from dynamic visualizations. Thus, the study addresses the following research question: How do students use EGST to construct meanings for horizontal function translation when supported by a dynamic visualization? To investigate this question, the study draws on a case study approach (Yin, 2018), examining how two preservice mathematics teachers engaged with the digital task.

3.1. Participants

The participants were two first-year female preservice mathematics teachers (PSMTs), referred to here by the pseudonyms Anna and Tamara, from a private university in Indonesia. In the Indonesian context, PSMTs enter university immediately after completing secondary school, where function transformations are part of the national secondary mathematics curriculum and are covered in the government-published textbook (Tohir et al., 2022). Prior to the task examined in this study, both participants had engaged in several EGST-oriented activities, including work on vertical translations of functions. The first author served as the teacher-researcher (TR).

3.2. Task design

The task used in this study was the Seeing the y-axis (SY) task. The SY task is part of a larger digital task sequence that connects function modeling and function transformations with a climate-change context (Kristanto & Lavicza, 2025). Given the inherent complexity of horizontal function translation, we deliberately focused on interpreting dynamically linked graphs rather than constructing the translated graph. This approach was intended to foreground the coordination of relationships between quantities and the identification of invariant relationships across corresponding points. In contrast, other transformations in the sequence, such as vertical translation, were approached through graph construction.

The goal of the SY task is to guide students in discovering the principle of horizontal function translation by examining the relationship between an original graph and its horizontally translated graph, together with the corresponding algebraic representations. To support this goal, the task is framed around a problem that asks students to identify the strategy used by a fictional character, Karuna, to make the y-axis visible. To guide students in identifying this strategy, they are provided with the original graph $y = T_2(x)$, which models temperature anomalies from 1880 to 2022, along with an interactive dynamic visualization that demonstrates how Karuna rescales the x quantity in T_2 so that the resulting graph, $y = T_3(x)$, begins at the y-axis or $x = 0$. The visualization was designed with three purposes. First, the visualization shows how the graphs of T_2 and T_3 are generated by depicting each as the trace of a moving point whose position is constrained by the quantities represented in the graph, which is intended to prompt students to interpret the graphs using EGST. Second, the visualization guides students' attention by highlighting corresponding pairs of points, one on the T_2 graph and one on the T_3 graph, that move simultaneously. Third, the visualization is interactive, allowing students to control its flow and enabling exploratory learning. Together, these design considerations aim to support students in using EGST to construct meanings related to horizontal function translation.

Figure 1 shows the visualization at two different moments.

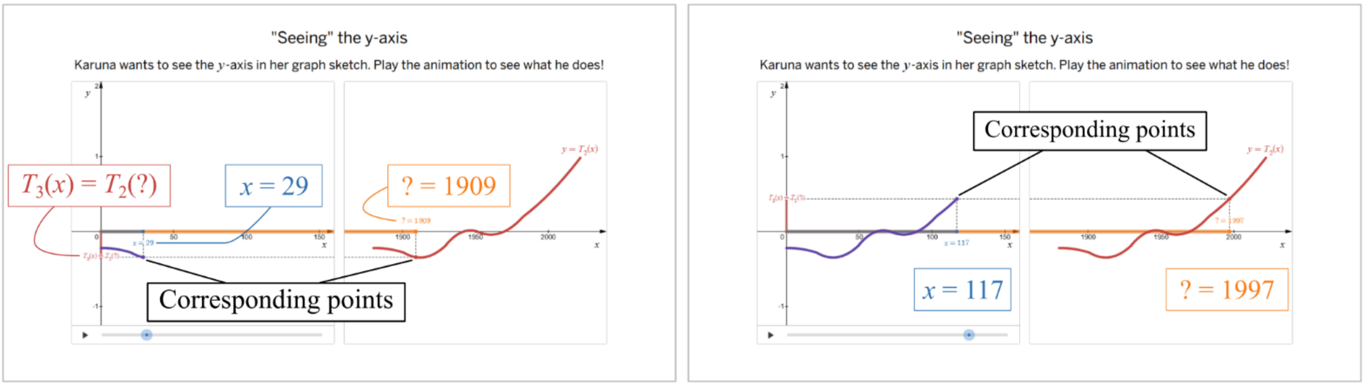

Figure 1. Two moments from the dynamic visualization in the Seeing the y-axis task

By interacting with the visualization, students can see how corresponding points on the T_3 and T_2 graphs move from left to right. They can also control the playhead, moving it forward or backward, to observe the effect on the points. For example, students can set the playhead so that the input of T_3 is 29, corresponding to an input of 1909 on T_2, with equal output values, i.e., $T_3(29) = T_2(1909)$ (Figure 1, left panel).

They can then move the playhead so that the input of T_3 is 117, corresponding to an input of 1997 on T_2, again with equal outputs, i.e., $T_3(117) = T_2(1997)$ (Figure 1, right panel). By observing how each point on T_3 corresponds to a point on T_2 throughout the dynamic visualization, students are expected to notice that the points share the same height and that the difference in their x-coordinates remains constant at 1880. Put differently, the two functions yield the same output whenever their inputs differ by 1880. From these observations, students are expected to infer the general relationship $T_3(x) = T_2(x + 1880)$, record it, and provide an explanation in the text boxes, as shown in Figure 2.

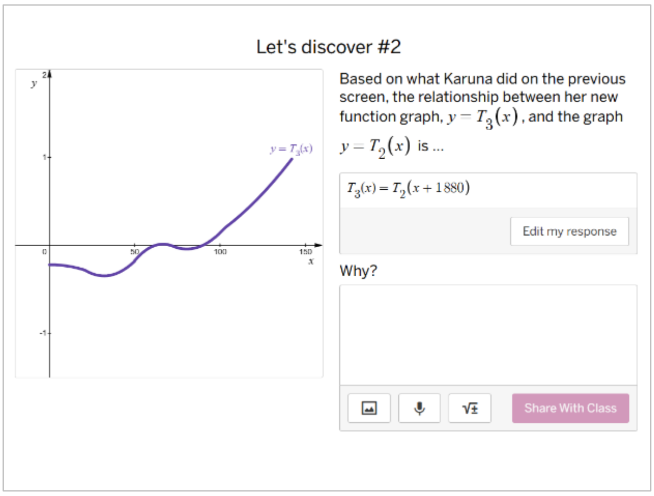

Figure 2. Text boxes for entering the inferred relationship and explanation

3.3. Analysis

The data source for this study was a video recording of Anna and Tamara as they worked on the SY task. The recording captured their dialogue, gestures, and interactions with the digital task environment. The video was transcribed to support further analysis. Consistent with a teaching experiment methodology (Steffe & Thompson, 2000), we conducted a retrospective analysis of the data. The first author began by identifying instances that offered insight into Anna's and Tamara's reasoning and then applied both generative and convergent approaches (Clement, 2000) to develop initial models of the students' mathematics. To interpret their mathematical activity, we drew on key components of Paoletti et al.'s (2023) EGST framework, including M.S. (situational quantitative and covariational reasoning), M.R. (reasoning with graphical representations of covarying quantities), and M.E. (emergent graphical shape thinking). The students' interactions with the task were also analyzed to investigate how the design supported the meanings they constructed.

The initial model was then discussed among all authors. The discussions focused on examining whether the model aligned with the students' sense-making by proposing and evaluating alternative explanations of their learning activity (Yin, 2018). This process led to a viable model that accounts for how the students used EGST to develop meanings for horizontal function translation through the SY task.

4. FINDINGS

In this section, we present Anna and Tamara's progression as they worked through the SY task. First, we describe their initial meanings, which led them to reverse the intended relationship between the two functions. Second, we illustrate how their attention to specific input correspondences, supported by instructional scaffolding, helped them refine this relationship. Lastly, we show how these developments enabled them to generalize the relationship between the functions in a way that reflects a coherent meaning for the horizontal translation.

4.1. Students' initial meanings and the reversal of the intended relationship

At the beginning of their work on the SY task, Anna and Tamara discussed how to approach the problem. Tamara was the first to propose a solution. Anna disagreed with her proposal, yet their exchange helped surface two important observations about T_3. They noted that the domain of T_3 spans from 0 to 142, and Anna pointed out that the inputs of T_2 and T_3 differ by roughly a thousand units. Their initial discussion is presented in Table 1.

Speaker	Transcript excerpt
Tamara	T_3 is… T_2, minus…
Anna	What?
Tamara	142.
Anna	142, right, from here to here [*points from the beginning to the end of T_3's domain on the x-axis*].
Tamara	Mm-hm. Or minus one…
Anna	No. I don't think so.
Tamara	That's y, $T_3(x)$.
Anna	Mm-hm. So the difference is… this one is from one thousand something… [*points to the left end of the T_2 graph*].
Tamara	Yes.
Anna	This one is from zero [*points to the left end of the T_3 graph*]. From zero, right, that's the value.
Tamara	Uh-huh.

Table 1. Initial discussion of Anna and Tamara

There are two central points in Anna and Tamara's discussion in Table 1. First, they shared an understanding of what the task required them to do, namely, to express T_3 in terms of T_2. Second, their discussion focused on the difference in the x-coordinates of corresponding points on the graphs of T_2 and T_3, initially focusing on the domain of functions (i.e., 142) before turning to the difference between corresponding x-values with the same y-value. Their search for this difference was grounded in their ability to interpret the graphs (M.R.).

Building on this, Anna searched for a more precise difference between those corresponding x-coordinates. She did so by comparing the left endpoints of the two graphs, i.e., 1880 and 0, and determining their separation, which she found to be 1880. This activity shows that she was able to extract quantitative information from the graphs, particularly from their left endpoints. We therefore infer that she was engaging in situational quantitative reasoning (M.S.). With this in place, she formed an initial conjecture, expressing it tentatively as:

> So T_3 is… T_2 minus… so that the result is zero. 1900. 1880?

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We see two possible interpretations of what Anna may have meant. One possibility is that she intended $T_3(x) = T_2(x) - 1880$. Alternatively, she may have been expressing the idea that, to obtain the same T_3 and T_2 values, the x-value for T_3 is equal to the x-value for T_2 minus 1880. In symbolic terms, this can be written as $T_3(x - 1880) = T_2(x)$. We consider the second interpretation more plausible because it aligns with the reasoning that both students used afterwards when testing Anna's initial conjecture.

After formulating her initial conjecture, Anna used the interactivity of the dynamic visualization to test it. She moved the playhead back and forth and stopped when the visualization displayed corresponding points with x-coordinates 103.5 on the T_3 graph and 1983.5 on the T_2 graph. She then asked Tamara to compute the difference between these two x-coordinates, saying:

> Try this. This one [*points to 1983.5 on the T_2 graph*] minus this one [*points to 103.5 on the T_3 graph*]. How much?

Tamara calculated the difference between the two x-coordinates using her calculator, obtained 1880, and reported it to Anna. In response, Anna moved the playhead again, appearing to test her conjecture with a second pair of x-values visually. After a moment, she stated her conjecture that "T_2 … x minus one thousand nine …, uh, 1880" to the TR.

Anna and Tamara's activities in testing Anna's initial conjecture provide additional evidence that she was determining the difference between the x-coordinates of corresponding points. They were arguing that $T_3(x - 1880) = T_2(x)$, but were not yet aware that this was one way to represent this relationship. In testing the conjecture, Anna used the interactive features of the visualization, which displayed pairs of corresponding points whose heights were always equal. Anna, assisted by Tamara, then determined the difference between their corresponding x-coordinates and found this difference to be 1880.

Throughout the process of formulating and testing the initial conjecture, there was no evidence that Anna or Tamara referred to the heights or y-coordinates of the corresponding points. This does not imply that they failed to attend to them; rather, we suspect they did not mention them because the dynamic visualization made it clear that the points always shared the same height. Anna's engagement in both M.S. and M.R. while examining the pattern relating corresponding points on T_2 and T_3 graphs was supported by the cueing and interactive features of the dynamic visualization. Her involvement in these two components of EGST may have helped her build reasoning toward M.E.

Anna's engagement in M.S. is particularly interesting for one reason. Her M.S. engagement operated independently of the context underlying the construction of T_2, which represents the relationship between time (in years) and temperature anomaly (in °C). When she extracted information from points on this graph, especially their x-coordinates, she did not explicitly interpret them as times (in years) but instead treated them as real numbers in general. She proceeded similarly when extracting information from points on the T_3 graph. Her strategy of treating the x-coordinates of these points as real numbers detached from their contextual meaning, in a way, supported her in identifying the relationship between the corresponding x-coordinates on the T_2 and T_3 graphs. This approach was efficient for her at that moment to achieve her goal of defining T_3 in terms of T_2. She understood that this goal did not require her to make sense of these rules in terms of the quantities in the original context (M.R. without explicit M.S.).

4.2. Refining the relationship through input correspondences

In the preceding section, we described how Anna and Tamara initially approached the SY task by expressing T_2 in terms of T_3. In response, the TR directed their attention to the hint provided in the visualization, which displayed "$T_3(x) = T_2(?)$" (Figure 1). Using this cue, the TR encouraged them to express T_3 in terms of T_2 rather than the reverse, since T_2 was the given function. The TR then prompted them to use the dynamic visualization to record several pairs of corresponding points from the graphs of T_2 and T_3 and to determine the relationships between the quantities represented by those points. After examining several pairs in the visualization, Anna recorded her observations, as shown in Figure 3.

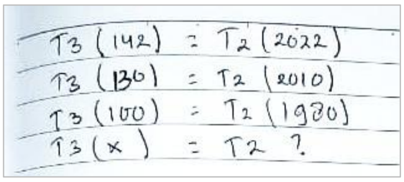

Figure 3. Anna's notes showing the relationship between T_3 and T_2

The notes shown in Figure 3 prompted Anna and Tamara to find a relationship between the inputs of T_3 and those of T_2. After a while, Anna noticed, "So every T… T_3 x [*meaning the input of T_3*] increases by 10, then T_2 x [*meaning the input of T_2*] also increases by 10." Based on this observation, she concluded and stated it aloud that $T_3(x) = T_2(x)$.

Although Anna's final statement was not valid, she was reasoning covariationally as she sought a relationship between the corresponding inputs of T_3 and T_2. In doing so, she identified that a change in one quantity was always accompanied by an equal change in the other. We infer that this perceived sameness in the magnitude of change across the two quantities led her to conclude that $T_3(x) = T_2(x)$. In this moment, her way of reasoning was not yet productive for explicitly expressing the input of T_2 in terms of the input of T_3.

Noticing that Anna and Tamara were still struggling to express the input of T_2 in terms of the input of T_3, TR invited them to revisit the problem. In this discussion, TR adopted a different approach by encouraging them to determine how each input of T_3 maps to a corresponding input of T_2. He did this by asking them to find the value of the T_2 input for a given value of the T_3 input. This discussion is presented in Table 2.

Speaker	Transcript excerpt
TR	If x is zero [*pointing to the x-value of T_3*], here [*pointing to the corresponding x-value of T_2*]?
Anna	Zero [*answered spontaneously*].
Tamara	Zero [*answered spontaneously, as if parroting Anna*].
Anna	1880 [*looks at the T_2 graph and corrects her answer*].
Tamara	1880 [*looks at the T_2 graph also and corrects her answer*].
TR	[*Moves the playhead so the x-value of T_3 is 19*] If here is 19, then?
Tamara	1999, 1899 [*looks at the corresponding x-value of T_2*].
TR	If I cover this [*closing the T_2 graph, then moves the playhead so x of T_3 is 35*], if here is 35, what would it be approximately?
Anna	1880 plus 35… [*muttering*].
Anna	[*Appears to calculate in her notebook, then states the result*] 1915!
TR	How do you do it?
Tamara	Add.
TR	With?
Tamara	With, this one [*points to the graph, but not specifically*]. What I mean is… The 1980 one [*points to the left end of the T_2 graph, meaning she actually refers to 1880*].
Anna	Yes.

Table 2. Discussion revealing how T_3 inputs map to T_2 inputs

At the end of the discussion shown in Table 2, Anna and Tamara determined that for any input x of T_3, the corresponding input of T_2 is x + 1880. Reaching this conclusion was supported by their meaning that the correspondence between the inputs of T_3 and T_2 could be viewed as an input-output assignment. In this case, the input is the value from T_3, and the output is the associated value from T_2. Using this perspective, they initially identified outputs by directly observing the visualization. Finally, TR prevented them from seeing the output directly by covering the portion of the screen that displayed the T_2 input (Figure 4). This move prompted them to formulate a rule that would allow them to compute the output once the input was known.

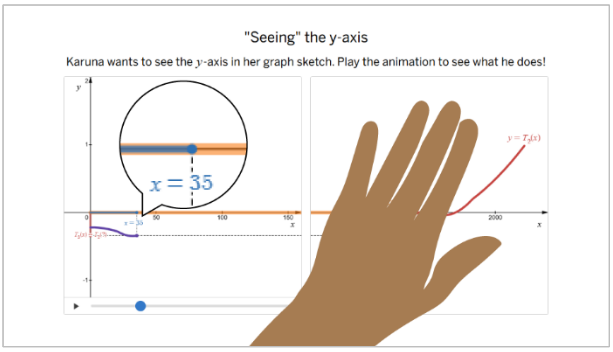

Figure 4. Illustration of the TR covering the T_2 input

4.3. Generalizing the relationship between the functions in the horizontal translation

In the previous section, Anna and Tamara had identified a rule that maps the inputs of T_3 to the corresponding inputs of T_2. Building on this, TR asked them to test the rule using the pairs Anna had previously recorded (Figure 3). They applied the rule to those numerical pairs and confirmed that it still held. TR then posed new problems (i.e., "For $T_3(10)$, what is it equal to? For $T_3(50)$, what is it equal to in T_2?") which they answered correctly without relying on the dynamic visualization, instead using their rule. Hence, we infer that each student could connect the rule to the relationship between T_3 and T_2.

The TR then asked Anna and Tamara to generalize their finding by asking, "$T_3(x)$ equals?" Anna responded "1880 plus x," indicating $T_2(1880 + x)$. This suggests that Anna, at least, identified the relationship between T_3 and T_2. This interpretation is supported by their written response in the text box (Figure 2), where they stated "$T_3(x) = T_2(1880 + x)$". In addition to stating this relationship, they also justified: "Since Karuna wanted to see the y-axis, he started from the point 0, which led to the function $T_3(x) = T_2(1880 + x)$."

We argue that Anna and Tamara's conclusion about the relationship between T_3 and T_2 emerged from the meanings they had already developed for the two graphs and for the quantities represented in them. As described earlier, the visualization provided in the SY task supported their engagement in the kinds of reasoning that move toward M.E. as they conceived both graphs. This conception, combined with the visualization that directed their attention to corresponding point pairs on T_3 and T_2 graphs, helped them identify the relationship between the two functions, even though this relationship was not trivial for them. The visualization allowed them to notice two key patterns in the corresponding points: first, the heights or y-coordinates always matched, and second, there was an invariant pattern in the x-coordinates. By synthesizing these two relationships, they concluded that $T_3(x) = T_2(1880 + x)$. They then recontextualized this conclusion in terms of the original problem. They reasoned that, to see the y-axis, the new graph needed to begin at $x = 0$, which led them to express the functional relationship as $T_3(x) = T_2(1880 + x)$.

5. DISCUSSION AND CONCLUSION

We have used the case of Anna and Tamara to describe how students progressed in constructing meanings for horizontal function translation. Considering their progression, we argue that they held a meaning for this equation by conceiving, graphically, the height of T_3 at any value x as matching the height of T_2 at the corresponding value 1880 + x. Thus, their discovery of the horizontal translation rule was supported by what Paoletti et al. (2023) describe as reasoning with graphical representations. Oehrtman et al. (2008) describe this kind of meaning as a powerful way of making sense of horizontal function translation and as reinforcing a process view of functions.

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The interactivity of the dynamic visualization supported Anna and Tamara in exploring and identifying the relationship between a function and its horizontal translation. The interactive features enabled them to formulate an initial conjecture based on their observations of the visualization. In addition, they used these features to generate alternative cases to test whether those cases were consistent with their initial conjecture. The interactive features were also leveraged by the TR to provide scaffolding that helped Anna and Tamara revise their initial conjecture. This aligns with findings from previous studies suggesting that interactive dynamic visualizations can offer opportunities for exploratory learning (Bodemer et al., 2004; Wichmann & Timpe, 2015).

Our study shows that Anna and Tamara's construction of quantitative meaning for horizontal function translation is not straightforward. This is evident in their initial conjecture, which described the known function in terms of its translated version, the opposite of what we intended. We suspect this happened because of the conceptual complexity of horizontal translation (and horizontal transformations in general) described in the introduction. In such transformations, students must express the transformed function in relation to the original, but they must do so by describing the original function's input in terms of the transformed function's input. In other words, the direction of the relationship between the functions is reversed relative to the relationship between their respective inputs, a conceptual shift that can pose challenges for students. This possible explanation extends those offered in prior studies that have identified sources of students' difficulties in learning horizontal function transformations (Yim & Lee, 2022; Zazkis et al., 2003).

Generalizing the pattern between the corresponding inputs of the original function and its translated version was also not straightforward for Anna and Tamara. They first reasoned covariationally to understand how these input values related, eventually noticing that the changes in the two inputs had the same magnitude. They were then supported in repeatedly determining the input of the original function, given the input of the translated function. This process led them to identify what Smith (2008) refers to as a correspondence relationship. We therefore argue that opportunities to discern how one value depends on another, provided repeatedly across varied input pairs, supported their discovery of this correspondence relationship. This is consistent with findings from previous studies (Chimoni et al., 2023; Ureña et al., 2024).

Our study carries several implications. Our findings suggest that, in this case, students' reasoning toward EGST while interpreting the graphs of an original function and its horizontal translation supports the development of productive meanings for this transformation. This appears to contrast with approaches that introduce horizontal translation by treating the graph as a static shape to be shifted (i.e., through static graphical shape thinking), a method that can make the transformation more counterintuitive for learners.

We note that while the students likely had learned about function transformations, neither directly connected their prior activity to the SY task, as their prior instruction on the topic likely involved static graphical shape thinking; thinking of the graph as shifting is not a productive way to address this task. Instead, our approach focused on maintaining the same underlying relationship but with a change in reference point. Future research could build on this work by exploring instructional designs that foreground changes in the reference point of the independent variable (e.g., redefining the origin to represent a different reference year) as a quantitative way to understand horizontal translation. For instance, tasks that ask students to coordinate a redefinition of the vertical axis with corresponding algebraic expressions, or to select among alternative graphs representing the same relationship under different reference frames, may further support students in linking graphical, algebraic, and contextual meanings of horizontal shifts.

Before closing this article, we acknowledge several limitations of the study. First, the meanings for horizontal function transformation that we have described are based on the reasoning of two students; other students may construct different meanings. Future research could investigate how students from other populations develop meanings for this transformation when supported by EGST. Second, the meanings constructed by Anna and Tamara in this study arose from their interpretations of the graphs presented in the dynamic visualization. It would be valuable to examine how such meanings might emerge not from interpreting given graphs, but from constructing the graph of the translated function based on the original graph.

Our study extends ongoing lines of inquiry into how students can be supported in making sense of horizontal function translation, how EGST can foster students' understanding of graphs and related mathematical ideas, and how students learn with dynamic visualizations. From this study, we are persuaded that EGST can support students in developing productive meanings for horizontal function translation. Future research can investigate whether EGST is similarly productive for supporting students' meaning-making for other function transformations or function operations more generally.

ACKNOWLEDGEMENTS

The authors would like to thank the participants for their time and thoughtful engagement in this study. This study received partial support from OeAD-GmbH in cooperation with ASEA-UNINET, as it forms part of Y.D.K.'s PhD study funded through the Ernst Mach Grant. T.P. also acknowledges support from the National Science Foundation under grant NSF DRL-2142000.

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BIOGRAPHICAL NOTES

Yosep Dwi Kristanto is currently a PhD student at Johannes Kepler University (JKU) Linz, whose research focuses on supporting students' covariational and graphical reasoning to develop mathematical ideas and ideas beyond mathematics.

Teo Paoletti is a researcher at the University of Delaware whose recent work involves designing digital tasks to support students in developing meanings for various function classes, graphs, and inequalities.

Houssam El-Kasti is a mathematics educator at Qatar University with expertise in mathematics education and educational technology, contributing effectively to work in higher education.

Zsolt Lavicza is a professor of STEM education research methods at JKU Linz, working on projects related to technology integration and serving as the head of the PhD programme in STEAM Education.

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