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A THEORY OF DIRAC INTEGRAL SPACES AND THE ISOMORPHISM BETWEEN THE CATEGORY OF SUCH SPACES AND THE CATEGORY OF LEBESGUE MEASURE SPACES

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Chapter 1

Introduction

In his book published in 1930, Dirac [14] employed intuitive concepts of integrals to formulate the principles of Quantum Mechanics. He never defined the integrals, but he postulated certain conditions that they had to satisfy so that one could describe the evolution of dynamical systems in Quantum Mechanics.

In 1902, Lebesgue [22] discovered the modern Theory of Integration of real-valued functions, based on the Measure Theory. Daniell [12][13], in 1917, introduced the construction of the integrals using positive linear functionals defined on a vector lattice. Later in 1948 Stone [31] clarified the relation between the Lebesgue integral and the Daniell integral by proving that they are equivalent. Bochner [4] generalized the Lebesgue Integration Theory to the case of Banach-space-valued functions in 1933. At about the same time Kolmogorov [21] used the Lebesgue Integration Theory to prove the Strong Law of Large Numbers and thus put the Theory of Probability and Statistics on a precise mathematical footing.

In [5][6] Bogdan generalized and simplified the development of the Lebesgue-Bochner integral. In these papers the theory was constructed by means of extensions of primitive structures to richer ones. In [7][8][9] Bogdan showed that one could axiomatize the theory of spaces, which include the classical Lebesgue L_p spaces of real-valued functions, and develop the theory directly from the axioms.

In [10] Bogdan presented a new approach to the Theory of Probability via

Algebraic Categories. In this paper he showed that Probability Theory involves five isomorphic categories, namely the category of probability measures, the category of expectation spaces, the category of distributions, the category of characteristic functions, and the category of Baire expectation spaces.

Kolmogorov's Strong Law of Large Numbers thus can be translated into the language of any of the categories involved. The law provides the theoretical foundations for statistical experiments. Since Quantum Mechanics involves statistical experiments, it is clear that the underlying integrals should have the properties of the Lebesgue integrals. In classical Probability Theory the domains of the Lebesgue integrals consist of real-valued functions. In Quantum Mechanics, however, the domains of the underlying integrals should consist of complex-valued functions.

To describe a state of a dynamical system in Quantum Mechanics, Dirac utilized a vector which he called *ket* vector and denoted by $|v>$. He assumed that the set V of all ket vectors forms a vector space with respect to addition and scalar multiplication. The set V^* of all linear functionals from V into C is also a vector space. The members of V^* were called *bra* vectors. He introduced the notation $<\phi|$ to denote a bra vector and $<\phi|v>$ the image of a ket vector $|v>$ under a bra vector $<\phi|$, which is a complex number. Dirac assumed that

1. There exists a one-to-one correspondence between V and V^*, i.e. there exists a map $\sigma : V \rightarrow V^*$ which is one-one and onto. Let $\sigma(|v>) = <v|$ for every $|v> \in V$. He called $<v|$ the bra vector corresponding to the ket vector $|v>$.

2. The map σ is conjugate linear, that is $\sigma(|v> + |w>) = <v| + <w|$ and $\sigma(\lambda\ |v>) = \overline{\lambda}\ <v|$ for any ket vectors $|v>$ and $|w>$ and any complex number λ.

The scalar product of the ket vector $|v>$ by the ket vector $|w>$ is defined as the complex number $<w|v>$, where $<w|$ is the bra vector corresponding to the ket vector $|w>$. Dirac assumed that

1. $<v|v> \geq 0; \quad <v|v>= 0 \quad \Leftrightarrow \quad |v>= 0.$

2. $<w|v> = \overline{<v|w>}.$

He made further assumption that V is a Hilbert space, i.e. it is complete with respect to the norm $\|\cdot\|$ defined by the formula

$$\| \, |v> \| = \sqrt{<v|v>}$$

for all $|v> \in V$. Analogous properties of the bra vectors also led him to the assumption that the space V^* is complete.

Using the basic kets $|\xi>$ where the ξ's are discrete, he expressed the scalar product of ket vectors as a summation:

$$<w|v> = \sum_{\xi} <w|\xi> \overline{<v|\xi>}.$$

For continuous ξ the scalar product can be written in the form of an integral:

$$<w|v> = \int <w|\xi> \overline{<v|\xi>} \, d\xi.$$

Dirac never gave an explicit characterization of his integral, but from his assumption that both V and V^* are complete we can conclude that it should have the properties of the Lebesgue integral.

In this dissertation we develop a theory of such integrals which we call Dirac integrals. From the above discussion it is obvious that the underlying space of functions of a Dirac integral should be an algebra of complex-valued functions which is closed under involution (conjugation). In addition we also stipulate that this algebra should be closed under pointwise convergence and should contain constant functions. The theory is established from a set of axioms which we postulate as the defining properties of the Dirac Integral Space. Then we prove that the category of Dirac Integral Spaces (DIS) is isomorphic to the category of Lebesgue Measure Spaces (LMS).

This study provides a precise mathematical framework for the axiomatic development of Quantum Mechanics and allows us to prove many theorems which were derived heuristically by Dirac. The possibility of such development is suggested by the results of Bogdan [7][10]. A recent monograph of Kelley and Srinivasan [20] developes an approach to the Lebesgue Integral Spaces from the axioms of the integrals defined on real-valued functions.

The content of this dissertation is organized as follows. After the Introduction in Chapter 1, we present some important results on the Baire algebra of complex-valued functions in Chapter 2. We are especially interested in a certain kind of such Baire algebra, namely the space of all compositors $u: C^T \to C$. We prove that this space of compositors is precisely the Baire space $B(C^T, C)$. In this Chapter we also introduce the notion of a Baire algebra morphism φ from a Baire algebra A into a Baire algebra A', and show that every such morphism commutes with any compositor.

In Chapter 3 we investigate the relation between rings of subsets of a space X and algebras of complex-valued functions defined on X. We prove that if V is a σ-algebra of subsets of X, then the space $M(V, C)$ of V-measurable complex-valued functions defined on X forms a Baire algebra. Conversely, if L is a Baire algebra of functions in C^X, then $V = \text{trace}(L)$ is a σ-algebra of subsets of X such that $L = M(V, C)$.

The Dirac Integral Space theory is developed in Chapter 4. Starting from the axioms of the Dirac Integral Space we develop the theory of such spaces. In this Chapter we establish the convergence theorems and prove the topological completeness of the space of summable functions.

Chapter 5 is devoted to generating a Dirac Integral Space from a Lebesgue Measure Space. We begin with a positive linear functional defined on the space $S(W, C)$ of simple functions, which we extend to a richer space of functions. We then further extend this functional to the space of summable functions and prove that this extension is indeed a Dirac Integral.

The Category of Dirac Intergal Spaces (DIS) and the Category of Lebesgue Measure Spaces (LMS) are investigated in Chapter 6. Using the Theorems which have been proved in the previous chapters, we construct functors $F\colon \mathrm{DIS} \to \mathrm{LMS}$ and $G\colon \mathrm{LMS} \to \mathrm{DIS}$ such that $F \circ G$ and $G \circ F$ are both identity functors. In other words, we prove that the two categories are isomorphic.

Chapter 2

Baire Algebras of Functions

Let X be an abstract set and C the field of complex numbers. Denote by C^X the space of all functions from X into C. A subset A of the space C^X is called an *algebra* if A is closed under addition of functions, muliplication of functions and scalar multiplication. An algebra A is said to be *closed under pointwise convergence* if whenever f_n is a sequence of functions in A such that $f_n(x) \longrightarrow f(x)$ for every $x \in X$, then we have $f \in A$. As usual we denote the conjugate of a complex number z by $\bar{z}$. We say that an algebra A is *closed under involution* if $f \in A$ implies $\bar{f} \in A$, where $\bar{f}(x) = \overline{f(x)}$ for every $x \in X$. An algebra A in C^X will be called *Baire Algebra* if it is closed under pointwise convergence and under involution and contains all constant functions. It is obvious that C^X itself is a Baire algebra and it is easy to verify that the intersection of any collection of Baire algebras is also a Baire algebra.

If f is a function from X to Y and g is a function from Y to Z, then the composition $g \circ f$ of the two functions is a function from X to Z defined by the formula

$$(g \circ f)(x) = g(f(x))$$

for every $x \in X$.

Let Re and Im denote functions from C into the field of real numbers R such that

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2} \quad \text{and} \quad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i}$$

for every $z \in C$. Then it is obvious that for every function f belonging to a Baire algebra A in C^X we have

$$(\mathrm{Re} \circ f) \in A \quad \text{and} \quad (\mathrm{Im} \circ f) \in A. \tag{2.1}$$

Proposition 2.1 *If B is any nonempty subset of C^X, then there exists the smallest Baire algebra A in C^X containing B.*

Proof. Let $\mathcal{A}$ be the family of all Baire algebras in C^X containing B. Then $\mathcal{A}$ is non-empty, since $C^X \in \mathcal{A}$. By the remark in the above paragraph, $D = \bigcap_{A \in \mathcal{A}} A$ is a Baire algebra and therefore D is the smallest Baire algebra in C^X containing B. ∎

2.1 The Space of Compositors

Let T be any nonempty set and C^T denote the set of all elements of the form $z = (z_t)_{t \in T}$, where $z_t \in C$ for every $t \in T$. Let $\{f_t\colon\ t \in T\}$ be a subset of C^X. By $(f_t)_{t \in T}$ we mean a function f from X into C^T such that for every $x \in X$

$$f(x) \ = \ (f_t(x))_{t \in T}.$$

The function $f = (f_t)_{t \in T}$ is said to be generated by the set $\{f_t\colon\ t \in T\}$.

Let $\mathrm{Comp}(T)$ denote the space of all functions $u\colon C^T \to C$ such that for every Baire algebra A and every function $f = (f_t)_{t \in T}$ generated by a subset $\{f_t\colon\ t \in T\}$ of A, we have

$$(u \circ f) \in A.$$

The elements of the space $\mathrm{Comp}(T)$ are called *compositors of order T*.

If $\alpha \in T$, then a function $p_\alpha\colon C^T \to C$ defined by the relation

$$p_\alpha(z) = z_\alpha$$

for every $z = (z_t)_{t\in T} \in C^T$, is called *the projection onto the α-th coordinate.*

We recall that if E is a subset of X then the *characteristic function* c_E of the set E is defined by

$$c_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

for every $x \in X$.

Lemma 2.2 *Let T be any nonempty set and $f = (f_t)_{t\in T}$ be the function generated by a subset $\{f_t\colon\ t \in T\}$ of C^X. Then for every functions u and v from $C^{(C^T)}$ and any complex number λ, the following statements hold:*

1. $(u+v)\circ f \ = \ (u\circ f)+(v\circ f)$.
2. $(uv)\circ f \ = \ (u\circ f)(v\circ f)$.
3. $(\lambda u)\circ f \ = \ \lambda(u\circ f)$.
4. $\bar{u}\circ f = \overline{u\circ f}$.
5. *If u_n is a sequence of functions from C^T into C converging pointwise to u, then $u_n\circ f$ is a sequence of functions from X into C converging pointwise to the function $u\circ f$.*
6. *If c_X is the characteristic function of the set X, then $c_{(C^T)}\circ f = c_X$.*
7. *If $p_\alpha\colon C^T \to C$ for some $\alpha \in T$ is the projection onto the α-th coordinate, then $p_\alpha \circ f = f_\alpha$.*

Proof. For every $x \in X$ we have

$$\begin{aligned} ((u+v)\circ f)(x) &= (u+v)(f(x)) \\ &= u(f(x)) + v(f(x)) \\ &= (u\circ f)(x) + (v\circ f)(x) \\ &= ((u\circ f)+(v\circ f))(x) \end{aligned}$$

which proves (1). Statements (2) and (3) are proved in similar ways.

To prove (4), we observe that

$$\begin{aligned}(\overline{u} \circ f)(x) &= \overline{u}(f(x)) \\ &= \overline{u(f(x))} \\ &= \overline{(u \circ f)(x)} \\ &= \overline{(u \circ f)}(x)\end{aligned}$$

for every $x \in X$.

Let u_n be a sequence of functions from C^T into C such that $u_n(z) \longrightarrow u(z)$ for every $z = (z_t)_{t \in T} \in C^T$. Then $(u_n \circ f)(x) = u_n(f(x)) \longrightarrow u(f(x)) = (u \circ f)(x)$ for every $x \in X$, which proves (5).

If $c_{(C^T)}$ is the characteristic function of the space C^T, then $c_{(C^T)}(z) = 1$ for every $z \in C^T$. Thus for every $x \in X$, we have $(c_{(C^T)} \circ f)(x) = c_{(C^T)}(f(x)) = 1 = c_X(x)$. This proves (6).

Let $p_\alpha : C^T \to C \quad (\alpha \in T)$ be the projection onto the α-th coordinate. Then

$$\begin{aligned}(p_\alpha \circ f)(x) &= p_\alpha(f(x)) \\ &= p_\alpha((f_t(x))_{t \in T}) \\ &= f_\alpha(x)\end{aligned}$$

for every $x \in X$. Hence $p_\alpha \circ f = f_\alpha$. ∎

Proposition 2.3 *The space* Comp(T) *of compositors of order* T *forms a Baire algebra containing the set* $P = \{p_\alpha \colon \alpha \in T\}$ *of all the projections.*

Proof. Let A be any Baire algebra contained in the space C^X and $\{f_t \colon t \in T\}$ be any subset of A generating the function $f = (f_t)_{t \in T}$. For any two functions u and v in Comp(T), the functions $(u \circ f)$ and $(v \circ f)$ are both in the Baire algebra

A. From Lemma 2.2 and the fact that A is an algebra, the functions $(u+v)\circ f$, and $(uv)\circ f$, and $(\lambda u)\circ f$ are all in the algebra A. This means that the functions $(u+v)$, (uv) and (λu) are in Comp(T). Thus the space Comp(T) is an algebra.

Let u_n be a sequence of functions in Comp(T) such that $u_n(z) \longrightarrow u(z)$ for every $z=(z_t)_{t\in T}\in C^T$. Then $(u_n\circ f)\longrightarrow(u\circ f)$. Since $(u_n\circ f)\in A$ and A is closed under pointwise convergence, we conclude that $(u\circ f)\in A$. Thus $u\in$ Comp(T), which shows that the space Comp(T) is closed under pointwise convergence.

If $u\in$ Comp(T), then $(u\circ f)\in A$ and, since A is closed under involution, $\overline{(u\circ f)}\in A$. But $\overline{u\circ f}=\overline{u}\circ f$. So $(\overline{u}\circ f)\in A$, hence $\overline{u}\in$ Comp(T). Thus Comp(T) is closed under involution.

The function $c_{(C^T)}\circ f=c_X \in A$, since the algebra A contains all constant functions. Thus $c_{(C^T)}\in$ Comp(T). Since the space Comp(T) is an algebra, we conclude that it contains all constant functions. Thus we have proved that the space Comp(T) of compositors of order T is a Baire algebra.

Now, let $p_\alpha\colon C^T\to C$ $(\alpha\in T)$ be any projection. Then $p_\alpha\circ f=f_\alpha\in A$. Hence $p_\alpha\in$ Comp(T) for every $\alpha\in T$. This shows that the space Comp(T) contains all the projections. ∎

Theorem 2.4 *The space* Comp(T) *of compositors of order T is the smallest Baire algebra in $C^{(C^T)}$ containing the set $P=\{p_\alpha\colon\ \alpha\in T\}$ of all the projections.*

Proof. Let Q be the smallest Baire algebra in $C^{(C^T)}$ containing the set P of all the projections. Such Baire algebra exists by Proposition 2.1. It is the intersection of all Baire algebras in $C^{(C^T)}$ containing the set P. Since the space Comp(T) is a Baire algebra in $C^{(C^T)}$ containing the set P, we must have

$$Q\subset \text{Comp}(T). \tag{2.2}$$

Conversely, given any function $u \in \text{Comp}(T)$ and any Baire algebra A in $C^{(C^T)}$ containing the set P of pojections, we have $(u \circ (p_t)_{t\in T}) \in A$. But

$$\begin{aligned}(u \circ (p_t)_{t\in T})(z) &= u((p_t)_{t\in T}(z)) \\ &= u((z_t)_{t\in T}) \\ &= u(z)\end{aligned}$$

for every $z = (z_t)_{t\in T} \in C^T$, which implies that $u \in A$. So u belongs to every Baire algebra A containing P, which means that $u \in Q$. Thus we have the inclusion

$$\text{Comp}(T) \subset Q. \tag{2.3}$$

Hence from (2.2) and (2.3) follows $\text{Comp}(T) = Q$. ∎

Corollary 2.5 *If A is a Baire algebra in $C^{(C^T)}$ containing all the projections and $A \subset \text{Comp}(T)$, then $A = \text{Comp}(T)$.*

Proof. The above conclusion follows from the hypothesis and the fact that $\text{Comp}(T) \subset A$, since $\text{Comp}(T)$ is the smallest of all such Baire algebras. ∎

2.2 Baire Spaces of Functions

Let X and Y be topological Hausdorff spaces and $\mathcal{C}(X,Y)$ denote the space of all continuous functions from X into Y. Consider the family $\mathcal{F}$ of all sets S of functions in Y^X such that S contains $\mathcal{C}(X,Y)$ and S is closed under pointwise convergence. Let

$$B(X,Y) = \bigcap_{S\in\mathcal{F}} S.$$

This set $B(X,Y)$ will be called *Baire Space*, and its members are called *Baire functions*. It is easy to see that $B(X,Y)$ contains $\mathcal{C}(X,Y)$ and is closed under pointwise convergence. This means that the Baire space is the smallest class of functions in the space Y^X containing all continuous functions and being closed under pointwise convergence.

In this section we are going to show that Comp(T) is the Baire space of functions in $C^{(C^T)}$. For that we need several Lemmas.

Proposition 2.6 *Let X, Y and Z be topological Hausdorff spaces. If $f \in B(X,Y)$ and $g \in B(Y,Z)$, then $(g \circ f) \in B(X,Z)$.*

Proof. Let g be any continuous function in $C(Y,Z)$, and

$$P_g = \{f \in Y^X : (g \circ f) \in B(X,Z)\}.$$

If $f \in C(X,Y)$, then $(g \circ f) \in C(X,Z)$; therefore $(g \circ f) \in B(X,Z)$, which means that $f \in P_g$. Thus $C(X,Y) \subset P_g$. Take any sequence f_n in P_g, which converges pointwise to some function f. Then $g \circ f_n$ is a sequence of functions in $B(X,Z)$. But by the continuity of g we get $(g \circ f_n)(x) = g(f_n(x)) \longrightarrow g(f(x)) = (g \circ f)(x)$ for every $x \in X$. Hence $(g \circ f) \in B(X,Z)$, which means that $f \in P_g$. So P_g is a class of functions in Y^X which contains all continuous functions and is closed under pointwise convergence. From this it follows that $B(X,Y) \subset P_g$. So if we take any function $g \in C(Y,Z)$ and $f \in B(X,Y)$ we have $(g \circ f) \in B(X,Z)$.

Now, take any function $f \in B(X,Y)$, and consider the set

$$Q_f = \{g \in Z^Y : (g \circ f) \in B(X,Z)\}.$$

Using similar arguments we can show that Q_f is a set of functions in Z^Y containing all continuous functions and being closed under pointwise convergence. Therefore $B(Y,Z) \subset Q_f$. So for every functions $f \in B(X,Y)$ and $g \in B(Y,Z)$ we have $(g \circ f) \in B(X,Z)$. ∎

Let X, Y, Z and U be topological Hausdorff spaces, and $W = Z \times U$. On W we introduce the product (Tychonoff) topology, namely the smallest topology with respect to which all the projections are continuous. Then W is also a topological Hausdorff space.

If $g \in Z^X$ and $h \in U^X$, then by (g,h) we mean a function from X into W defined by

$$(g,h)(x) = (g(x), h(x))$$

for every $x \in X$. Function (g,h) is continuous if g and h are. If g_n and h_n are sequences of functions in Z^X and U^X respectively, such that $g_n(x) \longrightarrow g(x)$ and $h_n(x) \longrightarrow h(x)$ for every $x \in X$, then (g_n, h_n) is a sequence in W^X which converges pointwise to the function (g,h). Now let us prove the following Proposition.

Proposition 2.7 *If* $f \in B(W,Y)$, $g \in B(X,Z)$, *and* $h \in B(X,U)$, *then* $f \circ (g,h) \in B(X,Y)$.

Proof. Let f and h be any continuous functions in $C(W,Y)$ and $C(X,U)$ respectively. Consider the set

$$P_{f,h} = \{g \in Z^X : \ f \circ (g,h) \in B(X,Y)\}.$$

Using similar arguments as in the proof of Proposition 2.6, we get $B(X,Z) \subset P_{f,h}$. From this it follows that for every functions $f \in C(W,Y)$, $h \in C(X,U)$ and $g \in B(X,Z)$, we have $f \circ (g,h) \in B(X,Y)$.

Now, for any functions $f \in C(W,Y)$ and $g \in B(X,Z)$ let

$$Q_{f,g} = \{h \in U^X : \ f \circ (g,h) \in B(X,Y)\}.$$

Then $B(X,U) \subset Q_{f,g}$. Consequently, for every $f \in C(W,Y)$, $g \in B(X,Z)$ and $h \in B(X,U)$ we have $f \circ (g,h) \in B(X,Y)$.

Finally, we let

$$S_{g,h} = \{f \in Y^W : \ f \circ (g,h) \in B(X,Y)\}$$

for any functions $g \in B(X,Z)$ and $h \in B(X,U)$, and get $B(W,Y) \subset S_{g,h}$, which completes the proof. ∎

Now let us introduce the product topology on the space C^T i.e. the smallest topology in which all the projections p_α ($\alpha \in T$) are continuous. Let $\mathcal{C}(C^T, C)$ denote the set of all continuous functions from C^T into C. We shall prove that $\mathcal{C}(C^T, C) \subset \text{Comp}(T)$. To do so we need one of the Corollaries of the Stone-Weierstrass Theorem and a Lemma. Let us first state the theorem.

Theorem 2.8 (Stone-Weierstrass) *Let X be a compact Hausdorff space. If A is a closed subalgebra of $\mathcal{C}(X,R)$ which separates points and contains a non-zero constant function, then $A = \mathcal{C}(X,R)$.*

If A is a subset of $\mathcal{C}(X,R)$, then the smallest complex algebra containing A is the algebra of all polynomials of functions from A with complex coefficients. One of the Corollaries of the Stone-Weierstrass Theorem says:

Corollary 2.9 *If A is a subset of $\mathcal{C}(X,R)$ which separates points and B is the smallest complex algebra containing A, then $\overline{B} = \mathcal{C}(X,C)$.*

Using these results we now prove the following Lemma.

Lemma 2.10 *If $f: C \to C$ is a continuous function and $p_\alpha: C^T \to C$ is the projection onto the α-th coordinate, then $(f \circ p_\alpha) \in \text{Comp}(T)$.*

Proof. Let $f: C \to C$ be continuous and $D_n = \{z \in C: |z| \leq n\}$. Then D_n is a compact subset of C, and the function f is continuous on D_n for every $n \in N$. Let $S = \{\text{Re}, \text{Im}\}$. Then $S \subset \mathcal{C}(D_n, R)$ and S separates points of D_n. Hence, by the Corollary of the Stone-Weierstrass Theorem, the function f can be uniformly approximated on D_n by a sequence of polynomials of members of S with complex coefficients. So for every $n \in N$ there exists a polynomial q_n of members of S such that

$$|f(z) - q_n(z)| < 1/n$$

for all $z \in D_n$. Thus, as $n \to \infty$, the sequence $q_n(z)$ converges to $f(z)$ for every $z \in C$. Consequently, if $p_\alpha: C^T \to C$ is the projection onto the α-th coordinate, then $q_n(p_\alpha(z)) \longrightarrow f(p_\alpha(z))$ for every $z \in C^T$, i.e.

$$(q_n \circ p_\alpha)(z) \longrightarrow (f \circ p_\alpha)(z)$$

for every $z \in C^T$. Notice that $q_n \circ p_\alpha$ is a polynomial of $\mathrm{Re} \circ p_\alpha$ and $\mathrm{Im} \circ p_\alpha$, which belong to Comp$(T)$. So $(q_n \circ p_\alpha) \in$ Comp(T) for every $n \in N$, and therefore $(f \circ p_\alpha) \in$ Comp(T). ∎

Proposition 2.11 *The space* Comp(T) *contains all continuous functions from* C^T *into* C.

Proof. Let $u: C^T \to C$ be a continuous function. The disk $D_m = \{z \in C\colon\ |z| \leq m\}$ is compact in C, and so the set D_m^T is compact in C^T for every positive integer m. Thus the function u is continuous on the compact set D_m^T.

We observe that the projections p_α $(\alpha \in T)$ are continuous on C^T and separate points of C^T. Let $B = \{\mathrm{Re} \circ p_\alpha,\ \mathrm{Im} \circ p_\alpha\colon\ \alpha \in T\}$. Then $B \subset C(C^T, R)$ and B separates points of C^T. So, by the Corollary of the Stone-Weierstrass Theorem, the function u can be uniformly approximated on D_m^T by a sequence of polynomials of members of B with complex coefficients. This means that for every positive integer m there exists such polynomial q_m satisfying the inequality

$$|u(z) - q_m(z)| < 1/m \tag{2.4}$$

for all $z \in D_m^T$.

Now, for each positive integer n consider function g_n defined by

$$g_n(z) = \begin{cases} z & \text{if } |z| \le n \\ \frac{nz}{|z|} & \text{if } |z| > n \end{cases}$$

for every $z \in C$. Then g_n is a continuous function from C into D_n, and $g_n(z) \longrightarrow z$ for every $z \in C$ when n tends to infinity. By Lemma 2.10, we get $(g_n \circ p_\alpha) \in \text{Comp}(T)$ for every $n \in N$ and $\alpha \in T$.

Let $f_n = (g_n \circ p_\alpha)_{\alpha \in T}$. Then f_n is a continuous function from C^T into D_n^T. If $m > n$, then $f_n(z) \in D_m^T$ for every $z \in C^T$. So, by (2.4), we get

$$|u(f_n(z)) - q_m(f_n(z))| < 1/m$$

for all $m > n$ and $z \in C^T$, i.e.

$$|(u \circ f_n)(z) - (q_m \circ f_n)(z)| < 1/m$$

for all $m > n$ and $z \in C^T$. This means that, for any fixed n, the sequence $(q_m \circ f_n)$ converges to $(u \circ f_n)$ as $m \to \infty$. But $q_m \circ f_n$ is a polynomial of $\text{Re} \circ p_\alpha \circ f_n = \text{Re} \circ (g_n \circ p_\alpha)$ and $\text{Im} \circ p_\alpha \circ f_n = \text{Im} \circ (g_n \circ p_\alpha)$, which belong to $\text{Comp}(T)$ for every $n \in N$ and $\alpha \in T$. Hence $(q_m \circ f_n) \in \text{Comp}(T)$ for every positive integer m and n, and consequently $(u \circ f_n) \in \text{Comp}(T)$ for every $n \in N$.

Notice that for every $z = (z_t)_{t \in T} \in C^T$ we have

$$\begin{aligned} f_n(z) &= (g_n \circ p_\alpha)_{\alpha \in T}(z) \\ &= (g_n(p_\alpha(z)))_{\alpha \in T} \\ &= (g_n(z_\alpha))_{\alpha \in T}. \end{aligned}$$

Since $g_n(z_\alpha) \longrightarrow z_\alpha$ when n tends to infinity, it follows that $f_n(z) \longrightarrow (z_\alpha)_{\alpha \in T} = z$ for every $z \in C^T$. Thus, by the continuity of u, we get $u(f_n(z)) \longrightarrow u(z)$ for every $z \in C^T$, which means that $(u \circ f_n) \longrightarrow u$. But $\text{Comp}(T)$ is closed under pointwise convergence. Hence $u \in \text{Comp}(T)$. ∎

Corollary 2.12 *Every Baire algebra A of functions in the space C^X is closed under composition with any continuous function $u: C^T \to C$, in the sense that $(u \circ f) \in A$ for every function $f = (f_t)_{t \in T}$ generated by a subset $\{f_t\colon\ t \in T\}$ of the algebra A.*

Proof. Let A be a Baire algebra in the space C^X, and $u: C^T \to C$ a continuous function. By Proposition 2.11, the function u is contained in the space $\mathrm{Comp}(T)$. Hence, by the definition of the space of compositors, we have $(u \circ f) \in A$ for every function $f = (f_t)_{t \in T}$ generated by a subset $\{f_t\colon\ t \in T\}$ of the algebra A. ∎

Theorem 2.13 *The space* $\mathrm{Comp}(T)$ *is the Baire space of functions in $C^{(C^T)}$, which means that the space of compositors of order T is the smallest class of functions in $C^{(C^T)}$ which contains all continuous functions from C^T into C and is closed under pointwise convergence.*

Proof. We need to prove that $\mathrm{Comp}(T) = B(C^T, C)$. By Proposition 2.11 and the fact that $\mathrm{Comp}(T)$ is closed under pointwise convergence we immediately get $B(C^T, C) \subset \mathrm{Comp}(T)$.

Our next step is proving that $B(C^T, C)$ is a Baire algebra containing all the projections. Take any function $f \in B(C^T, C)$ and any complex number λ. Notice that function g_λ, defined by $g_\lambda(z) = \lambda z$ for every $z \in C$, is continuous, and therefore $g_\lambda \in B(C, C)$. So, by Proposition 2.6, we get $(g_\lambda \circ f) \in B(C^T, C)$. But $(g_\lambda \circ f)(z) = g_\lambda(f(z)) = \lambda f(z) = (\lambda f)(z)$ for every $z \in C^T$. Hence $\lambda f \in B(C^T, C)$, which proves that $B(C^T, C)$ is closed under scalar multiplication. Similarly, by taking function $g: C \to C$ defined by $g(z) = \bar{z}$ for every $z \in C$, we prove that $B(C^T, C)$ is closed under involution.

Now take any two functions g and h in $B(C^T, C)$. We observe that function $f: C^2 \to C$, defined by $f(z_1, z_2) = z_1 + z_2$ for every $(z_1, z_2) \in C^2$, is continuous, and therefore $f \in B(C^2, C)$. So, by Proposition 2.7, we get $f \circ (g, h) \in B(C^T, C)$. But $(f \circ (g,h))(z) = f(g(z), h(z)) = g(z) + h(z) = (g+h)(z)$ for every $z \in C^T$. Hence $(g+h) \in B(C^T, C)$, which shows that $B(C^T, C)$ is closed under addition. To show that $B(C^T, C)$ is closed under multiplication, we use the function $f: C^2 \to C$ defined by $f(z_1, z_2) = z_1 \cdot z_2$ for every $(z_1, z_2) \in C^2$.

Constant functions are continuous, so they are Baire functions. By definition a Baire space is closed under pointwise convergence. So $B(C^T, C)$ is a Baire algebra. Since $B(C^T, C)$ contains all continuous functions from C^T into C and the projections p_α $(\alpha \in T)$ are continuous, we conclude that $B(C^T, C)$ contains all the projections. Thus, by Corollary 2.5, we get $B(C^T, C) = \text{Comp}(T)$. ∎

2.3 Baire Algebra Spanned by a Set of Functions

If F is a non-empty subset of C^X, then Proposition 2.1 guarantees the existence of the smallest Baire algebra in C^X containing F. Such Baire algebra is said to be *spanned* by the set F. In this section we shall characterize the Baire algebra spanned by a set of functions using the compositors.

Proposition 2.14 *Let u_n be a sequence of real-valued functions in* Comp(T). *If $u(z) = \sup \{u_n(z): n \in N\} < \infty$ for every $z \in C^T$, then $u \in$* Comp(T). *Similarly, if $u(z) = \inf \{u_n(z): n \in N\} < \infty$ for every $z \in C^T$, then $u \in$* Comp(T).

Proof. Let $v_n(z) = \sup \{u_j(z): j = 1, 2, \ldots, n\} = \max \{u_j(z): j = 1, 2, \ldots, n\}$ for every $z \in C^T$. Then

$$v_n = w_n \circ (u_1, u_2, \ldots, u_n)$$

where w_n is a function from R^n into R defined by

$$w_n(x_1, x_2, \ldots, x_n) = \max \{x_1, x_2, \ldots, x_n\}$$

for every $x_1, x_2, \ldots, x_n \in R$, and $(u_1, u_2, \ldots, u_n)$ is a function from C^T into R^n such that $(u_1, u_2, \ldots, u_n)(z) = (u_1(z), u_2(z), \ldots, u_n(z))$ for every $z \in C^T$. Notice that the function w_n is continuous for every $n \in N$, since w_2 is continuous and $w_n(x_1, x_2, \ldots, x_n) = w_2(w_{n-1}(x_1, \ldots, x_{n-1}), x_n)$.

Let τ_n be a function from C^n into C such that

$$\tau_n(z_1, z_2, \ldots, z_n) = w_n(\mathrm{Re}(z_1), \mathrm{Re}(z_2), \ldots, \mathrm{Re}(z_n))$$

for every $z_1, z_2, \ldots, z_n \in C$. The function τ_n is continuous, since the functions w_n and Re are. Therefore τ_n is a compositor of order n. So $\tau_n \circ (u_1, u_2, \ldots, u_n) \in$ Comp(T), since the functions $u_1, u_2, \ldots, u_n$ belong to the Baire algebra Comp(T). But

$$\begin{aligned} (\tau_n \circ (u_1, u_2, \ldots, u_n))(z) &= \tau_n(u_1(z), u_2(z), \ldots, u_n(z)) \\ &= w_n(\mathrm{Re}(u_1(z)), \mathrm{Re}(u_2(z)), \ldots, \mathrm{Re}(u_n(z))) \\ &= w_n(u_1(z), u_2(z), \ldots, u_n(z)) \\ &= (w_n \circ (u_1, u_2, \ldots, u_n))(z) \\ &= v_n(z) \end{aligned}$$

for every $z \in C^T$. Thus $v_n \in$ Comp(T). Since $v_n(z) \longrightarrow u(z)$ for every $z \in C^T$, it follows that $u \in$ Comp(T).

The second part of the Proposition can be proved in a similar way. ∎

Let A_n be a subset of the space X for every $n \in N$, and let

$$\bigvee_{n \in N} c_{A_n}(x) = \sup\{c_{A_n}(x) \colon n \in N\}$$

and

$$\bigwedge_{n \in N} c_{A_n}(x) = \inf\{c_{A_n}(x) \colon n \in N\}$$

for every $x \in X$. If $A = \bigcup_{n\in N} A_n$ and $B = \bigcap_{n\in N} A_n$, then

$$c_A = \bigvee_{n\in N} c_{A_n} \quad \text{and} \quad c_B = \bigwedge_{n\in N} c_{A_n}.$$

We also notice that for every function $f : X \to Y$ and every subset A in Y we have

$$c_{f^{-1}(A)} = c_A \circ f$$

since

$$\begin{aligned} c_{f^{-1}(A)}(x) = 1 &\iff x \in f^{-1}(A) \\ &\iff f(x) \in A \\ &\iff c_A(f(x)) = 1 \\ &\iff (c_A \circ f)(x) = 1 \end{aligned}$$

for every $x \in X$.

Lemma 2.15 *Let L be the set of all elements $z = (z_n)_{n\in N} \in C^N$ such that the sequence z_n of complex numbers converges. Then the characteristic function of the set L is a compositor of order N, i.e. $c_L \in \mathrm{Comp}(N)$.*

Proof. Since every convergent sequence is a Cauchy sequence, we have

$$\begin{aligned} L &= \left\{ z = (z_n)_{n\in N} \,\middle|\, (\forall_{n\in N})(\exists_{m\in N})(\forall_{k\geq m})(\forall_{l\geq m})\ |z_k - z_l| < \frac{1}{n} \right\} \\ &= \bigcap_{n\in N} \bigcup_{m\in N} \bigcap_{k\geq m} \bigcap_{l\geq m} \left\{ z = (z_n)_{n\in N} \,\middle|\, |z_k - z_l| < \frac{1}{n} \right\} \\ &= \bigcap_{n\in N} \bigcup_{m\in N} \bigcap_{k\geq m} \bigcap_{l\geq m} \left\{ z = (z_n)_{n\in N} \,\middle|\, |p_k(z) - p_l(z)| < \frac{1}{n} \right\} \\ &= \bigcap_{n\in N} \bigcup_{m\in N} \bigcap_{k\geq m} \bigcap_{l\geq m} \left\{ z = (z_n)_{n\in N} \,\middle|\, f_{kl}(z) \in (-\infty, \frac{1}{n}) \right\} \\ &= \bigcap_{n\in N} \bigcup_{m\in N} \bigcap_{k\geq m} \bigcap_{l\geq m} f_{kl}^{-1}\left((-\infty, \frac{1}{n}) \right) \end{aligned}$$

where $f_{kl}(z) = |p_k(z) - p_l(z)|$ for every $z \in C^N$. Notice that f_{kl} is a continuous function for every k and l in N. The characteristic function of the set L is

$$\begin{aligned} c_L &= \bigwedge_{n\in N} \bigvee_{m\in N} \bigwedge_{k\geq m} \bigwedge_{l\geq m} c_{f_{kl}^{-1}(-\infty,1/n)} \\ &= \bigwedge_{n\in N} \bigvee_{m\in N} \bigwedge_{k\geq m} \bigwedge_{l\geq m} (c_{(-\infty,1/n)} \circ f_{kl}). \end{aligned} \tag{2.5}$$

Let a be any fixed real number. For each $n \in N$ consider function $g_n : R \to R$ defined by

$$g_n(x) = \begin{cases} 1 & \text{if } x \leq (a - \frac{1}{n}) \\ n(a-x) & \text{if } (a - \frac{1}{n}) < x < a \\ 0 & \text{if } x \geq a \end{cases}$$

for every $x \in R$. It is easy to see that the function g_n is continuous for each $n \in N$, and that $g_n(x) \longrightarrow c_{(-\infty,a)}(x)$ for every $x \in R$ as $n \to \infty$. So the composite function $g_r \circ f_{kl}$ is continuous for every $r, k, l \in N$ and therefore it belongs to the space Comp(N). But $g_r(f_{kl}(z)) \longrightarrow c_{(-\infty,1/n)}(f_{kl}(z))$ for every $z \in C^N$, which means that $(g_r \circ f_{kl}) \longrightarrow (c_{(-\infty,1/n)} \circ f_{kl})$ as $r \to \infty$. Thus we conclude that $(c_{(-\infty,1/n)} \circ f_{kl}) \in$ Comp(N). And consequently, by equation (2.5) and Proposition 2.14, we get $c_L \in$ Comp(N). ∎

Lemma 2.16 *If $v: C^N \to C$ is a function defined by*

$$v\left((z_n)_{n\in N}\right) = \begin{cases} \lim_{n\to\infty} z_n & \textit{if } z_n \textit{ converges} \\ 0 & \textit{if } z_n \textit{ does not converge} \end{cases}$$

then $v \in$ Comp(N).

Proof. Let L be the set of all elements $z = (z_n)_{n\in N}$ in C^N such that the sequence z_n of complex numbers converges. By Lemma 2.15, the characteristic function $c_L \in$ Comp(N). Let $u_n = c_L \cdot p_n$, where $p_n : C^N \to C$ is the projection onto the n-th coordinate. It is easy to see that $u_n \in$ Comp(N) for every $n \in N$ and that $u_n(z) \longrightarrow v(z)$ for every $z = (z_n)_{n\in N} \in C^N$. Hence $v \in$ Comp(N). ∎

Theorem 2.17 *If $F = \{f_t\colon\ t \in T\}$ is a subset of C^X and $f = (f_t)_{t\in T}$ is the function generated by F, then the smallest Baire algebra in C^X containing the set F is*

$$B(F) \;=\; \{g:\ g = u \circ f,\ u \in \mathrm{Comp}(T)\}.$$

Proof. Let $\mathcal{A}$ be the family of all Baire algebras in C^X containing the set F and let $G = \bigcap_{A\in\mathcal{A}} A$. We want to show that $B(F) = G$.

For every $g \in B(F)$, we have $g = u \circ f$ for some $u \in \mathrm{Comp}(T)$. Thus, by definiton of the space $\mathrm{Comp}(T)$, we get $g \in A$ for every $A \in \mathcal{A}$. Hence $g \in G$ and so

$$B(F) \subset G. \tag{2.6}$$

From Lemma 2.2 and the fact that the space $\mathrm{Comp}(T)$ is a Baire algebra, it is easy to show that $B(F)$ is an algebra, closed under involution and containing all constant functions.

Let g_n be a sequence of functions in $B(F)$ such that $g_n(x) \longrightarrow g(x)$ for every $x \in X$. Then $g_n = u_n \circ f$ for some sequence u_n in $\mathrm{Comp}(T)$. Let $u = (u_n)_{n\in N}$ and v be the function defined in Lemma 2.16. Since $v \in \mathrm{Comp}(N)$ and u_n belongs to the Baire algebra $\mathrm{Comp}(T)$ for every $n \in N$, we get

$$w = v \circ u \in \mathrm{Comp}(T).$$

By the definition of function v and the pointwise convergence of the sequence g_n, we have $v((g_n(x))_{n\in N}) = \lim_{n\to\infty} g_n(x)$ for every $x \in X$, i.e.

$$\begin{aligned}
\lim_{n\to\infty} g_n(x) &= v\left((u_n(f(x)))_{n\in N}\right) \\
&= v\left(u(f(x))\right) \\
&= \left((v \circ u) \circ f\right)(x) \\
&= (w \circ f)(x).
\end{aligned}$$

So we must have $g = w \circ f$, where $w \in \text{Comp}(T)$. Hence $g \in B(F)$, which shows that $B(F)$ is closed under pointwise convergence.

Now, take any function $f_\alpha \in F$ $(\alpha \in T)$. The projection $p_\alpha \in \text{Comp}(T)$ and so $p_\alpha \circ f = f_\alpha \in B(F)$. Hence $F \subset B(F)$. So we conclude that $B(F) \in \mathcal{A}$, and therefore

$$G \subset B(F). \tag{2.7}$$

Thus from (2.6) and (2.7) we conclude that $B(F) = G$, i.e. $B(F)$ is the smallest Baire algebra in C^X containing the set F. ∎

2.4 Baire Algebra Morphisms

Let A and A' be Baire algebras in C^X and $C^{X'}$ respectively. A map $\varphi: A \to A'$ is called *Baire algebra morphism* if it preserves the Baire algebra structure, i.e. for every functions f and g from A and any complex number λ, the following conditions are satisfied:

1. $\varphi(f+g) = \varphi(f) + \varphi(g)$.
2. $\varphi(\lambda f) = \lambda\varphi(f)$.
3. $\varphi(fg) = \varphi(f)\varphi(g)$.
4. If f_n is a sequence of functions in A converging pointwise to function f, then $\varphi(f_n)$ is a sequence of functions in A' converging pointwise to function $\varphi(f)$.
5. $\varphi(\bar{f}) = \overline{\varphi(f)}$.
6. $\varphi(c_X) = c_{X'}$.

Theorem 2.18 *Let A and A' be Baire algebras in C^X and $C^{X'}$ respectively and φ: $A \to A'$ be a Baire algebra morphism. Then for every function $u \in$* Comp(T)

and every function $f = (f_t)_{t\in T}$ *generated by a subset* $\{f_t\colon\ t \in T\}$ *of* A*, we have*

$$\varphi(u \circ f) \ = \ u \circ f'$$

where $f' = (\varphi(f_t))_{t\in T}$.

Proof. Let $U = \{u \in \mathrm{Comp}(T)\colon\ \varphi(u \circ f) = u \circ f'\}$. To prove the proposition, it is sufficient to show that $U = \mathrm{Comp}(T)$. We do this by showing that U is a Baire algebra in $C^{(C^T)}$ containing all the projections.

Take any two elements u and v in U. Then $(u + v) \in \mathrm{Comp}(T)$ and $\varphi(u \circ f) = u \circ f'$ and $\varphi(v \circ f) = v \circ f'$, and so

$$\begin{aligned} \varphi((u+v) \circ f) &= \varphi((u \circ f) + (v \circ f)) \\ &= \varphi(u \circ f) + \varphi(v \circ f) \\ &= (u \circ f') + (v \circ f') \\ &= (u+v) \circ f'. \end{aligned}$$

Thus $(u + v) \in U$. Similarly, we can prove that $(uv) \in U$ and $(\lambda u) \in U$ for every complex number λ. Hence U is an algebra.

If $u \in U$, then $\bar{u} \in \mathrm{Comp}(T)$ and

$$\begin{aligned} \varphi(\bar{u} \circ f) &= \varphi(\overline{u \circ f}) \\ &= \overline{\varphi(u \circ f)} \\ &= \overline{u \circ f'} \\ &= \bar{u} \circ f'. \end{aligned}$$

Thus $\bar{u} \in U$, i.e. U is closed under involution.

The characteristic function $c_{(C^T)} \in U$, since it belongs to $\mathrm{Comp}(T)$ and

$$\varphi(c_{(C^T)} \circ f) \ = \ \varphi(c_X)$$

$$
\begin{aligned}
&= c_{X'} \\
&= c_{(C^T)} \circ f'.
\end{aligned}
$$

Since U is an algebra, it must contain all constant functions.

Let u_n be a sequence in U such that $u_n(g) \longrightarrow u(g)$ for every $g \in C^{(C^T)}$. Since the sequence u_n is in $\text{Comp}(T)$ and the space $\text{Comp}(T)$ is closed under pointwise convergence, we have $u \in \text{Comp}(T)$. By Lemma 2.2, the sequence $(u_n \circ f)$ converges pointwise to $(u \circ f)$, and therefore $\varphi(u_n \circ f)$ is a sequence in A' converging pointwise to $\varphi(u \circ f)$. In other word, since $u_n \in U$, the sequence $(u_n \circ f')$ converges to $\varphi(u \circ f)$. But, using Lemma 2.2 again, the sequence $(u_n \circ f')$ must converge to $(u \circ f')$. Thus $\varphi(u \circ f) = u \circ f'$, which means that $u \in U$. So U is closed under pointwise convergence.

Finally, U also contains all of the projections $p_\alpha : C^T \to C \quad (\alpha \in T)$, since $p_\alpha \in \text{Comp}(T)$ and

$$
\begin{aligned}
\varphi(p_\alpha \circ f) &= \varphi(f_\alpha) \\
&= p_\alpha \circ (\varphi(f_t))_{t \in T} \\
&= p_\alpha \circ f'
\end{aligned}
$$

for every $\alpha \in T$.

Since $U \subset \text{Comp}(T)$ and U is a Baire algebra in $C^{(C^T)}$ containing all the projections, by Corollary 2.5, we get $U = \text{Comp}(T)$. ∎

Chapter 3

Rings of Sets and Algebras of Functions

Let V be a family of subsets of an abstract set X, and $S(V)$ be the collection of those subsets A of X with the property that there exist disjoint sets $A_1, A_2, \ldots, A_n$ in V such that

$$A = \bigcup_{i=1}^{n} A_i.$$

The members of $S(V)$ are called *simple sets* and we say that $S(V)$ is the collection of simple sets generated by the family V. It is obvious that every set in V is simple, in other words $V \subset S(V)$.

A family V of subsets of X is called a *prering* if for every two sets A and B in V we have the sets $A \setminus B$ and $A \cap B$ are both in $S(V)$. If V is closed under difference and union of sets, i.e. $(A \setminus B) \in V$ and $(A \cup B) \in V$ for every A and B in V, then V is called a *ring*. If V is a ring, then it is also closed under intersection, since $A \cap B = A \setminus (A \setminus B) \in V$ for every A and B in V. Hence every ring of sets is a prering.

Proposition 3.1 *Let V be a prering. If $A \in V$ and $B_1, B_2, \ldots, B_n$ are disjoint sets in V, then $A \setminus (B_1 \cup B_2 \cup \cdots \cup B_n) \in S(V)$ for every positive integer n.*

Proof. We prove by induction. If $n = 1$, it is clear that $(A \setminus B_1) \in S(V)$ since V is a prering. Assume that the statement is true for $n = k$. Let $B_1, B_2, \ldots, B_k, B_{k+1}$

be disjoint sets in V. Notice that

$$D = A \setminus (B_1 \cup \cdots \cup B_k \cup B_{k+1}) = (A \setminus (B_1 \cup \cdots \cup B_k)) \setminus B_{k+1}.$$

By our assumption, $A \setminus (B_1 \cup \cdots \cup B_k) \in S(V)$, which means that there exist disjoint sets $E_1, \ldots, E_m$ in V such that $A \setminus (B_1 \cup \cdots \cup B_k) = \cup_{i=1}^m E_i$. Thus $D = (\cup_{i=1}^m E_i) \setminus B_{k+1} = \cup_{1=1}^m (E_i \setminus B_{k+1}) = \cup_{i=1}^m F_i$, where $F_i = E_i \setminus B_{k+1} \in S(V)$ for every $i \in \{1, 2, \ldots, m\}$, since V is a prering. So we get disjoint sets F_i (since sets E_i are disjoint), each of which is a finite union of disjoint sets in V. Thus D is a finite union of disjoint sets in V. So $D \in S(V)$. ∎

3.1 Simple Functions

Let Y be any non-empty set and 0 be a fixed element in Y (called the *scaling element*). A function $s: X \to Y$ is called *simple* with respect to a family V of subsets of X if there exist disjoint sets $A_1, A_2, \ldots, A_n$ in V and $y_1, y_2, \ldots, y_n$ in Y such that

$$s(x) = \begin{cases} y_i & \text{if } x \in A_i \\ 0 & \text{if } x \notin \bigcup_{i=1}^n A_i \end{cases}$$

for all $x \in X$. Let $S(V, Y)$ denote the set of all simple functions $s \in Y^X$ generated by the family V.

If $s_1, s_2, \ldots, s_n$ are simple functions from $S(V, Y)$, then by $(s_1, s_2, \ldots, s_n)$ we mean a function $s: X \to Y^n$ defined by

$$s(x) = (s_1(x), s_2(x), \ldots, s_n(x))$$

for all $x \in X$. The function $s = (s_1, s_2, \ldots, s_n)$ is said to be generated by simple functions $s_1, s_2, \ldots, s_n$. We say that the set $S(V, Y)$ of simple functions is closed under composition with a function $u: Y^n \to Y$ if $(u \circ s) \in S(V, Y)$ for every function $s = (s_1, s_2, \ldots, s_n)$ generated by simple functions $s_1, s_2, \ldots, s_n \in S(V, Y)$.

A family V of subsets of X is said to have *finite refinement property* if for every finite subfamily V_1 of V there exists a finite subfamily V_2 of pairwise disjoint sets from V such that every member of V_1 is a union of members of V_2. The family V_2 is called a finite refinement of V_1.

Proposition 3.2 *For a fixed family V of subsets of a space X the following statements are equivalent:*

1. *For every set Y with a scaling element 0, the set $S(V,Y)$ of simple functions is closed under composition with every function $u\colon Y^n \to Y$ vanishing at zero (i.e. $u(0,0,\ldots,0)=0$).*
2. *The collection $S(V)$ of simple sets is a ring.*
3. *The family V of subsets of X is a prering.*
4. *The family V has finite refinement property.*

Proof. $1 \Rightarrow 2$: Let $Y=\{0,1\}$ where 0 is the scaling element. Then $A \in S(V) \iff c_A \in S(V,Y)$. Take any two sets A and B in $S(V)$. Then their characteristic functions c_A and c_B are both in $S(V,Y)$. Consider function $u\colon Y^2 \to Y$ defined by $u(x,y) = \max\,\{x,y\}$ for all $x,y \in Y$. It is clear that $u(0,0)=0$. Thus, by our hypothesis, $u \circ (c_A,c_B) \in S(V,Y)$. But $(u \circ (c_A,c_B))(x) = \max\,\{c_A(x), c_B(x)\} = c_{A\cup B}(x)$ for every $x \in X$. Hence $c_{A\cup B} \in S(V,Y)$, and so $(A\cup B) \in S(V)$.

Next we consider function $u\colon Y^2 \to Y$ defined by $u(x,y) = x(1-y)$ for every $x,y \in Y$. From our hypothesis it follows that $u \circ (c_A,c_B) \in S(V,Y)$. But for every $x \in X$ we have $(u \circ (c_A,c_B))(x) = c_A(x)(1-c_B(x)) = c_{A\setminus B}(x)$. Hence $c_{A\setminus B} \in S(V,Y)$, and thus $(A\setminus B) \in S(V)$. So $S(V)$ is a ring.

$2 \Rightarrow 3$: Assume that $S(V)$ is a ring. Take any two sets A and B in V. Then A and B are both in $S(V)$. Since $S(V)$ is a ring, both $A\setminus B$ and $A\cap B$ belong to $S(V)$. So V is a prering.

$3 \Rightarrow 4$: Let V be a prering. We shall prove (4) by induction. Every subfamily of V consisting of only one set has itself as a finite refinement. Assume that subfamily $H_n = \{A_1, \ldots, A_n\}$ of V has a finite refinement, say $W_0 = \{B_1, \ldots, B_m\}$ where $B_i \in V$ and $B_i \cap B_j = \emptyset$ if $i \neq j \quad (i, j = 1, 2, \ldots, m)$. Consider subfamily $H_{n+1} = \{A_1, \ldots, A_n, A_{n+1}\}$ of V. We observe that

$$\begin{aligned} A_{n+1} &= [A_{n+1} \setminus (\cup_{i=1}^m B_i)] \cup [A_{n+1} \cap (\cup_{i=1}^m B_i)] \\ &= D \cup E \end{aligned}$$

where $D = A_{n+1} \setminus (\cup_{i=1}^m B_i)$ and $E = A_{n+1} \cap (\cup_{i=1}^m B_i)$. By Proposition 3.1, we get $D \in S(V)$, which means that D is a union of members of some finite collection W_1 of disjoint sets in V. And $E = A_{n+1} \cap (\cup_{i=1}^m B_i) = \cup_{i=1}^m (A_{n+1} \cap B_i) = \cup_{i=1}^m E_i$ where $E_i = A_{n+1} \cap B_i \in S(V)$ for every $i \in \{1, 2, \ldots, m\}$ since V is a prering. Thus each E_i is a union of members of some finite collection V_i of disjoint sets in V. Let $W_2 = \cup_{i=1}^m V_i$. Then $W = W_0 \cup W_1 \cup W_2$ is a finite refinement of the collection H_{n+1}.

$4 \Rightarrow 1$: We now assume that V has finite refinement property. Take any function $u: Y^n \to Y$ such that $u(0, 0, \ldots, 0) = 0$ and any simple functions $s_1, s_2, \ldots, s_n$ from the space $S(V, Y)$. We need to prove that $(u \circ s) \in S(V, Y)$, where $s = (s_1, s_2, \ldots, s_n)$. For each $i \in \{1, 2, \ldots, n\}$ there exist a finite index set $\Gamma_i = \{1, 2, \ldots, m_i\}$, and disjoint sets $A_{ij} \quad (j \in \Gamma_i)$ in V, and elements $y_{ij} \quad (j \in \Gamma_i)$ in Y such that

$$s_i(x) = \begin{cases} y_{ij} & \text{if } x \in A_{ij} \\ 0 & \text{if } x \notin \bigcup_{j \in \Gamma_i} A_{ij}. \end{cases}$$

Let $H = \{A_{ij}\colon i = 1, 2, \ldots, n; j \in \Gamma_i\}$. Then H is a finite subfamily of V. By our assumption, H has a finite refinement, say $B = \{B_1, \ldots, B_k\}$. Each $A_{ij} \in H$ is a union of disjoint sets from the collection B. Let $z_{it} = y_{ij}$ if $B_t \subset A_{ij}$ and $B_t \neq \emptyset$, otherwise let $z_{it} = 0$. Notice that

$$s_i(x) = \begin{cases} z_{it} & \text{if } x \in B_t \\ 0 & \text{if } x \notin \bigcup_{t=1}^{k} B_t. \end{cases}$$

Hence

$$(u \circ s)(x) = u(s_1(x), \ldots, s_n(x)) = \begin{cases} u(z_{1t}, z_{2t}, \ldots, z_{nt}) = v_t & \text{if } x \in B_t \\ u(0, 0, \ldots, 0) = 0 & \text{if } x \notin \bigcup_{t=1}^{k} B_t \end{cases}$$

which shows that $(u \circ s) \in S(V, Y)$. ∎

A partially ordered set is called a *lattice* if every pair of elements in it has a least upper bound and a greatest lower bound. Let f and g be real-valued functions defined on X. By $f \wedge g$ and $f \vee g$ we mean functions defined as follows

$$(f \wedge g)(x) = \min \{f(x), g(x)\}$$

$$(f \vee g)(x) = \max \{f(x), g(x)\}$$

for every $x \in X$. It is clear that a set S of real-valued functions defined on X is a lattice if $(f \wedge g) \in S$ and $(f \vee g) \in S$ for every pair of functions f and g in S.

Theorem 3.3 *If V is a prering of subsets of X, and C is the field of complex numbers, then the set $S(V, C)$ of simple functions generated by the family V is an algebra of functions closed under involution. Moreover, the set $S(V, R)$ of real-valued simple functions is a lattice.*

Proof. Consider function $u: C^2 \to C$ defined by $u(z_1, z_2) = z_1 + z_2$ for every complex numbers z_1 and z_2. By Proposition 3.2, we get $u \circ (s_1, s_2) \in S(V, C)$ for every $s_1, s_2 \in S(V, C)$. But $(u \circ (s_1, s_2))(x) = u(s_1(x), s_2(x)) = s_1(x) + s_2(x) = (s_1 + s_2)(x)$ for every $x \in X$. Hence $(s_1 + s_2) \in S(V, C)$, i.e. $S(V, C)$ is closed under addition. Similarly, by considering function $u: C^2 \to C$ defined by $u(z_1, z_2) = z_1 \cdot z_2$ for every $z_1, z_2 \in C$, we prove that $S(V, C)$ is closed under multiplication. To show that it is closed under scalar multiplication and involution, we use functions $u: C \to C$ defined by $u(z) = \lambda z$ and $u(z) = \bar{z}$ for all $z \in C$, respectively.

For every pair of functions s_1 and s_2 from $S(V, R)$, the functions $u_1 = s_1 \wedge s_2$ and $u_2 = s_1 \vee s_2$ from R^2 into R vanish at zero. From this it follows that the set $S(V, R)$ is a lattice. ∎

3.2 Measurable Functions

A ring V of subsets of a space X is called an *algebra* if $X \in V$. If a ring V is closed under countable union, i.e.

$$\bigcup_{n=1}^{\infty} A_n \in V$$

for every countable collection $\{A_n\}$ of sets from V, then V is called a *σ-ring*. A ring V is called a *δ-ring* if it is closed under countable intersection, i.e.

$$\bigcap_{n=1}^{\infty} A_n \in V$$

for every countable collection $\{A_n\}$ of sets from V. A σ-ring V is called a *σ-algebra* if $X \in V$. If V is a σ-ring and $\{A_n\}$ a countable collection of sets in V, then $\bigcap_{n=1}^{\infty} A_n = A_1 \setminus (A_1 \setminus \bigcap_{n=1}^{\infty} A_n) = A_1 \setminus \bigcup_{n=1}^{\infty}(A_1 \setminus A_n) \in V$. Hence every σ-ring is a δ-ring.

Every δ-ring V is *closed under dominated countable union*, in the sense that if $\{A_n\}$ is a countable collection of sets from V, and the set $B \in V$ is such that $A_n \subset B$ for every $n \in N$, then $\bigcup_{n=1}^{\infty} A_n \in V$. This is true since $\bigcup_{n=1}^{\infty} A_n = B \setminus (B \setminus \bigcup_{n=1}^{\infty} A_n) = B \setminus \bigcap_{n=1}^{\infty}(B \setminus A_n)$.

If V is a family of subsets of X, then by V^σ we mean the collection of all countable unions of sets in V, that is

$$V^\sigma = \left\{A\colon\ A = \bigcup_{n\in N} A_n,\ A_n \in V\right\}.$$

Proposition 3.4 *If V is a δ-ring of subsets of X, then V^σ is the smallest σ-ring containing V.*

Proof. From the definition of V^σ it is clear that the family V is contained in V^σ. Take any two sets A and B from the family V^σ. Then $A = \bigcup_n A_n$ and $B = \bigcup_m B_m$, where $A_n \in V$ and $B_m \in V$ for every n and m in N. Therefore

$$\begin{aligned} A \setminus B &= (\bigcup_n A_n) \setminus (\bigcup_m B_m) \\ &= \bigcup_n (A_n \setminus \bigcup_m B_m) \\ &= \bigcup_n \bigcap_m (A_n \setminus B_m). \end{aligned}$$

Since V is a δ-ring, the set $\bigcap_m (A_n \setminus B_m) \in V$ for every $n \in N$. So $(A \setminus B) \in V^\sigma$.

Let A_n be a countable collection of sets from V^σ. Then for every $n \in N$

$$A_n = \bigcup_m A_{nm}$$

where $A_{nm} \in V$ for every $n, m \in N$. Thus

$$\bigcup_n A_n = \bigcup_n \bigcup_m A_{nm} \in V^\sigma$$

since it is a countable union of sets from V. Thus V^σ is a σ-ring.

Suppose that W is another σ-ring containing V. If $A \in V^\sigma$, then $A = \bigcup_n A_n$ where $A_n \in V$ for every $n \in N$. Since $V \subset W$, we must have $A_n \in W$ for every $n \in N$. So $A = \bigcup_n A_n \in W$, because W is a σ-ring. Hence $V^\sigma \subset W$, which proves that V^σ is the smallest σ-ring containing V. ∎

In the following Proposition, by $\overline{G}$ we mean the closure of the set G.

Proposition 3.5 *Let V be δ-ring of subsets of a space X and f be a complex-valued function defined on X. Then the following statements are equivalent:*

1. *For every open set G in C such that $0 \notin \overline{G}$, the inverse image $f^{-1}(G)$ of the set G is in V.*

2. *For every closed set F in C such that $0 \notin F$, the inverse image $f^{-1}(F)$ of the set F is in V.*

Proof. $1 \Rightarrow 2$: Assume that $f^{-1}(G) \in V$ for every open set G in C such that $0 \notin \overline{G}$. Let F be any closed set in C such that $0 \notin F$. Consider the distance function $d: C \to R^+$ defined by $d(z) = \inf \{|z - w|: w \in F\}$ for all $z \in C$. Then the function d is continuous on C. Notice that

$$\begin{aligned} F &= \{z \in C: d(z) = 0\} \\ &= \{z \in C: (\forall_{n \in N})\ d(z) < 1/n\} \\ &= \bigcap_{n \in N} d^{-1}(-\infty, 1/n) \\ &= \bigcap_{n \in N} G_n, \end{aligned}$$

where $G_n = d^{-1}(-\infty, 1/n)$. Since d is continuous, the set G_n is open in C for every $n \in N$. We also notice that $G_{n+1} \subset \overline{G_{n+1}} \subset G_n$ for every $n \in N$. Since $0 \notin F$, there exists $n_0 \in N$ such that $0 \notin G_{n_0}$. Thus $0 \notin G_n$, and therefore $0 \notin \overline{G_{n+1}}$, for all $n > n_0$. So we have $F = \bigcap_{n > n_0 + 1} G_n$ and $0 \notin \overline{G_n}$ for all $n > n_0 + 1$. And consequently

$$\begin{aligned} f^{-1}(F) &= f^{-1}\left(\bigcap_{n > n_0 + 1} G_n \right) \\ &= \bigcap_{n > n_0 + 1} f^{-1}(G_n) \in V, \end{aligned}$$

since by our assumption $f^{-1}(G_n) \in V$ for all $n > n_0 + 1$ and V is a δ-ring.

$2 \Rightarrow 1$: Now let us assume that $f^{-1}(F) \in V$ for every closed set F in C with $0 \notin F$. Take any open set G in C such that $0 \notin \overline{G}$. Let $F = G^c$ and d be the corresponding distance function as defined above. Then

$$G = \{z \in C | \ z \notin F\}$$

$$\begin{aligned} &= \{z \in C \mid d(z) > 0\} \\ &= \{z \in C \mid (\exists_{m,n\in N})\ \frac{1}{m} \leq d(z) \leq n\} \\ &= \bigcup_{m,n\in N} d^{-1}[\frac{1}{m}, n] \\ &= \bigcup_{m,n\in N} F_{mn} \end{aligned}$$

where $F_{mn} = d^{-1}[\frac{1}{m}, n]$. Since function d is continuous on C, the set F_{mn} is closed for every m and n in N. Also $F_{mn} \subset G \subset \overline{G}$, which implies that $0 \notin F_{mn}$, and $f^{-1}(F_{mn}) \subset f^{-1}(\overline{G})$, for every m and n in N. By our assumption $f^{-1}(F_{mn}) \in V$ for every $m, n \in N$, and also $f^{-1}(\overline{G}) \in V$. Since V is a δ-ring (which is closed under dominated countable union), we get $\bigcup_{m,n\in N} f^{-1}(F_{mn}) = f^{-1}(\bigcup_{m,n\in N} F_{mn}) = f^{-1}(G) \in V$. ∎

Let $M(V, C)$ denote the set of all functions $f \in C^X$ such that $f^{-1}(G) \in V$ for every open set G in C with $0 \notin \overline{G}$. The members of $M(V, C)$ are called *measurable functions* with respect to the family V. If V is a ring of subsets of X, then it is easy to verify that every simple function $s \in S(V, C)$ is measurable with respect to V.

Proposition 3.6 *If V is a σ-ring of subsets of X, then the set $M(V, C)$ of measurable functions is closed under pointwise convergence on X.*

Proof. Let f_n be a sequence of functions in $M(V, C)$ converging pointwise to a function f, and G be an open set in C such that $0 \notin \overline{G}$. Let $F = G^c$ and d be the distance function defined by $d(z) = \inf\{|z - w| : w \in F\}$ for every $z \in C$. Since $z \in F \iff d(z) = 0$, we get

$$\begin{aligned} G &= \{z \in C \mid z \notin F\} \\ &= \{z \in C \mid d(z) > 0\} \end{aligned}$$

$$\begin{aligned} &= \{z \in C \mid (\exists k \in N)\ d(z) > \frac{1}{k}\} \\ &= \bigcup_{k=1}^{\infty} d^{-1}(\frac{1}{k}, \infty) \\ &= \bigcup_{k=1}^{\infty} G_k \end{aligned}$$

where $G_k = d^{-1}(\frac{1}{k}, \infty)$ is an open set for every $k \in N$, since d is continuous. Notice that $\overline{G_k} = \{z \in C\colon\ d(z) \geq \frac{1}{k}\} \subset G$ for every positive integer k. We claim that

$$f^{-1}(G) = \{x \in X \mid (\exists_{k\in N})(\exists_{m\in N})(\forall_{n\geq m})\ f_n(x) \in G_k\}. \tag{3.1}$$

Take any $x \in f^{-1}(G)$. Then $f(x) \in G$, which means that $f(x) \in G_k$ for some positive integer k. Since G_k is open, there exists some positive number r such that the ball $B(f(x); r)$ is contained in G_k. But $f_n(x) \longrightarrow f(x)$ for every $x \in X$. So there exists a positive integer m such that $f_n(x) \in B(f(x); r)$ for all $n \geq m$. Thus $(\exists_{k\in N})(\exists_{m\in N})(\forall_{n\geq m})\ f_n(x) \in G_k$. Conversely, take any $x \in X$ such that $f_n(x) \in G_k$ for some k and m in N and all $n \geq m$. Since $f_n(x) \longrightarrow f(x)$, we must have $f(x) \in \overline{G_k} \subset G$. Thus $x \in f^{-1}(G)$. This completes the proof of (3.1), and so

$$f^{-1}(G) = \bigcup_{k\in N} \bigcup_{m\in N} \bigcap_{n\geq m} f_n^{-1}(G_k),$$

where G_k is open and $0 \notin \overline{G_k}$ for every $k \in N$. Since $f_n \in M(V, C)$, we must have $f_n^{-1}(G_k) \in V$ for every $n \in N$. But V is a σ-ring. So we conclude that $f^{-1}(G) \in V$, which means that $f \in M(V, C)$. ∎

If W is a lattice of subsets of a space X, then

$$V = \{A \setminus B\colon\ A \in W,\ B \in W\}$$

is a prering closed under intersection. The collection of all open sets G in the complex plane C whose closures do not contain the origin is obviously a lattice. Let K be the family of all subsets in C which are of the form $G \setminus H$ where G and H

are open sets in C such that $0 \notin \overline{G}$ and $0 \notin \overline{H}$. Then K is a prering closed under intersection. Denote by $S(K, C)$ the set of all complex-valued simple functions with respect to the family K.

Lemma 3.7 *If $e: C \to C$ is the identity map, i.e. $e(z) = z$ for every $z \in C$, then there exists a sequence u_n of simple functions in $S(K, C)$ such that $u_n(z) \longrightarrow e(z)$ and $|u_n(z)| \nearrow |e(z)|$ for every $z \in C$.*

Proof. For each $n \in N$ let $S_n = \{z \in C \colon \frac{1}{2n} \leq |z| \leq n\}$. Then S_n is compact, and so there exist $z_1, z_2, \ldots, z_{k_n} \in S_n$ such that

$$S_n \subset \bigcup_{j=1}^{k_n} B(z_j; \frac{1}{4n}).$$

Let $G_j^n = B(z_j; \frac{1}{4n})$, and let $A_1^n = G_1^n$ and $A_j^n = (G_j^n \setminus (G_1^n \cup \cdots \cup G_{j-1}^n)) \cap S_n$ for $j = 2, \ldots, k_n$. Then for each $n \in N$ the sets A_j^n are mutually disjoint, $A_j^n \subset G_j^n$, $A_j^n \in K$ for every $j \in \{1, 2, \ldots, k_n\}$, and

$$S_n = \bigcup_{j=1}^{k_n} A_j^n.$$

Let $D_1 = \{A_j^1 \colon j = 1, 2, \ldots, k_1\}$ and

$$D_n = \{D \cap A_j^n \colon D \in D_{n-1},\ j = 1, 2, \ldots, k_n\} \cup \{A_j^n \setminus S_{n-1} \colon j = 1, 2, \ldots, k_n\}$$

for $n \geq 2$. Then by induction we have the following facts:

1. Each D_n is a finite family of mutually disjoint sets. So for every $n \in N$ let us write $D_n = \{D_j^n \colon j \in \Gamma_n\}$ for some finite index set Γ_n.

2. For every $n \in N$ we have $S_n = \bigcup_{j \in \Gamma_n} D_j^n$.

3. Each D_j^n is the union of members of the family D_{n+1}.

4. The family D_n is a subset of K for every $n \in N$.

For each $n \in N$ and each $j \in \Gamma_n$ let $w_j^n \in \overline{D_j^n}$ such that

$$|w_j^n| = \inf\ \{|z|:\ z \in \overline{D_j^n}\}.$$

We now define a function $u_n: C \to C$ by setting

$$u_n(z) = \sum_{j \in \Gamma_n} w_j^n c_{D_j^n}(z)$$

for every $z \in C$. It is clear that $u_n \in S(K, C)$ for every $n \in N$. We claim that $|u_n(z) - z| < \frac{1}{n}$ for all $z \in C$ such that $|z| \leq n$.

If $|z| < \frac{1}{2n}$ then $z \notin S_n$ and therefore $u_n(z) = 0$. Thus $|u_n(z) - z| = |z| < \frac{1}{2n} < \frac{1}{n}$. If $\frac{1}{2n} \leq |z| \leq n$ then $z \in S_n$. So $z \in D_j^n$ for some $j \in \Gamma_n$, and therefore $u_n(z) = w_j^n$. Since $w_j^n \in \overline{D_j^n} \subset \overline{A_j^n} \subset \overline{G_j^n}$, we get

$$\begin{aligned}
|u_n(z) - z| &= |w_j^n - z| \\
&\leq |w_j^n - z_j| + |z_j - z| \\
&< \frac{1}{4n} + \frac{1}{4n} \\
&= \frac{1}{2n} < \frac{1}{n}.
\end{aligned}$$

Thus for each $z \in C$ we have $|u_n(z) - z| < \frac{1}{n}$ if $n \geq |z|$, which means that $u_n(z) \longrightarrow e(z)$ for every $z \in C$.

We still have to show that

$$|u_n(z)| \leq |u_{n+1}(z)| \tag{3.2}$$

for every $n \in N$ and every $z \in C$. Take any $n \in N$ and any complex number $z \in C$. If $z \notin S_{n+1}$, then $u_n(z) = u_{n+1}(z) = 0$ and so inequality (3.2) holds. If $z \in S_{n+1} \setminus S_n$, then $u_n(z) = 0$ and thus inequality (3.2) also holds. If $z \in S_n$, then $z \in D_k^n$ for some $k \in \Gamma_n$. So

$$\begin{aligned}
|u_n(z)| &= |w_k^n| \\
&= \inf\ \{|z|:\ z \in \overline{D_k^n}\}
\end{aligned}$$

where $w_k^n \in \overline{D_k^n}$. Let Γ'_{n+1} be a subset of Γ_{n+1} such that $D_k^n = \bigcup_{j\in\Gamma'_{n+1}} D_j^{n+1}$. So $z \in D_m^{n+1}$ for some $m \in \Gamma'_{n+1}$, and therefore

$$\begin{aligned} |u_{n+1}(z)| &= |w_m^{n+1}| \\ &= \inf\{|z|\colon z \in \overline{D_m^{n+1}}\} \end{aligned}$$

where $w_m^{n+1} \in \overline{D_m^{n+1}}$. But notice that

$$\begin{aligned} \overline{D_k^n} &= \overline{\bigcup_{j\in\Gamma'_{n+1}} D_j^{n+1}} \\ &= \bigcup_{j\in\Gamma'_{n+1}} \overline{D_j^{n+1}}. \end{aligned}$$

Thus $\overline{D_m^{n+1}} \subset \overline{D_k^n}$, which implies that $|u_n(z)|$ is also a lower bound of the set $\{|z|\colon z \in \overline{D_m^{n+1}}\}$. Since $|u_{n+1}(z)|$ is the greatest lower bound of the set, we get $|u_n(z)| \leq |u_{n+1}(z)|$ for every $n \in N$ and every $z \in C$. ∎

Lemma 3.8 *Let V be a δ-ring of subsets of X. If $f \in M(V,C)$ and $u \in S(K,C)$, then $(u \circ f) \in S(V,C)$.*

Proof. Since $u \in S(K,C)$, we can find complex numbers $z_1, \ldots, z_n$ and disjoint sets $A_1, \ldots, A_n \in K$ such that

$$u(z) = \sum_{j=1}^{n} z_j c_{A_j}(z)$$

for every $z \in C$. And so

$$\begin{aligned} (u \circ f)(x) &= \sum_{j=1}^{n} z_j c_{A_j}(f(x)) \\ &= \sum_{j=1}^{n} z_j c_{f^{-1}(A_j)}(x) \end{aligned}$$

for every $x \in X$. For $j = 1, \ldots, n$ let $A_j = G_j \setminus H_j$ for some open sets G_j and H_j whose closures do not contain 0. Since $f \in M(V,C)$, we have $f^{-1}(A_j) =$

$f^{-1}(G_j \setminus H_j) = f^{-1}(G_j) \setminus f^{-1}(H_j) \in V$. The sets $f^{-1}(A_j)$ are mutually disjoint, since the sets A_j are. Thus $(u \circ f) \in S(V, C)$. ∎

Lemma 3.9 *A function $f \in C^X$ is measurable with respect to a σ-ring V of subsets of X if and only if there exists a sequence s_n in $S(V, C)$ such that $s_n(x) \longrightarrow f(x)$ for every $x \in X$.*

Proof. Let $f \in M(V, C)$, and u_n be the sequence of functions from Lemma 3.7. Then by Lemma 3.8 the sequence $s_n = u_n \circ f \in S(V, C)$. And for every $x \in X$ we have $s_n(x) = u_n(f(x)) \longrightarrow e(f(x)) = f(x)$.

Conversely, suppose that s_n is a sequence of functions in $S(V, C)$, converging pointwise to function $f \in C^X$. Then $s_n \in M(V, C)$ for every $n \in N$, and so by Proposition 3.6 we get $f \in M(V, C)$. ∎

Theorem 3.10 *If V is a σ-algebra of subsets of X, then the set $M(V, C)$ of measurable functions with respect to V forms a Baire algebra. Moreover, the set $M(V, R)$ of real-valued measurable functions is a lattice.*

Proof. Take any two functions f and g in $M(V, C)$. By Lemma 3.9, there exist sequences s_n and t_n in $S(V, C)$ such that $s_n(x) \longrightarrow f(x)$ and $t_n(x) \longrightarrow g(x)$ for every $x \in X$. Since $S(V, C)$ is an algebra, we must have $(s_n + t_n) \in S(V, C)$, and therefore $(s_n + t_n) \in M(V, C)$ for every $n \in N$. But the sequence $s_n + t_n$ converges pointwise to $f + g$, and therefore by Proposition 3.6 we get $(f + g) \in M(V, C)$. So we proved that $M(V, C)$ is closed under addition. Using similar arguments we prove that $M(V, C)$ is closed under multiplication, scalar multiplication and involution.

Since $c_X(x) = 1$ for every $x \in X$, we have

$$c_X^{-1}(G) = \begin{cases} X & \text{if } 1 \in G \\ \emptyset & \text{if } 1 \notin G \end{cases}$$

for every open set G in C. But $X \in V$, since V is a σ-algebra. Hence $c_X^{-1}(G) \in V$ for every open set G in C, which means that $c_X \in M(V,C)$. Since $M(V,C)$ is an algebra, it must contain all constant functions. So we have proved that $M(V,C)$ is a Baire algebra.

Take any two functions f and g from $M(V,R)$ and consider functions u and v from C^2 into R defined by

$$u(z_1,z_2) = \min\{|z_1|,|z_2|\}$$

$$v(z_1,z_2) = \max\{|z_1|,|z_2|\}$$

for every $(z_1,z_2) \in C^2$. Then both functions u and v are continuous. Since $M(V,C)$ is a Baire algebra, by Corollary 2.12, we get $u \circ (f,g) \in M(V,C)$ and $v \circ (f,g) \in M(V,C)$. But $u \circ (f,g) = f \wedge g$ and $v \circ (f,g) = f \vee g$ and both are real-valued functions. This completes the proof that the set $M(V,R)$ is a lattice. ∎

Proposition 3.11 *If V is a σ-algebra of subsets of X, then $M(V,C)$ is the smallest Baire algebra of functions in C^X containing the set $F = \{c_A\colon A \in V\}$.*

Proof. The Baire algebra $M(V,C)$ contains the set $F = \{c_A\colon A \in V\}$, since every characteristic function in F is simple, and therefore measurable, with respect to V. Suppose that M_1 is another Baire algebra of functions in C^X containing the set F. Take any function $f \in M(V,C)$. By Lemma 3.9, there is a sequence s_n in $S(V,C)$ converging pointwise to f. So

$$s_n = \sum_{j=1}^{k_n} z_j c_{A_j}$$

where $A_j \in V$ for $j \in \{1,2,\ldots,k_n\}$. Since $c_{A_j} \in F$ (and therefore in M_1) for $j \in \{1,2,\ldots,k_n\}$, and M_1 is an algebra, we must have $s_n \in M_1$ for every $n \in N$. Hence $f \in M_1$, since M_1 is a Baire algebra. So we have proved that $M(V,C) \subset M_1$, which means that $M(V,C)$ is the smallest Baire algebra containing the set F. ∎

3.3 The Trace of a Space of Functions

Let L be a collection of functions in C^X. Then we define

$$\text{trace}(L) = \{A \subset X:\ c_A \in L\}.$$

If L is an algebra of functions in C^X and $V = \text{trace}(L)$, then $S(V, C) \subset L$.

Theorem 3.12 *If L is an algebra of functions in C^X which is closed under pointwise convergence on X, then $V = \text{trace}(L)$ is a σ-ring of subsets of X.*

Proof. Take any two sets A and B in V. Then $c_A \in L$ and $c_B \in L$. Notice that $c_{A\setminus B} = c_{A\setminus(A\cap B)} = c_A - c_{A\cap B} = c_A - c_A \cdot c_B \in L$, since L is an algebra. Thus $(A \setminus B) \in V$. Since $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$, we have $c_{A\cup B} = c_{A\setminus B} + c_A \cdot c_B + c_{B\setminus A} \in L$. Thus $(A \cup B) \in V$.

Now let $\{A_n\}$ be any countable collection of sets in V and $A = \bigcup_{n=1}^{\infty} A_n$. Let $B_n = A_1 \cup \cdots \cup A_n$ for every $n \in N$. Then $B_n \in V$, and therefore $c_{B_n} \in L$, for every $n \in N$. Moreover, $A = \bigcup_{n=1}^{\infty} B_n$, and $B_n \subset B_{n+1}$ for every $n \in N$. We claim that the sequence c_{B_n} converges pointwise to c_A. Take any $x \in X$. If $x \in A$, then there exists some $n_0 \in N$ such that $x \in B_{n_0}$. So $x \in B_n$, and therefore $c_{B_n}(x) = 1$, for all $n > n_0$. Thus $c_{B_n}(x) \longrightarrow c_A(x)$ if $x \in A$. If $x \notin A$, then $x \notin B_n$, and therefore $c_{B_n}(x) = 0$, for all $n \in N$. Thus $c_{B_n}(x) \longrightarrow c_A(x)$ if $x \notin A$. So we have proved that $c_{B_n}(x) \longrightarrow c_A(x)$ for every $x \in X$. Since L is closed under pointwise convergence everywhere on X, we must have $c_A \in L$. Hence $A = \bigcup_{n=1}^{\infty} A_n \in V$, which completes the proof that $V = \text{trace}(L)$ is a σ-ring. ∎

Corollary 3.13 *If L is a Baire algebra of functions in C^X, then* trace(L) *is a σ-algebra of subsets of X.*

Proof. By Theorem 3.12, trace(L) is a σ-ring. Since L contains all constant functions, we must have $c_X \in L$. Hence $X \in \text{trace}(L)$. ∎

Proposition 3.14 *If L is an algebra of functions in C^X which is closed under pointwise convergence and under involution, then L is closed under composition with every continuous function $u: C^n \to C$ vanishing at zero.*

Proof. Let $u: C^n \to C$ be a continuous function vanishing at zero. For $m \in N$ consider the compact set

$$D_m = \{(z_1, \dots, z_n) \in C^n: \ |z_j| \leq m, \ j = 1, 2, \dots, n\}.$$

Let $p_j: C^n \to C$ be the projection onto the j-th coordinate. Then the set

$$B = \{\mathrm{Re} \circ p_j, \ \mathrm{Im} \circ p_j: \ j = 1, 2, \dots, n\}$$

is a subset of $\mathcal{C}(C^n, R)$ and separates points of C^n. So by the Corollary of the Stone-Weierstrass Theorem the function u can be uniformly approximated on D_m by a sequence of polynomials of members of B with complex coefficients. Thus for every $m \in N$ there exists such polynomial q_m such that

$$|u(z) - q_m(z)| < 1/m$$

for all $z \in D_m$. From this we see that the sequence $q_m(z)$ converges to $u(z)$ for every $z \in C^n$.

Let $f_1, f_2, \dots, f_n$ be any n functions from L. Then $h_m = q_m \circ (f_1, f_2, \dots, f_n)$ is a sequence of functions in L (because L is an algebra closed under involution) converging pointwise to the function $u \circ (f_1, f_2, \dots, f_n)$. Since L is closed under pointwise convergence, we get $u \circ (f_1, f_2, \dots, f_n) \in L$. ∎

Proposition 3.15 *If L is an algebra of functions in C^X which is closed under pointwise convergence and under involution, and G is an open set in C such that $0 \notin \overline{G}$, then $(c_G \circ f) \in L$ for every function $f \in L$.*

Proof. Take any function $f \in L$. Let $F = G^c$ and d be the distance function defined by $d(z) = \inf\{|z - w|: w \in F\}$ for every $z \in C$. Function d is continuous on C. For each $n \in N$ consider function d_n defined by $d_n(z) = \min\{nd(z), 1\}$. Then function d_n is continuous on C and vanishes at zero (since $0 \in F$) for every $n \in N$. By Proposition 3.14, we get $(d_n \circ f) \in L$ for every $n \in N$. We claim that $d_n(z) \longrightarrow c_G(z)$ for every $z \in C$. If $z \in G$, then $d(z) > 0$, and so there exists some $n_0 \in N$ such that $d_n(z) = 1$ for all $n > n_0$. Thus $d_n(z) \longrightarrow c_G(z)$ if $z \in G$. If $z \notin G$, then $d(z) = 0$, and so $d_n(z) = 0$ for all $n \in N$. Hence $d_n(z) \longrightarrow c_G(z)$ if $z \notin G$.

So we have a sequence $d_n \circ f$ of functions in L, which converges pointwise to $c_G \circ f$. Thus $(c_G \circ f) \in L$, since L is closed under pointwise convergence. ∎

Theorem 3.16 *If L is an algebra of functions in C^X which is closed under pointwise convergence and under involution, and $V = \text{trace}(L)$, then $M(V, C) = L$.*

Proof. Take any function $f \in M(V, C)$ and let s_n be the sequence of functions in $S(V, C)$ converging pointwise to f. Then $s_n \in L$ for every $n \in N$ since $S(V, C) \subset L$. Since L is closed under pointwise convergence, we get $f \in L$. So

$$M(V, C) \subset L. \tag{3.3}$$

Now we take any function $f \in L$. Let u_n be the sequence of functions from Lemma 3.7. As in the proof of Lemma 3.8, we have

$$u_n \circ f = \sum_{j=1}^{k_n} w_j^n c_{f^{-1}(A_j^n)}$$

where the sets A_j^n are mutually disjoint and each one is the difference of two open sets in C whose closures do not contain 0. Suppose that A is any of such sets A_j^n, then $A = G \setminus H$ for some open sets G and H in C such that $0 \notin \bar{G}$ and $0 \notin \bar{H}$.

Hence

$$\begin{aligned} c_A \circ f &= c_{G\setminus H} \circ f \\ &= c_{G\setminus(G\cap H)} \circ f \\ &= (c_G - c_G \cdot c_H) \circ f \\ &= c_G \circ f - (c_G \circ f)(c_H \circ f). \end{aligned}$$

By Proposition 3.15, both $c_G \circ f$ and $c_H \circ f$ are in L. Thus $(c_A \circ f) \in L$; or equivalently $c_{f^{-1}(A)} \in L$, and so $f^{-1}(A) \in V$. Hence $(u_n \circ f) \in S(V,C)$, and therefore $(u_n \circ f) \in M(V,C)$ for every $n \in N$. Since the sequence $u_n \circ f$ converges pointwise to $e \circ f = f$, and $M(V,C)$ is closed under pointwise convergence, we must have $f \in M(V,C)$. Thus

$$L \subset M(V,C). \tag{3.4}$$

From (3.3) and (3.4) we conclude that $M(V,C) = L$. ∎

Chapter 4

Dirac Integral Spaces

Let A be an algebra of functions in the space C^X and L a subset of A. We say that L is a *solid* subset of A if whenever $f \in A$, and $g \in L$, and $|f(x)| \leq |g(x)|$ for every $x \in X$, we have $f \in L$.

Proposition 4.1 *If L is a solid subset of a Baire algebra A, then $f \in L$ implies $|f| \in L$.*

Proof. Take any function $f \in L$. Then $f \in A$. Since A is a Baire algebra, by Corollary 2.12 it is closed under composition with continuous functions. Thus $|f| \in A$. But L is a solid subset of A. So we must have $|f| \in L$. ∎

A function $f \in C^X$ is called *non-negative* if $f(x) \in R$ and $f(x) \geq 0$ for every $x \in X$. A function $\int$ from a linear space L of functions in C^X into the field of complex numbers C is called a *linear functional* if for every functions f and g from the space L and any complex number λ we have

1. $\int(f+g) = \int f + \int g$.

2. $\int \lambda f = \lambda \int f$.

A linear functional $\int : L \to C$ is called *positive* if $\int f \geq 0$ for every non-negative function $f \in L$.

A quintuple $(X, C, A, L, \int)$ will be called *Dirac Integral Space* if the following axioms are satisfied:

1. The set A is a Baire algebra of functions in C^X.
2. The set L is a solid linear subspace of A.
3. The map $\int$ is a positive linear functional from L into C.
4. If f_n is a sequence of functions in L such that $\sum_n |f_n(x)| < \infty$ for every $x \in X$, the series $\sum_n f_n(x)$ converges to $f(x)$ for every $x \in X$, and the series $\sum_n \int |f_n|$ is also convergent, then $f \in L$ and $\int f = \sum_n \int f_n$.

The members of the space L are called *summable* (*integrable*) *functions*, and the functional $\int$ will be called *Dirac integral.* Before deriving important theorems within this theory let us give some examples of Dirac Integral Spaces.

Example 1. Let X be any non-empty set and $x_0 \in X$. If we take $A = L = C^X$ and define $\int f = f(x_0)$ for every $f \in L$, then $(X, C, A, L, \int)$ is a Dirac Integral Space.

Example 2. Let X be any non-empty set and $A = C^X$. Notice that the collection $D = \{J \subset X\colon\ J \text{ is finite}\}$ is a directed set with respect to the set inclusion relation $\subset$. Let L be the set of all functions $f \in A$ such that

$$\lim_{J \in D, \subset} \sum_{x \in J} f(x) \quad \text{exists.}$$

Define

$$\int f = \lim_{J \in D, \subset} \sum_{x \in J} f(x)$$

for every $f \in L$. Then $(X, C, A, L, \int)$ is a Dirac Integral Space.

Example 3. Again we take any non-empty set X and $A = C^X$. Let g be a fixed non-negative real-valued function defined on X. Denote by L the set of all functions $f \in A$ such that

$$\lim_{J \in D, \subset} \sum_{x \in J} g(x) f(x) \quad \text{exists.}$$

If we define

$$\int f = \lim_{J \in D, \subset} \sum_{x \in J} g(x) f(x)$$

for every $f \in L$, then $(X, C, A, L, \int)$ is a Dirac Integral Space.

Example 4. Consider the integral developed by V. M. Bogdan in his paper [5]. He starts with a positive volume v defined on a prering V of subsets of a space X, that is a set-function $v: V \to [0, \infty)$ satisfying the condition: $v(\bigcup_n E_n) = \sum_n v(E_n)$ for every countable collection $\{E_n\}$ of disjoint sets from V such that $\bigcup_n E_n \in V$. For every simple function $s \in S(V, C)$ define $\int s = \sum_{i=1}^k z_i v(E_i)$ and $\|s\| = \int |s|$ if $s = \sum_{i=1}^k z_i c_{E_i}$ for some complex numbers $z_1, \ldots, z_k$ and disjoint sets $E_1, \ldots, E_k$ from V. Then $\int$ is a positive linear functional on $S(V, C)$. A sequence t_n is called *basic* if there exists a sequence $s_n \in S(V, C)$ and a positive constant M such that $t_n = s_1 + s_2 + \cdots + s_n$ and $\|s_n\| \leq M 4^{-n}$ for every $n \in N$. A subset E of the space X is called a *null-set* if for every $\varepsilon > 0$ there exists a countable collection $\{E_n\}$ of sets from V such that $E \subset \bigcup_n E_n$ and $\sum_n v(E_n) < \varepsilon$. A condition $p(x)$ depending on a parameter $x \in X$ is said to be satisfied *almost everywhere* (*a.e.*) on X if there exists a null-set E such that $p(x)$ is true for all $x \in (X \setminus E)$. Let L be the set of all functions $f \in C^X$ such that there exists a basic sequence t_n converging pointwise almost everywhere to f. For every $f \in L$ define $\int f = \lim_n \int t_n$ if t_n is a basic sequence converging pointwise almost everywhere to f. Consider the set A' of all functions $f \in C^X$ such that $f^{-1}(C \setminus \{0\}) \subset \bigcup_{n \in N} E_n$ for some sequence E_n of sets from V and $c_E \frac{f}{1+|f|} \in L$ for every $E \in V$. Let A be the set of all functions $f \in C^X$ such that $c_E f \in A'$ for every $E \in V$. Again the quintuple $(X, C, A, L, \int)$ is a Dirac Integral Space. The facts that A is a Baire algebra and L is a solid linear subspace of A follow from Theorems 1, 2 and 3 of paper [6]. Using Theorems 1, 4 and 5 of paper [5] we can prove that conditions 3 and 4 of the Dirac Integral Space are also satisfied.

Example 5. We can also derive a Dirac Integral Space from a given one. Suppose that $(X, C, A, L, \int)$ is a Dirac Integral Space. Take any function $f_0 \in A$ and let $X' = f_0(X)$. If we let $A' = \{u \in C^{X'}\colon\ u \circ f_0 \in A\}$ and $L' = \{u \in A'\colon\ u \circ f_0 \in L\}$ and $\int' u = \int (u \circ f_0)$ for every $u \in L'$, then $(X', C, A', L', \int')$ is also a Dirac Integral Space.

Example 6. Let Γ be any index set and $(X_j, C, A_j, L_j, \int_j)$ a Dirac Integral Space for each $j \in \Gamma$. Let $\widetilde{X_j} = X_j \times \{j\}$ for each $j \in \Gamma$. Then the sets $\widetilde{X_j}$ are mutually disjoint and each $\widetilde{X_j}$ is in one-to-one correspondence with X_j. We shall identify these spaces. Consider the space $X = \bigcup_{j \in \Gamma} X_j$. Let $A = \{f \in C^X\colon\ c_{X_j} f \in A_j\}$ and

$$L = \left\{ f \in A\colon\ c_{X_j} f \in L_j \ \ \& \ \ \sum_{j \in \Gamma} \int_j \left| c_{X_j} f \right| < \infty \right\}$$

and

$$\int f = \sum_{j \in \Gamma} \int_j c_{X_j} f.$$

Then $(X, C, A, L, \int)$ is also a Dirac Integral Space.

Example 7. Suppose that $(X, C, A_j, L_j, \int_j)$ is a Dirac Integral Space for each $j \in \Gamma$. If we take $A = \bigcap_{j \in \Gamma} A_j$ and $L = \{f \in \bigcap_{j \in \Gamma} L_j\colon\ \sum_{j \in \Gamma} \int_j |f| < \infty\}$ and $\int f = \sum_{j \in \Gamma} \int_j f$, then $(X, C, A, L, \int)$ is also a Dirac Integral Space.

4.1 Convergence Theorems

In this section we are going to verify important theorems in the integration theory, namely the Monotone Convergence Theorem and the Dominated Convergence Theorem. To accomplish this we need several Propositions and Lemmas. We begin by proving that the Dirac integral is a monotone functional, in the sense described in the following Proposition.

Proposition 4.2 *If f and g are real-valued functions in L, and $f(x) \leq g(x)$ for every $x \in X$, then $\int f \leq \int g$.*

Proof. Let f and g be real-valued functions in L such that $f(x) \leq g(x)$ for every $x \in X$. Then $(g - f)$ is a non-negative function in L. Since $\int$ is a positive linear functional, we must have $\int(g - f) \geq 0$. Hence $\int g - \int f \geq 0$ and thus $\int f \leq \int g$. ∎

Lemma 4.3 *If $f \in L$ and φ: $C \to R$ is a linear map over the field of reals, then*

1. *The composite function $(\varphi \circ f) \in L$, and $\int(\varphi \circ f) = \varphi(\int f)$.*
2. *For every $z \in C$, $\sup \{\varphi(z) : |\varphi| \leq 1\} = |z|$.*

Proof. Let $f \in L$. Then $f \in A$ and thus $g = \mathrm{Re} \circ f \in A$. Since L is solid in A and $|g(x)| \leq |f(x)|$ for every $x \in X$, we have $g \in L$. Similarly, we get $h = \mathrm{Im} \circ f \in L$. Now, let φ: $C \to R$ be a linear map. Then

$$\begin{aligned}(\varphi \circ f)(x) &= \varphi(f(x)) \\ &= \varphi(g(x) + ih(x)) \\ &= g(x)\varphi(1) + h(x)\varphi(i)\end{aligned}$$

for every $x \in X$. So $\varphi \circ f = \varphi(1) \cdot g + \varphi(i) \cdot h$, and thus by the linearity of L, we get $(\varphi \circ f) \in L$. By the linearity of $\int$ we also have

$$\begin{aligned}\int(\varphi \circ f) &= \varphi(1) \int g + \varphi(i) \int h \\ &= \varphi(\int g) + \varphi(i \int h) \\ &= \varphi(\int(g + ih)) \\ &= \varphi(\int f).\end{aligned}$$

For every $z \in C$ we have $|\varphi(z)| \leq |\varphi||z|$. Thus $|\varphi(z)| \leq |z|$ for all φ such that $|\varphi| \leq 1$. Hence

$$\sup\, \{\varphi(z) \colon\ |\varphi| \leq 1\} \leq |z| \tag{4.1}$$

for every $z \in C$.

If $z = (x, y)$ is any complex number such that $|z| \leq 1$ and $v = (\varphi(1), \varphi(i))$, then

$$\begin{aligned}
|\varphi(z)| &= |\varphi(x + iy)| \\
&= |\varphi(x) + \varphi(iy)| \\
&= |(\varphi(1))x + (\varphi(i))y| \\
&= |v \bullet z| \\
&= |v||z||\cos\alpha| \\
&\leq |v|.
\end{aligned}$$

Thus $|\varphi| = \sup\, \{|\varphi(z)| \colon\ |z| \leq 1\} \leq |v| = |(\varphi(1), \varphi(i))|$ for every linear map $\varphi \colon C \to R$.

Now let $z = (x, y)$ be any complex number in C, and $\varphi \colon C \to R$ be a map defined by $\varphi(w) = \frac{1}{|z|}(x\mathrm{Re}(w) + y\mathrm{Im}(w))$ for every $w \in C$. Map φ is linear, since maps Re and Im are. Since $|\varphi| \leq |(\varphi(1), \varphi(i))| = |(\frac{x}{|z|}, \frac{y}{|z|})| = \frac{1}{|z|}|z| = 1$ and $\varphi(z) = \frac{1}{|z|}(x^2 + y^2) = \frac{1}{|z|}|z|^2 = |z|$, we must have

$$|z| \leq \sup\, \{\varphi(z) \colon\ |\varphi| \leq 1\}. \tag{4.2}$$

Hence, from (4.1) and (4.2), we conclude that $\sup\, \{\varphi(z) \colon\ |\varphi| \leq 1\} = |z|$. ∎

Proposition 4.4 *For every $f \in L$, the inequality $|\int f| \leq \int |f|$ holds.*

Proof. Let $f \in L$ and $\varphi \colon C \to R$ be a linear map such that $|\varphi| \leq 1$. Then for every $x \in X$ we have $|\varphi \circ f|(x) = |\varphi(f(x))| \leq |\varphi||f(x)| \leq |f(x)| = |f|(x)$, i.e.

$|\varphi \circ f| \leq |f|$. Thus, by applying the first part of Lemma 4.3, we get $\varphi(\int f) = \int(\varphi \circ f) \leq \int |\varphi \circ f| \leq \int |f|$, which means that $\int |f|$ is an upper-bound of the set $\{\varphi(\int f)\colon |\varphi| \leq 1\}$. But, by using the second part of Lemma 4.3, we have $\sup\{\varphi(\int f)\colon |\varphi| \leq 1\} = |\int f|$. Hence $|\int f| \leq \int |f|$. ∎

A real-valued function $\|\cdot\|$ defined on a linear space L is called a *semi-norm* on L if it satisfies the following conditions:

1. $\|f\| \geq 0$ for every $f \in L$.

2. $\|f + g\| \leq \|f\| + \|g\|$ for every f and g in L.

3. $\|\lambda f\| = |\lambda| \|f\|$ for every $\lambda \in C$ and $f \in L$.

Let us define $\|f\| = \int |f|$ for every f in the space L of summable functions and prove the following Proposition.

Proposition 4.5 *The function $\|\cdot\|$ is a semi-norm on L.*

Proof.

1. Since $\int$ is a positive linear functional, we immediately get $\|f\| = \int |f| \geq 0$ for every $f \in L$.

2. For every f and g in L and every $x \in X$ we have $|f+g|(x) = |f(x) + g(x)| \leq |f(x)| + |g(x)| = (|f| + |g|)(x)$. Thus by Proposition 4.2 we get $\int |f+g| \leq \int(|f| + |g|) = \int |f| + \int |g|$, which shows that $\|f+g\| \leq \|f\| + \|g\|$.

3. For every $\lambda \in C$, $f \in L$ and $x \in X$, we have $|\lambda f|(x) = |(\lambda f)(x)| = |\lambda f(x)| = |\lambda||f(x)| = |\lambda||f|(x)$. Hence, by the linearity of $\int$, we get $\int |\lambda f| = \int |\lambda||f| = |\lambda| \int |f|$, which means that $\|\lambda f\| = |\lambda| \|f\|$. ∎

Proposition 4.6 *If $f \in L$ and f_n is a sequence of functions in L such that $\|f_n - f\| \longrightarrow 0$, then $\int f_n \longrightarrow \int f$.*

Proof. Let $\varepsilon > 0$ be given. Since $\|f_n - f\| \longrightarrow 0$, there exists a positive integer n_0 such that $\|f_n - f\| < \varepsilon$ for all $n > n_0$. But $|\int f_n - \int f| = |\int (f_n - f)| \le \int |f_n - f| = \| f_n - f \| < \varepsilon$ for all $n > n_0$, which means that $\int f_n \longrightarrow \int f$. ∎

A sequence f_n of real-valued functions defined on X is said to be *increasing* if $f_n(x) \le f_{n+1}(x)$ for every $n \in N$ and $x \in X$. The sequence f_n is called *decreasing* if $f_n(x) \ge f_{n+1}(x)$ for every $n \in N$ and $x \in X$. It is called a *monotone* sequence if it is either increasing or decreasing.

Theorem 4.7 (Monotone Convergence Theorem) *If f_n is a monotone sequence of real-valued functions in L such that $f_n(x) \longrightarrow f(x)$ for every $x \in X$, then the following statements are equivalent:*

1. *The sequence $\int f_n$ is bounded.*
2. *The function $f \in L$ and $\int f_n \longrightarrow \int f$.*
3. *The function $f \in L$ and $\|f_n - f\| \longrightarrow 0$.*
4. *The sequence $\|f_n\|$ is bounded.*

Proof. Let us assume that the sequence f_n is increasing. For decreasing sequences the proof is analogous.

(1) $\Rightarrow$ (2): Assume that the sequence $\int f_n$ of numbers is bounded. Let $g_n = f_{n+1} - f_n$ for every n. Then $g_n \in L$ and $g_n(x) \ge 0$ for every n and every $x \in X$. So

$$\begin{aligned}
\sum_n |g_n(x)| &= \sum_n g_n(x) \\
&= \lim_{n\to\infty} \sum_{j=1}^{n} g_j(x) \\
&= \lim_{n\to\infty} (f_{n+1}(x) - f_1(x)) \\
&= f(x) - f_1(x) \\
&= (f - f_1)(x) < \infty
\end{aligned}$$

for every $x \in X$. By the linearity of $\int$ we also have

$$\sum_{j=1}^{n} \int g_j = \int f_{n+1} - \int f_1$$

which is bounded since the sequence $\int f_n$ is bounded. Thus the sequence $\sum_{j=1}^{n} \int g_j$ is convergent, and so

$$\sum_n \int |g_n| = \sum_n \int g_n < \infty.$$

Now, using axiom (4) of Dirac Integral Space, we get $(f - f_1) \in L$ and

$$\int (f - f_1) = \sum_n \int g_n.$$

Hence $f = (f - f_1) + f_1 \in L$ and

$$\begin{aligned} \int f - \int f_1 &= \lim_n \sum_{j=1}^{n-1} \int g_j \\ &= \lim_n \left(\int f_n - \int f_1 \right) \\ &= \left(\lim_n \int f_n \right) - \int f_1 \end{aligned}$$

which implies that $\lim_n \int f_n = \int f$.

(2) $\Rightarrow$ (3): Let $f \in L$ and $\int f_n \longrightarrow \int f$. Then $(\int f - \int f_n) \longrightarrow 0$. But $\|f_n - f\| = \int |f_n - f| = \int (f - f_n) = \int f - \int f_n$. Hence $\|f_n - f\| \longrightarrow 0$.

(3) $\Rightarrow$ (4): Suppose that $f \in L$ and $\|f_n - f\| \longrightarrow 0$. This means that the sequence $\|f_n - f\|$ is bounded. But $\|f_n\| \leq \|f_n - f\| + \|f\|$. Hence the sequence $\|f_n\|$ is also bounded.

(4) $\Rightarrow$ (1): Let $\|f_n\|$ be a bounded sequence. Then the sequence $\int f_n$ is also bounded, since $|\int f_n| \leq \int |f_n| = \|f_n\|$. ∎

Theorem 4.8 (Dominated Convergence Theorem) *If $g \in L$ and f_n is a sequence of functions in L such that $|f_n| \leq |g|$ for every $n \in N$ and f_n converges pointwise to some function f, then $f \in L$ and $\|f_n - f\| \longrightarrow 0$ and $\int f_n \longrightarrow \int f$.*

Proof. We first consider the case where f_n and g are real-valued functions in L. Then f must also be real-valued. For every pair of positive integers n and m let

$$h_{nm}(x) = \sup\ \{f_j(x)\colon\ n \leq j < n+m\}$$

for every $x \in X$. The function h_{nm} is well-defined since $-|g(x)| \leq f_j(x) \leq |g(x)|$ for every $j \in N$ and $x \in X$. We observe that $h_{nm} \in L$ for every $n, m \in N$. Notice that as $m \to \infty$ the sequence h_{nm} converges increasingly to the function h_n defined by the formula

$$h_n(x) = \sup\ \{f_j(x)\colon\ n \leq j\}$$

for every $x \in X$. Since $h_{nm}(x) \leq |g(x)|$ for every $n, m \in N$ and every $x \in X$, we have $\int h_{nm} \leq \int |g| < \infty$ for every $n, m \in N$. So the sequence $\int h_{nm}$ is bounded. Thus by the Monotone Convergence Theorem we get $h_n \in L$ for every $n \in N$. But the sequence h_n converges decreasingly to the function f and the sequence $\int h_n$ is bounded. Applying the Monotone Convergence Theorem again we get $f \in L$ and $\lim_n \int h_n = \int f$.

Similarly, if for every $n, m \in N$ we define

$$k_{nm}(x) = \inf\ \{f_j(x)\colon\ n \leq j < n+m\}$$

and

$$k_n(x) = \inf\ \{f_j(x)\colon\ n \leq j\}$$

for every $x \in X$, then k_n is an increasing sequence of functions in L converging pointwise to f, and the sequence $\int k_n$ is bounded. Thus $\lim_n \int k_n = \int f$. We also notice that

$$k_n(x) \leq f_n(x) \leq h_n(x)$$

and

$$k_n(x) \leq f(x) \leq h_n(x)$$

which imply that

$$|f_n(x) - f(x)| \leq h_n(x) - k_n(x)$$

for every $n \in N$ and every $x \in X$. So

$$\|f_n - f\| = \int |f_n - f| \;\leq\; \int h_n - \int k_n.$$

Hence $\lim_n \|f_n - f\| \leq \lim_n \int h_n - \lim_n \int k_n = \int f - \int f = 0$, which shows that $\lim_n \|f_n - f\| = 0$.

In the case where f_n, g and f are complex-valued functions, we see that $|g| \in L$ and $|\mathrm{Re} \circ f_n| \leq |f_n| \leq |g|$ and $|\mathrm{Im} \circ f_n| \leq |f_n| \leq |g|$ for every $n \in N$. Since L is solid in A, we get $\mathrm{Re} \circ f_n \in L$ and $\mathrm{Im} \circ f_n \in L$ for every $n \in N$. From the continuity of the functions Re and Im we have

$$\mathrm{Re} \circ f_n \longrightarrow \mathrm{Re} \circ f \quad \text{and} \quad \mathrm{Im} \circ f_n \longrightarrow \mathrm{Im} \circ f.$$

By the above discussion on real-valued functions we obtain

$$\mathrm{Re} \circ f \in L \quad \text{and} \quad \|\mathrm{Re} \circ f_n - \mathrm{Re} \circ f\| \longrightarrow 0$$

and

$$\mathrm{Im} \circ f \in L \quad \text{and} \quad \|\mathrm{Im} \circ f_n - \mathrm{Im} \circ f\| \longrightarrow 0.$$

Thus $f = (\mathrm{Re} \circ f) + i(\mathrm{Im} \circ f) \in L$. Moreover

$$\begin{aligned} |f_n - f| &= |(\mathrm{Re} \circ f_n - \mathrm{Re} \circ f) + i(\mathrm{Im} \circ f_n - \mathrm{Im} \circ f)| \\ &\leq |\mathrm{Re} \circ f_n - \mathrm{Re} \circ f| + |\mathrm{Im} \circ f_n - \mathrm{Im} \circ f| \end{aligned}$$

which implies that

$$\begin{aligned} \|f_n - f\| &= \int |f_n - f| \\ &\leq \int |\mathrm{Re} \circ f_n - \mathrm{Re} \circ f| + \int |\mathrm{Im} \circ f_n - \mathrm{Im} \circ f| \\ &= \|\mathrm{Re} \circ f_n - \mathrm{Re} \circ f\| + \|\mathrm{Im} \circ f_n - \mathrm{Im} \circ f\|. \end{aligned}$$

Hence $\|f_n - f\| \longrightarrow 0$. And by Proposition 4.6 we get $\lim_n \int f_n = \int f$. ∎

4.2 Convergence Almost Everywhere

Let $L_0 = \{f \in L\colon \|f\| = 0\}$ and $V_0 = \text{trace}(L_0)$. The members of L_0 are called *null-functions* and the members of V_0 *null-sets*. A proposition $p(x)$, depending on $x \in X$, is said to hold *almost everywhere* (*a.e.*) on X if there exists a null-set E such that $p(x)$ is true for all $x \in (X \setminus E)$.

A subset B of an algebra A is called an *algebra-ideal* if B is a linear subspace of A, and B is closed under multiplication with functions in A, namely if $f \in A$ and $g \in B$ then $fg \in B$. We are going to prove that the set L_0 of null-functions is an algebra-ideal and a solid subset in the Baire algebra A, and is closed under pointwise convergence and under involution. Let us now establish the first step by proving the following Lemma.

Lemma 4.9 *The set L_0 is an algebra-ideal and is solid in the Baire algebra A.*

Proof. Take any two functions f and g in L_0. Then $\int |f| = \int |g| = 0$. For every $x \in X$ we have $|f(x) + g(x)| \leq |f(x)| + |g(x)|$. Thus $\int |f + g| \leq \int |f| + \int |g| = 0$, from which we conclude that $\int |f + g| = 0$. So $(f + g) \in L_0$. From the fact that $|\lambda f(x)| = |\lambda||f(x)|$ for every $x \in X$ and $\lambda \in C$, we obtain $\int |\lambda f| = |\lambda| \int |f| = 0$. Thus $\lambda f \in L_0$. Hence L_0 is a linear subspace of A.

Let $f \in A$ and $g \in L_0$. For every positive integer n consider function $h_n\colon C \to C$ defined by

$$h_n(z) = \begin{cases} z & \text{if } |z| \leq n \\ (nz)/|z| & \text{if } |z| > n \end{cases}$$

for every $z \in C$. Then function h_n is continuous, and $|h_n(z)| \leq n$ for every $n \in N$ and $z \in C$. Since the Baire algebra A is closed under composition with continuous functions, we must have $(h_n \circ f) \in A$, and hence the function $(h_n \circ f)g \in A$ for every $n \in N$. For every $x \in X$ we have $|(h_n \circ f)g|(x) = |h_n(f(x))g(x)| = |h_n(f(x))||g(x)| \leq n|g(x)| = |ng|(x)$. But $ng \in L_0$ (and therefore $ng \in L$), and

L is a solid subspace of A. So we get $(h_n \circ f)g \in L$, and hence the function $k_n = |(h_n \circ f)g| \in L$, for every $n \in N$. From the above inequality we also obtain $\int k_n \leq n \int |g| = 0$, which implies that $\int k_n = 0$ for every $n \in N$. Thus the sequence $\int k_n$ is bounded.

We observe that, for every $z \in C$, the sequence $|h_n(z)|$ converges increasingly to $|z|$ as n tends to infinity. This implies that $|h_n(f(x))| \nearrow |f(x)|$ for every $x \in X$. And so $k_n = |h_n \circ f||g| \nearrow |f||g| = |fg|$.

Now, applying the Monotone Convergence Theorem, we get $|fg| \in L$ and $\int k_n \nearrow \int |fg|$. But L is solid in A. So we must have $fg \in L$. Recall that the sequence $\int k_n = 0$ for every $n \in N$, from which we must conclude that $\int |fg| = 0$. Hence $fg \in L_0$.

To show that L_0 is solid in A, take any function $f \in A$ and $g \in L_0$ such that $|f(x)| \leq |g(x)|$ for every $x \in X$. Then $g \in L$ and $\int |g| = 0$. Since L is solid in A, we get $f \in L$. Hence $|f| \in L$; and $\int |f| \leq \int |g| = 0$, which implies that $\int |f| = 0$. So $f \in L_0$. ∎

Lemma 4.10 *If f_n is a sequence of non-negative real-valued functions in L_0, and $f(x) = \sup \{f_n(x) \colon n \in N\} < \infty$ for every $x \in X$, then $f \in L_0$.*

Proof. For each $n \in N$ and $x \in X$, let $g_n(x) = \sup \{f_1(x), \ldots, f_n(x)\}$ and $h_n(x) = f_1(x) + \cdots + f_n(x)$. Then $g_n \in A$ for every $n \in N$. We show this by induction. The function $g_2 \in A$, since $g_2 = \frac{1}{2}(f_1 + f_2 + |f_1 - f_2|)$. If $g_k \in A$, then $g_{k+1} \in A$, since $g_{k+1} = \frac{1}{2}(g_k + f_{k+1} + |g_k - f_{k+1}|)$.

For every $n \in N$, the function $h_n \in L_0$, since L_0 is a linear subspace of A. Notice that $0 \leq g_n(x) \leq h_n(x)$ for every $n \in N$ and $x \in X$. Since L_0 is solid in A, we get $g_n \in L_0$, and thus $\int g_n = 0$, for every $n \in N$. We also see that the sequence g_n converges increasingly to f. So, by the Monotone Convergence Theorem, we obtain $f \in L$ and $\int g_n \longrightarrow \int f$. Hence $\int f = 0$, which shows that $f \in L_0$. ∎

Proposition 4.11 *The set L_0 is an algebra-ideal and a solid subset in the Baire algebra A, and is closed under pointwise convergence and under involution.*

Proof. We have proved that L_0 is an algebra-ideal and is solid in A. Now take any sequence f_n of functions in L_0 which converges pointwise to some function $f \in A$. Let $g_n = |f_n|$ for every $n \in N$, and $g = |f|$. Then $g_n \in L_0$ and $g_n(x) \longrightarrow g(x)$ for every $x \in X$. Hence $g(x) = \limsup_{n\to\infty} g_n(x)$ for every $x \in X$, i.e. $g(x) = \inf \{h_m(x)\colon\ m \in N\}$ where $h_m(x) = \sup \{g_n(x)\colon\ n \geq m\}$ for every $x \in X$. By Lemma 4.10, we get $h_m \in L_0$ for every $m \in N$. We also see that the sequence $h_m(x)$ converges decreasingly to $g(x)$ for every $x \in X$, and that $\int h_m = 0$ for every $m \in N$. Thus, by the Monotone Convergence Theorem, we obtain $g \in L$ and $\int h_m \longrightarrow \int g$. From this it follows that $\int g = 0$, and hence $\int |g| = 0$. Thus $g \in L_0$. But L_0 is solid in A and $g = |f|$. So $f \in L_0$, and thus L_0 is closed under pointwise convergence.

To show that L_0 is closed under involution, we take any function $f \in L_0$. Then $f \in L$ and $\int |f| = 0$. Since L is solid in A and $|\bar{f}| = |f|$, we have $\bar{f} \in L$. Moreover $\int |\bar{f}| = \int |f| = 0$. Hence $\bar{f} \in L_0$. ∎

The *support* of a function $f\colon X \to C$ is the set $\mathrm{supp}(f)$ of all $x \in X$ at which $f(x) \neq 0$. In other words

$$\mathrm{supp}(f) = \{x \in X\colon\ f(x) \neq 0\} = f^{-1}(C \setminus \{0\}).$$

In the following Proposition we shall see further characterization of the null-functions.

Proposition 4.12 *If $f \in A$, then the following conditions are equivalent:*

1. *f is a null-function*

2. $\mathrm{supp}(f)$ *is a null-set*

3. $f(x) = 0$ *a.e. on* X

Proof. (1) $\Rightarrow$ (2) : Let f be a null-function, i.e. $f \in L_0$. Since L_0 is an algebra of functions in C^X which is closed under pointwise convergence and under involution, by Theorem 3.16 we get $L_0 = M(V_0, C)$, where $V_0 = \text{trace}(L_0)$. For each positive rational number $r \in Q$ let $B_r = \{z \in C\colon |z| \leq r\}$. Then the set $G_r = C \setminus B_r$ is open in C, and $0 \notin \overline{G_r}$. So $f^{-1}(G_r) \in V_0$ for every positive rational number r, and consequently $\bigcup_{r \in Q} f^{-1}(G_r) \in V_0$ since, according to Theorem 3.12, V_0 is a σ-ring. But

$$\begin{aligned}
\bigcup_{r\in Q} f^{-1}(G_r) &= \bigcup_{r\in Q} f^{-1}(C \setminus B_r) \\
&= \bigcup_{r\in Q} (f^{-1}(C) \setminus f^{-1}(B_r)) \\
&= f^{-1}(C) \setminus \bigcap_{r\in Q} f^{-1}(B_r) \\
&= f^{-1}(C) \setminus f^{-1}(\bigcap_{r\in Q} B_r) \\
&= f^{-1}(C \setminus \bigcap_{r\in Q} B_r) \\
&= f^{-1}(C \setminus \{0\}) \\
&= \text{supp}(f)
\end{aligned}$$

which shows that $\text{supp}(f) \in V_0$.

(2) $\Rightarrow$ (3) : Assume that $\text{supp}(f) \in V_0$. It is clear that $f(x) = 0$ for all $x \in X \setminus \text{supp}(f)$, which means that $f(x) = 0$ almost everywhere on X.

(3) $\Rightarrow$ (1) : Now we assume that $f(x) = 0$ a.e. on X. Then there exists a null-set $E \in V_0$ such that $f(x) = 0$ for all $x \in X \setminus E$. So $c_{X\setminus E} f = 0$. Since $c_{X \setminus E} f = f - c_E f$, we get $f = c_E f$. But $c_E \in L_0$ (since $E \in V_0$), and L_0 is an algebra-ideal in A. This implies that $c_E f \in L_0$, and therefore $f \in L_0$. ∎

Proposition 4.13 *If* $f \in L$, $g \in A$, *and* $f(x) = g(x)$ *almost everywhere on* X, *then* $g \in L$, $\int g = \int f$, *and* $\|g\| = \|f\|$.

Proof. Take any functions $f \in L$ and $g \in A$ such that $f(x) = g(x)$ almost everywhere on X. This means that there exists a null-set $E \in V_0$ such that $f(x) = g(x)$ for all $x \in X \setminus E$. Hence $c_{X\setminus E} f = c_{X\setminus E} g$, or equivalently $f - c_E f = g - c_E g$. Thus

$$g = f - c_E f + c_E g. \tag{4.3}$$

But $c_E \in L_0$, and L_0 is an algebra-ideal in A. So both $c_E f$ and $c_E g$ are contained in L_0, and therefore in L. Hence from equation 4.3 we conclude that $g \in L$.

Since $c_E f \in L_0$, we have $\int |c_E f| = 0$. From this it follows that $\int c_E f = 0$, since $0 \leq |\int c_E f| \leq \int |c_E f|$. Similarly, we get $\int c_E g = 0$. Hence $\int g = \int f - \int c_E f + \int c_E g = \int f$.

Notice that, by Proposition 4.1, we have $|f - g| \in L$. Thus $c_E |f - g| \in L_0$, which means that $\int c_E |f - g| = 0$. From this we get

$$\begin{aligned} |\, \|f\| - \|g\| \,| &\leq \|f - g\| \\ &= \int |f - g| \\ &= \int c_E |f - g| = 0. \end{aligned}$$

Hence $\|f\| = \|g\|$. ∎

Proposition 4.14 *If f and g are real-valued functions in L such that $f(x) \leq g(x)$ almost everywhere on X, then $\int f \leq \int g$.*

Proof. Let f and g be functions in L such that $f(x) \leq g(x)$ a.e. on X. Then we can find a null-set $E \in V_0$ such that $f(x) \leq g(x)$ for all $x \in X \setminus E$. Thus $c_{X\setminus E} f \leq c_{X\setminus E} g$. And so, by Proposition 4.2, we get $\int c_{X\setminus E} f \leq \int c_{X\setminus E} g$, or equivalently $\int f - \int c_E f \leq \int g - \int c_E g$. But as in the proof of Proposition 4.13 we have $\int c_E f = \int c_E g = 0$. Hence $\int f \leq \int g$. ∎

Theorem 4.15 (Strong Monotone Convergence Theorem) *If $f \in A$ and f_n is a monotone sequence of real-valued functions in L with respect to the relation $\leq$ a.e. on X and $f_n(x) \longrightarrow f(x)$ a.e. on X, then the following statements are equivalent:*

1. *The sequence $\int f_n$ is bounded.*
2. *The function $f \in L$ and $\int f_n \longrightarrow \int f$.*
3. *The function $f \in L$ and $\|f_n - f\| \longrightarrow 0$.*
4. *The sequence $\|f_n\|$ is bounded.*

Proof. Suppose that f_n is a sequence of real-valued functions in L with the properties: (1) for every $n \in N$ there exists a null-set $E_n \in V_0$ such that $f_n(x) \leq f_{n+1}(x)$ for every $x \in X \setminus E_n$; (2) there exist a function $f \in A$ and a null-set $E_0 \in V_0$ such that $f_n(x) \longrightarrow f(x)$ for every $x \in X \setminus E_0$. Let

$$E = \bigcup_{n=0}^{\infty} E_n.$$

Then $E \in V_0$, since V_0 is a σ-ring. Let $g_n = c_{X\setminus E} f_n$ and $g = c_{X\setminus E} f$. Notice that g_n is a sequence of real-valued functions in L such that $g_n(x) \leq g_{n+1}(x)$ for every $n \in N$ and every $x \in X$, and $g_n(x) \longrightarrow g(x)$ for every $x \in X$. Since $g_n = f_n$ a.e. and $g = f$ a.e., by Proposition 4.13, we get $f \in L \Leftrightarrow g \in L$ and $\int g_n = \int f_n$ and $\|g_n\| = \|f_n\|$ and $\int g = \int f$. We also observe that $g_n - g = c_{X\setminus E} f_n - c_{X\setminus E} f = c_{X\setminus E}(f_n - f)$, which means that $g_n - g = f_n - f$ a.e. on X, and therefore $\|g_n - g\| = \|f_n - f\|$. Hence we can apply Theorem 4.7 to get Theorem 4.15.

Similar arguments can be applied to prove the theorem for the case where the sequence f_n is decreasing.∎

Theorem 4.16 (Strong Dominated Convergence Theorem) *Let $f \in A$ and $g \in L$. If f_n is a sequence of functions in L such that $|f_n| \leq |g|$ a.e. on X for* *every* $n \in N$ *and* $f_n(x) \longrightarrow f(x)$ *a.e. on* X*, then* $f \in L$ *and* $\|f_n - f\| \longrightarrow 0$ *and thus* $\int f_n \longrightarrow \int f$.

Proof. Use similar arguments as in the proof of Theorem 4.15 and apply Proposition 4.13 and Theorem 4.8. ∎

4.3 Measure Generated by the Dirac Integral Space

Let V be a σ-algebra of subsets of the space X. A set-function $\mu: V \rightarrow [0, \infty]$ is called a *measure* if

1. $\mu(\emptyset) = 0$.

2. $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$ for every countable collection $\{E_n\}$ of disjoint sets in V. In this case we say that μ is *countably additive.*

These two conditions imply the finite additivity of the set-function μ on V. Every measure μ defined on V is monotone, in the sense that if E and D are sets in V such that $E \subset D$ then $\mu(E) \leq \mu(D)$. This inequality follows from the fact that $\mu(D) = \mu(E \cup (D \setminus E)) = \mu(E) + \mu(D \setminus E) \geq \mu(E)$.

A triple (X, V, μ) will be called *Lebesgue Measure Space* if

1. The collection V is a σ-algebra of subsets of X.

2. The set-function μ is a measure defined on V.

The members of V are called *measurable sets.* In the folowing theorem we shall see that every Dirac Integral Space can generate a Lebesgue Measure Space.

Theorem 4.17 *Let* $(X, C, A, L, \int)$ *be a Dirac Integral Space. If* $V = \text{trace}(A)$, *and* μ *is the set-function defined on* V *by the formula*

$$\mu(E) = \begin{cases} \int c_E & \text{if } c_E \in L \\ \infty & \text{if } c_E \notin L \end{cases}$$

for every $E \in V$, then the triple (X, V, μ) forms a Lebesgue Measure Space. This Lebesgue Measure Space is said to be generated by the Dirac Integral Space.

Proof. From Corollary 3.13 we know that V is a σ-algebra of subsets of X. We need to prove that the set-function μ is a measure. From the definition of this function it is clear that $\mu(E) \geq 0$ for every $E \in V$, and that $\mu(\emptyset) = 0$.

Let $\{E_n\}$ be any countable collection of mutually disjoint sets in V, and let $E = \bigcup_{n=1}^{\infty} E_n$. First, consider the case when $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. In this case we have $\mu(E_n) < \infty$, and therefore $c_{E_n} \in L$, for every $n \in N$. Since the sets E_n are mutually disjoint, we get

$$c_E(x) = \sum_{n=1}^{\infty} c_{E_n}(x)$$

for every $x \in X$. Notice also that

$$\sum_{n=1}^{\infty} \int c_{E_n} = \sum_{n=1}^{\infty} \mu(E_n) < \infty.$$

Now, applying Axiom 4 of the Dirac Integral Space, we get $c_E \in L$ and

$$\int c_E = \sum_{n=1}^{\infty} \int c_{E_n}.$$

Thus

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E_n).$$

In the case when $\sum_{n=1}^{\infty} \mu(E_n) = \infty$ we claim that $\mu(E) = \infty$, and therefore $\mu(E) = \sum_{n=1}^{\infty} \mu(E_n)$. Suppose that $\mu(E) < \infty$. Then $c_E \in L$. Let $E_k = E_1 \cup \cdots \cup E_k$ for each $k \in N$. Then $c_{E_k} \leq c_E$ for every $k \in N$. Since L is a solid subspace of A, we must have $c_{E_k} \in L$ for each $k \in N$. By Proposition 4.2 we also obtain $\int c_{E_k} \leq \int c_E$, or equivalently $\sum_{n=1}^{k} \mu(E_n) \leq \mu(E)$, for every $k \in N$. So we conclude that

$$\sum_{n=1}^{\infty} \mu(E_n) \leq \mu(E) < \infty$$

which contradicts our assumption. ∎

4.4 Completeness of the Space of Summable Functions

Let (X, V, μ) be the Lebesgue Measure Space generated by the Dirac Integral Space $(X, C, A, L, \int)$. A set $E \in V$ is called a set of *finite measure* if $\mu(E) < \infty$. Set $E \in V$ is of finite measure $\Leftrightarrow c_E \in L \Leftrightarrow E \in \text{trace}(L)$.

If $\mu(E) = 0$, then the set E is called a set of *measure zero*. Set $E \in V$ is of measure zero $\Leftrightarrow \int c_E = 0 \Leftrightarrow c_E \in L_0 \Leftrightarrow E \in V_0 = \text{trace}(L_0)$. In oher words, a set $E \in V$ is of measure zero if and only if it is a null-set.

Lemma 4.18 *Let f_n be an increasing sequence of non-negative real-valued functions in L such that $\int f_n$ is bounded. If E is the set of all $x \in X$ at which the sequence $f_n(x)$ is not convergent, then E is a null-set.*

Proof. Let us first examine the set E. Since the sequence $\{f_n(x)\}$ of values is increasing, we have

$$\begin{aligned} E &= \{x \in X \mid \lim_{n\to\infty} f_n(x) = \infty\} \\ &= \{x \in X \mid (\forall_{k\in N})\ (\exists_{n\in N})\ \ f_n(x) > k\} \\ &= \bigcap_{k\in N} \bigcup_{n\in N} \{x \in X \mid f_n(x) > k\} \\ &= \bigcap_{k\in N} \bigcup_{n\in N} E_{n,k} \end{aligned}$$

where $E_{n,k} = f_n^{-1}(G_k)$ and $G_k = \{z \in C \mid \text{Re}(z) > k\}$. By Theorem 3.16, we know that $A = M(V, C)$, which implies that $f_n \in M(V, C)$ for every $n \in N$. Thus $E_{n,k} \in V$ for every positive integers n and k. But V is a σ-algebra. So we get $E \in V$.

For each $k \in N$, let $D_k = \bigcup_{n\in N} E_{n,k}$. Then $c_{E_{n,k}} \nearrow c_{D_k}$ as n goes to infinity. Let M be a positive real number such that $\int f_n \leq M$ for all $n \in N$. Then

$$\int c_{E_{n,k}} = \frac{1}{k} \int k c_{E_{n,k}}$$

$$\begin{aligned} &\leq \frac{1}{k}\int f_n c_{E_{n,k}} \\ &\leq \frac{1}{k}\int f_n \\ &\leq \frac{M}{k} \end{aligned}$$

which means that the sequence $\{\int c_{E_{n,k}}\}_{n\in N}$ is bounded for every $k \in N$. Applying the Monotone Convergence Theorem, we get $c_{D_k} \in L$ and $\int c_{E_{n,k}} \longrightarrow \int c_{D_k}$ for each $k \in N$. This convergence implies that $\int c_{D_k} \leq M/k$ for every $k \in N$. Notice that $E = \bigcap_{k\in N} D_k$. Thus, by the monotonicity of μ, we get $\mu(E) \leq \mu(D_k) = \int c_{D_k} \leq \frac{M}{k}$ for every $k \in N$. This implies that $\mu(E) = 0$, i.e. E is a null-set. ∎

Recall that the space L of summable functions is a semi-normed linear space, with the semi-norm $\|\cdot\|$ as defined on page 51. A sequence x_n in a semi-normed space $(X, \|\cdot\|)$ is called a *Cauchy sequence* if for every $\varepsilon > 0$ there exists a positive integer n_0 such that $\|x_m - x_n\| < \varepsilon$ for all $m, n > n_0$. A semi-normed linear space is said to be *complete* if every Cauchy sequence in it converges to some point belonging to the space. Now we are going to prove that the space L is complete. To do that we use the following Theorem, which is due to S. Banach [2].

Theorem 4.19 *A semi-normed space $(X, \|\cdot\|)$ is complete if and only if for every sequence x_n in X satisfying the condition $\sum_{n=1}^{\infty} \|x_n\| < \infty$, there exists an element $x \in X$ such that* $\lim_{n\to\infty} \|x - (x_1 + \cdots + x_n)\| = 0$.

Theorem 4.20 *If $(X, C, A, L, \int)$ is a Dirac Integral Space, then $(L, \|\cdot\|)$ is a complete semi-normed space.*

Proof. By Proposition 4.5, we already have that $(L, \|\cdot\|)$ is a semi-normed space. Let f_n be a sequence of functions in L such that $\sum_{n=1}^{\infty} \|f_n\| < \infty$. Let $g_n =$

$|f_1| + |f_2| + \cdots + |f_n|$. Then g_n is an increasing sequence of non-negative real-valued functions in L. Moreover

$$\begin{aligned} \int g_n &= \sum_{j=1}^{n} \int |f_j| \\ &= \sum_{j=1}^{n} \|f_j\| \\ &\le \sum_{j=1}^{\infty} \|f_j\| < \infty. \end{aligned}$$

Thus, by Lemma 4.18, the set E of all $x \in X$, at which the sequence $g_n(x)$ is not convergent, is a null-set. Now let $f: X \to C$ be a function defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in E \\ \sum_{n=1}^{\infty} f_n(x) & \text{if } x \notin E \end{cases}$$

for every $x \in X$. This function f is well defined, since if $x \notin E$ then the sequence $g_n(x)$ is convergent, i.e. the series $\sum_{n=1}^{\infty} f_n(x)$ is absolutely convergent, which implies that this series is convergent. Consider function $h_n = c_{X \setminus E} f_n$, i.e. $h_n = f_n$ almost everywhere for every $n \in N$. By Proposition 4.13 we get $h_n \in L$ and $\|h_n\| = \|f_n\|$ for every $n \in N$. We also notice that

$$\sum_{n=1}^{\infty} h_n(x) = f(x)$$

for every $x \in X$. Morover, the series

$$\sum_{n=1}^{\infty} \int |h_n| = \sum_{n=1}^{\infty} \|h_n\| = \sum_{n=1}^{\infty} \|f_n\| < \infty.$$

Applying the Axiom 4 of the Dirac Integral Space, we obtain $f \in L$ and $\int f = \sum_{n=1}^{\infty} \int h_n$.

From the above discussion we see that

$$\begin{aligned} \|f - (h_1 + \cdots + h_n)\| &= \|\sum_{j=1}^{\infty} h_j - (h_1 + \cdots + h_n)\| \\ &= \|\sum_{j=n+1}^{\infty} h_j\| \end{aligned}$$

$$= \int \Big| \sum_{j=n+1}^{\infty} h_j \Big|$$
$$\le \int \sum_{j=n+1}^{\infty} |h_j|. \tag{4.4}$$

Notice that

$$\sum_{j=n+1}^{\infty} |h_j| = \lim_{m\to\infty} \sum_{j=n+1}^{n+m} |h_j|$$

for each fixed positive integer n. But $s_m = \sum_{j=n+1}^{n+m} |h_j|$ is an increasing sequence of functions in L and

$$\begin{aligned} \int s_m &= \int \sum_{j=n+1}^{n+m} |h_j| \\ &= \sum_{j=n+1}^{n+m} \int |h_j| \\ &= \sum_{j=n+1}^{n+m} \|h_j\| \\ &= \sum_{j=n+1}^{n+m} \|f_j\| \\ &\le \sum_{j=1}^{\infty} \|f_j\| < \infty. \end{aligned}$$

Hence, by the Monotone Convergence Theorem, we get

$$\sum_{j=n+1}^{\infty} |h_j| \in L$$

and

$$\int \sum_{j=n+1}^{\infty} |h_j| = \lim_{m\to\infty} \int \sum_{j=n+1}^{n+m} |h_j|.$$

Since

$$\int \sum_{j=n+1}^{n+m} |h_j| = \sum_{j=n+1}^{n+m} \|f_j\|$$

for every $m \in N$, we must have

$$\int \sum_{j=n+1}^{\infty} |h_j| = \sum_{j=n+1}^{\infty} \|f_j\| \le \sum_{j=1}^{\infty} \|f_j\| < \infty.$$

Thus

$$\lim_{n\to\infty} \int \sum_{j=n+1}^{\infty} |h_j| = 0.$$

Hence, from inequality (4.4), it follows that

$$\|f - (h_1 + \cdots + h_n)\| \longrightarrow 0$$

as $n \to \infty$.

Now let $u_n = f - (h_1 + \cdots + h_n)$ and $v_n = f - (f_1 + \cdots + f_n)$. Then $u_n - v_n = (f_1 - h_1) + \cdots + (f_n - h_n) = 0$ almost everywhere since $f_n = h_n$ a.e. for every $n \in N$. Thus $u_n = v_n$ a.e., and therefore $\|u_n\| = \|v_n\|$, for every $n \in N$. This shows that $\|f - (f_1 + \cdots + f_n)\| \longrightarrow 0$ as n goes to infinity. By Theorem 4.19, we conclude that the space $(L, \|\cdot\|)$ is complete. ∎

Let L/L_0 be the *quotient space* generated by the space L_0 of null-functions, that is

$$L/L_0 \;=\; \{[f]:\; [f] = f + L_0,\; f \in L\}.$$

Then L/L_0 is a linear space with respect to addition and scalar multiplication defined as follows:

$$[f] + [g] = [f + g]$$

$$\lambda[f] = [\lambda f].$$

It is obvious that $f \in [f]$; and if $f \in L_0$, then $[f] = L_0$. It is also easy to see that $[f] + L_0 = [f]$ for every $[f] \in L/L_0$, which means that L_0 is the zero element in the linear space L/L_0. If g_1 and g_2 are any two functions in $[f]$, then $g_1 = f + h_1$ and $g_2 = f + h_2$ for some functions h_1 and h_2 in L_0. Thus the function $g_1 - g_2 = h_1 - h_2 \in L_0$. So, by Proposition 4.12, we get $(g_1 - g_2)(x) = 0$ almost everywhere, i.e. $g_1(x) = g_2(x)$ almost everywhere. Hence Proposition 4.13 implies that $\int g_1 = \int g_2$ and $\|g_1\| = \|g_2\|$.

Now on L/L_0 we define a complex-valued function $\mathcal{I}$ by

$$\mathcal{I}[f] = \int f$$

for every $[f] \in L/L_0$, and a norm $\sharp \cdot \sharp$ by

$$\sharp[f]\sharp = \|f\|$$

for every $[f] \in L/L_0$. By the above remark, these are good definitions, since they are independent of the representations which we choose from the residue class $[f]$.

Proposition 4.21 *The function $\mathcal{I}$ is a linear functional on L/L_0, and the inequality $|\mathcal{I}[f]| \le \sharp[f]\sharp$ holds for every $[f] \in L/L_0$.*

Proof. For every $[f]$ and $[g]$ in L/L_0 and every complex numbers λ and μ we have

$$\begin{aligned} \mathcal{I}(\lambda[f] + \mu[g]) &= \mathcal{I}[\lambda f + \mu g] \\ &= \int (\lambda f + \mu g) \\ &= \lambda \int f + \mu \int g \\ &= \lambda \mathcal{I}[f] + \mu \mathcal{I}[g] \end{aligned}$$

which shows that function $\mathcal{I}$ is linear. Moreover $|\mathcal{I}[f]| = |\int f| \le \int |f| = \|f\| = \sharp[f]\sharp$. ∎

A semi-norm $\|\cdot\|$ defined on a linear space L is called a *norm* if $\|f\| = 0$ only if $f = 0$. A complete normed linear space is also called a *Banach space.*

Theorem 4.22 *The space $(L/L_0, \sharp \cdot \sharp)$ is a Banach space.*

Proof. From the fact that $\|\cdot\|$ is a semi-norm it is easy to show that $\sharp \cdot \sharp$ is a semi-norm. Let $\sharp[f]\sharp = 0$. Then $\|f\| = 0$, which means that $f \in L_0$. So $[f] = L_0$, which is the zero element in L/L_0. Hence $\sharp \cdot \sharp$ is a norm.

Let $[f_n]$ be a Cauchy sequence in L/L_0. Given any $\varepsilon > 0$, we can find a positive integer n_0 such that $\sharp[f_m] - [f_n]\sharp < \varepsilon$ for all $m, n > n_0$. But $\sharp[f_m] - [f_n]\sharp = \sharp[f_m - f_n]\sharp = \|f_m - f_n\|$. This implies that f_n is a Cauchy sequence in L. Since L is complete, there exists a function $f \in L$ such that $\|f_n - f\| \longrightarrow 0$. Thus, for every $\varepsilon > 0$, there exists a positive integer m such that $\|f_n - f\| < \varepsilon$ for all $n > m$. But $\|f_n - f\| = \sharp[f_n - f]\sharp = \sharp[f_n] - [f]\sharp$. This implies that $\sharp[f_n] - [f]\sharp \longrightarrow 0$, which means that the sequence $[f_n]$ converges to $[f]$. Hence the normed linear space L/L_0 is complete. ∎

Chapter 5

Generating Dirac Integral from Lebesgue Measure

Let (X, V, μ) be a Lebesgue Measure Space as defined in Section 4.3 of Chapter 4. Since V is a σ-algebra of subsets of X, from Theorem 3.10 we know that the set $M(V, C,)$ of measurable functions with respect to V is a Baire algebra. We are going to construct a Dirac Integral Space with $M(V, C)$ as the underlying Baire algebra of functions.

Let W be the collection of all measurable sets with finite measure, i.e.

$$W = \{E \in V : \ \mu(E) < \infty\}.$$

It is easy to see that W is a δ-ring of subsets of X. So, by Theorem 3.3, the space $S(W, C)$ of simple functions generated by W is an algebra. Recall that a function s belongs to the space $S(W, C)$ if and only if there exist disjoint sets $E_1, E_2, \ldots, E_n$ from W and complex numbers $z_1, z_2, \ldots, z_n$ such that

$$s(x) = \sum_{i=1}^{n} z_i c_{E_i}(x)$$

for every $x \in X$.

5.1 Integral on Simple Functions

On the space $S(W, C)$ let us define a functional σ by

$$\sigma(s) = \sum_{i=1}^{n} z_i \mu(E_i)$$

if $s = \sum_{i=1}^{n} z_i c_{E_i} \in S(W,C)$ for some complex numbers $z_1, \ldots, z_n$ and disjoint sets $E_1, \ldots, E_n$ from W. We observe that the above representation of a simple function s is not unique. Suppose that for a simple function $s \in S(W,C)$ there exist families $\{E_1, \ldots, E_n\}$ and $\{D_1, \ldots, D_m\}$ of disjoint sets from W, and complex numbers $z_1, \ldots, z_n, w_1, \ldots, w_m$, such that

$$s(x) = \sum_{i=1}^{n} z_i c_{E_i}(x) = \sum_{j=1}^{m} w_j c_{D_j}(x)$$

for every $x \in X$. Let $H = \{E_i\colon\ i = 1, \ldots, n\} \cup \{D_j\colon\ j = 1, \ldots, m\}$. Then H is a finite subfamily of W. Since W is a prering, by Proposition 3.2, it has the finite refinement property. Let $B = \{B_1, \ldots, B_k\}$ be a finite refinement of H. Then every set E_i, as well as every set D_j, is a union of disjoint sets from B. Thus

$$s(x) = \begin{cases} y_t & \text{if } x \in B_t \\ 0 & \text{if } x \notin \bigcup_{t=1}^{k} B_t \end{cases}$$

where $y_t = z_i$ if $B_t \subset E_i$ for some i, and $y_t = w_j$ if $B_t \subset D_j$ for some j, else $y_t = 0$. Hence

$$\sigma(s) = \sum_{t=1}^{k} y_t \mu(B_t) = \sum_{i=1}^{n} z_i \mu(E_i) = \sum_{j=1}^{m} w_j \mu(D_j).$$

This shows that the functional σ is well defined, since its value $\sigma(s)$ does not depend on the different representations of the simple function s.

Proposition 5.1 *The functional σ is a positive linear functional on $S(W,C)$. If s_1 and s_2 are functions in $S(W,R)$ such that $s_1 \le s_2$, then $\sigma(s_1) \le \sigma(s_2)$.*

Proof. Take any two simple functions s_1 and s_2 in $S(W,C)$, and any complex numbers α and β. For each $i \in \{1,2\}$, there exist a finite index set Γ_i, disjoint sets $E_{ij} \in W$ $(j \in \Gamma_i)$, and complex numbers z_{ij} $(j \in \Gamma_i)$ such that

$$s_i(x) = \begin{cases} z_{ij} & \text{if } x \in E_{ij} \\ 0 & \text{if } x \notin \bigcup_{j \in \Gamma_i} E_{ij}. \end{cases}$$

Let $D = \{E_{ij}\colon\ i = 1, 2; j \in \Gamma_i\}$. Since W has the finite refinement property, there exists a finite refinement $B = \{B_1, B_2, \ldots, B_k\}$ of D. Every set $E_{ij} \in D$ is a union of disjoint sets from B. Thus

$$s_i(x) = \begin{cases} y_{it} & \text{if } x \in B_t \\ 0 & \text{if } x \notin \bigcup_{t=1}^{k} B_t \end{cases}$$

where $y_{it} = z_{ij}$ if $B_t \subset E_{ij}$ for some pair of indexes i and j, else $y_{it} = 0$. Therefore

$$(\alpha s_1 + \beta s_2)(x) = \begin{cases} \alpha y_{1t} + \beta y_{2t} & \text{if } x \in B_t \\ 0 & \text{if } x \notin \bigcup_{t=1}^{k} B_t. \end{cases}$$

And so

$$\begin{aligned} \sigma(\alpha s_1 + \beta s_2) &= \sum_{t=1}^{k} (\alpha y_{1t} + \beta y_{2t}) \mu(B_t) \\ &= \alpha \sum_{t=1}^{k} y_{1t} \mu(B_t) + \beta \sum_{t=1}^{k} y_{2t} \mu(B_t) \\ &= \alpha \sigma(s_1) + \beta \sigma(s_2) \end{aligned}$$

which shows that σ is a linear function.

Let $s \in S(W, R)$ and $s(x) \geq 0$ for every $x \in X$. Suppose that

$$s(x) = \sum_{i=1}^{n} r_i c_{E_i}(x)$$

for every $x \in X$. One may assume that sets E_j are not empty. Then $r_i \geq 0$ for every $i = 1, 2, \ldots, n$, and therefore

$$\sigma(s) = \sum_{i=1}^{n} r_i \mu(E_i) \geq 0.$$

This proves that the functional σ is positive.

If s_1 and s_2 are functions in $S(W, R)$ such that $s_1 \leq s_2$, then $(s_2 - s_1) \geq 0$. So $\sigma(s_2 - s_1) \geq 0$, which implies that $\sigma(s_1) \leq \sigma(s_2)$. ∎

A sequence E_n of sets is said to be *increasing* if $E_n \subset E_{n+1}$ for every $n \in N$. It is called *decreasing* if $E_n \supset E_{n+1}$ for every $n \in N$.

A linear functional φ defined on a linear subspace L of C^X is said to be *Daniell-continuous* if and only if for every decreasing sequence s_n of real-valued functions in L converging pointwise to zero, the sequence $\varphi(s_n)$ of values also converges to zero. To prove that the functional σ is Daniell-continuous we need the following Lemma.

Lemma 5.2 *If E_n is a decreasing sequence of sets in V such that $\mu(E_1) < \infty$ and $\bigcap_{n=1}^{\infty} E_n = \emptyset$, then $\lim_{n\to\infty} \mu(E_n) = 0$.*

Proof. For every positive integer n let $D_n = E_n \setminus E_{n+1}$. Then D_n is a sequence of disjoint sets in V, and $E_1 = \bigcup_{n=1}^{\infty} D_n$. Thus

$$\begin{aligned}
\mu(E_1) &= \mu(\bigcup_{n=1}^{\infty} D_n) \\
&= \sum_{n=1}^{\infty} \mu(D_n) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n-1} \mu(D_i) \\
&= \lim_{n\to\infty} \mu(\bigcup_{i=1}^{n-1} D_i) \\
&= \lim_{n\to\infty} \mu(E_1 \setminus E_n).
\end{aligned}$$

But $\mu(E_1) = \mu((E_1 \setminus E_n) \cup E_n) = \mu(E_1 \setminus E_n) + \mu(E_n)$, which implies that

$$\begin{aligned}
\lim_{n\to\infty} \mu(E_1 \setminus E_n) &= \lim_{n\to\infty} (\mu(E_1) - \mu(E_n)) \\
&= \mu(E_1) - \lim_{n\to\infty} \mu(E_n).
\end{aligned}$$

So we conclude that $\lim_{n\to\infty} \mu(E_n) = 0$. ∎

Proposition 5.3 *The linear functional σ as defined above is Daniell-continuous on the space $S(W, C)$ of simple functions.*

Proof. Let s_n be a decreasing sequence of simple functions in $S(W, R^+)$ such that s_n converges pointwise to zero. Suppose that $s_1(x) = \sum_{j=1}^{k} r_{1j} c_{E_{1j}}(x)$ for every $x \in X$, where $r_{1j} \geq 0$ and $E_{1j} \in W$ for every $j = 1, 2, \ldots, k$. Let

$$r = \max\{r_{1j}\colon\ j = 1, 2, \ldots, k\},$$

and $E = \{x \in X\colon\ s_1(x) > 0\}$. Then

$$E = \bigcup_{r_{1j} > 0} E_{1j} \in W$$

and $s_1(x) \leq r c_E(x)$ for every $x \in X$. We next consider two cases, namely $\mu(E) = 0$ and $\mu(E) > 0$.

If $\mu(E) = 0$, we must have $\sigma(s_1) = 0$, since $0 \leq \sigma(s_1) \leq \sigma(r c_E) = r\mu(E) = 0$. But $0 \leq s_n \leq s_1$ for every $n \in N$, which implies that $0 \leq \sigma(s_n) \leq \sigma(s_1)$. Thus $\sigma(s_n) = 0$ for every $n \in N$, and hence $\lim_{n\to\infty} \sigma(s_n) = 0$.

If $\mu(E) > 0$, we take any $\varepsilon > 0$, and let $\eta = \varepsilon/(2\mu(E))$. Let

$$E_n = \{x \in X\colon\ s_n(x) \geq \eta\}$$

for every $n \in N$. If $s_n = \sum_{j=1}^{m} r_{nj} c_{E_{nj}}$ where $\{E_{nj}\colon\ j = 1, \ldots, m\}$ are disjoint sets, then

$$E_n = \bigcup_{r_{nj} \geq \eta} E_{nj} \in W.$$

Moreover $E_{n+1} \subset E_n$ for every $n \in N$; and

$$\begin{aligned} \bigcap_{n=1}^{\infty} E_n &= \{x \in X\colon\ (\forall_{n \in N})\ \eta \leq s_n(x)\} \\ &= \emptyset \end{aligned}$$

since $s_n(x) \searrow 0$ for every $x \in X$ and $\eta > 0$. So, by Lemma 5.2, we get $\lim_{n\to\infty} \mu(E_n) = 0$. This means that there exists a positive integer n_0 such that

$$\mu(E_n) < \frac{\varepsilon}{2r}$$

for all $n > n_0$. Notice that $E_n \subset E$ for every $n \in N$. So

$$\begin{aligned} s_n &= c_E s_n \\ &= (c_{E\setminus E_n} + c_{E_n}) s_n \\ &= c_{E\setminus E_n} s_n + c_{E_n} s_n \end{aligned}$$

and from this it follows that

$$\begin{aligned} \sigma(s_n) &= \sigma(c_{E\setminus E_n} s_n) + \sigma(c_{E_n} s_n) \\ &\leq \sigma(\eta c_E) + \sigma(r c_{E_n}) \\ &= \eta\mu(E) + r\mu(E_n) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

which is true for all $n > n_0$. This shows that $\lim_{n\to\infty} \sigma(s_n) = 0$. ∎

Corollary 5.4 *If $s \in S(W,C)$ and s_n is a decreasing (increasing) sequence of real-valued functions in $S(W,C)$ which converges pointwise to s, then the sequence $\sigma(s_n)$ converges decreasingly (increasingly) to $\sigma(s)$.*

Proof. If s_n converges decreasingly to s, consider sequence $t_n = s_n - s$ (if s_n converges increasingly to s, consider sequence $t_n = s - s_n$), which is decreasing and converging pointwise to zero. By the Daniell-continuity of functional σ, the sequence $\sigma(t_n)$ converges decreasingly to zero. But $\sigma(t_n) = \sigma(s_n - s) = \sigma(s_n) - \sigma(s)$, which implies that the sequence $\sigma(s_n)$ converges decreasingly to $\sigma(s)$. ∎

If on $S(W,C)$ we define a functional $\|\cdot\|$ by $\|s\| = \sigma(|s|)$ for every $s \in S(W,C)$, then $\|\cdot\|$ is a semi-norm on $S(W,C)$. Moreover, if $s \in S(W,C)$ and $s = \sum_{i=1}^{n} z_i c_{E_i}$, then

$$|\sigma(s)| = |\sum_{i=1}^{n} z_i \mu(E_i)|$$

$$\begin{aligned} &\leq \sum_{i=1}^{n} |s_i| \mu(E_i) \\ &= \sigma(|s|). \end{aligned}$$

This shows that the inequality $|\sigma(s)| \leq \|s\|$ holds for every $s \in S(W, C)$.

5.2 Extension of the Integral

Let R^+ be the set of all non-negative real numbers. Denote by $D(W, R^+)$ the set of all functions $f: X \to R^+$ such that there exists an increasing sequence s_n of functions in $S(W, R^+)$ converging pointwise to f and the sequence $\sigma(s_n)$ of values is bounded. On $D(W, R^+)$ we define a functional ψ by

$$\psi(f) = \lim_{n \to \infty} \sigma(s_n)$$

for every $f \in D(W, R^+)$, where s_n is an increasing sequence in $S(W, R^+)$ converging pointwise to f and the sequence $\sigma(s_n)$ is bounded. To show that this funcional ψ is well-defined, we prove the following Lemma, which is due to P. J. Daniell.

Lemma 5.5 *If s_n and t_m are increasing sequences of functions in $S(W, R^+)$ such that* $\lim_n s_n(x) = \lim_m t_m(x)$ *for every $x \in X$, and the sequences $\sigma(s_n)$ and $\sigma(t_m)$ are bounded, then* $\lim_n \sigma(s_n) = \lim_m \sigma(t_m)$.

Proof. Let $s(x) = \lim_n s_n(x) = \lim_m t_m(x)$ for every $x \in X$. For every positive integers n and m we have $(s_n \wedge t_m) \leq s_n$, and therefore $\sigma(s_n \wedge t_m) \leq \sigma(s_n)$. But $(s_n \wedge t_m) \nearrow (s \wedge t_m) = t_m$, and therefore $\sigma(s_n \wedge t_m) \nearrow \sigma(t_m)$, for each $m \in N$ as $n \to \infty$. So we must have $\sigma(t_m) \leq \lim_n \sigma(s_n)$ for every $m \in N$. Hence $\lim_m \sigma(t_m) \leq \lim_n \sigma(s_n)$. Using similar arguments as above we can also get the inequality $\lim_n \sigma(s_n) \leq \lim_m \sigma(t_m)$, which completes the proof. ∎

Corollary 5.6 *If s_n and t_m are sequences of functions in $S(W, R^+)$ such that $\sum_n s_n(x) = \sum_m t_m(x)$ for every $x \in X$, and the series $\sum_n \sigma(s_n)$ and $\sum_m \sigma(t_m)$ are convergent, then $\sum_n \sigma(s_n) = \sum_m \sigma(t_m)$.*

Proof. Let $u_n = \sum_{i=1}^n s_i$ and $w_m = \sum_{i=1}^m t_i$. Then u_n and w_m are increasing sequences of functions in $S(W, R^+)$, and

$$\lim_n u_n(x) = \sum_{n=1}^{\infty} s_n(x) = \sum_{m=1}^{\infty} t_m(x) = \lim_m w_m(x)$$

for every $x \in X$. Moreover

$$\sigma(u_n) = \sum_{i=1}^{n} \sigma(s_i) \leq \sum_{i=1}^{\infty} \sigma(s_i) < \infty$$

and

$$\sigma(w_n) = \sum_{i=1}^{m} \sigma(t_i) \leq \sum_{i=1}^{\infty} \sigma(t_i) < \infty.$$

Thus, applying Lemma 5.5, we get $\lim_n \sigma(u_n) = \lim_m \sigma(w_m)$, that is $\sum_n \sigma(s_n) = \sum_m \sigma(t_m)$. ∎

From Lemma 5.5 it is obvious that the value of $\psi(f)$ does not depend on the sequence s_n converging to f. Thus the functional ψ is well-defined. It is also clear from the definition of the set $D(W, R^+)$ that $S(W, R^+) \subset D(W, R^+)$; and for every $s \in S(W, R^+)$ we have $\psi(s) = \sigma(s)$. Thus the functional ψ is an extension of σ from $S(W, R^+)$ onto $D(W, R^+)$.

Proposition 5.7 (a). *If f and g are any two functions in $D(W, R^+)$ and α and β are non-negative real numbers, then $(\alpha f + \beta g) \in D(W, R^+)$ and $\psi(\alpha f + \beta g) = \alpha\psi(f) + \beta\psi(g)$. (b). If f and g are functions in $D(W, R^+)$ such that $f \leq g$, then $\psi(f) \leq \psi(g)$.*

Proof. (a). Let s_n and t_n be sequences of simple functions from the space $S(W, R^+)$ converging increasingly to functions f and g, respectively, such that the sequences

$\sigma(s_n)$ and $\sigma(t_n)$ are bounded. For every $n \in N$ let $u_n = \alpha s_n + \beta t_n$. Then u_n is an increasing sequence of functions in $S(W, R^+)$ converging pointwise to the function $\alpha f + \beta g$. The sequence $\sigma(u_n)$ is bounded, since $\sigma(s_n)$ and $\sigma(t_n)$ are bounded. Hence $(\alpha f + \beta g) \in D(W, R^+)$. Moreover $\psi(\alpha f + \beta g) = \lim_n \sigma(u_n) = \alpha \lim_n \sigma(s_n) + \beta \lim_n \sigma(t_n) = \alpha\psi(f) + \beta\psi(g)$.

(*b*). Let f and g be functions in $D(W, R^+)$ such that $f \leq g$. Let s_n and t_m be increasing sequences in $S(W, R^+)$ converging pointwise to functions f and g respectively, such that the sequences $\sigma(s_n)$ and $\sigma(t_m)$ are bounded. For every positive integers n and m we have $(s_n \wedge t_m) \leq t_m$, and therefore

$$\sigma(s_n \wedge t_m) \leq \sigma(t_m). \tag{5.1}$$

But for each $n \in N$ we have

$$\lim_{m\to\infty}(s_n \wedge t_m) = s_n \wedge g = s_n$$

since $s_n \leq f \leq g$. Therefore

$$\lim_{m\to\infty} \sigma(s_n \wedge t_m) = \sigma(s_n).$$

From this and inequality (5.1) it follows that

$$\sigma(s_n) \leq \lim_{m\to\infty} \sigma(t_m)$$

for every $n \in N$. Hence

$$\lim_{n\to\infty} \sigma(s_n) \leq \lim_{m\to\infty} \sigma(t_m)$$

that is $\psi(f) \leq \psi(g)$. ∎

Proposition 5.8 *If f_n is an increasing sequence of functions in $D(W, R^+)$ such that f_n converges pointwise to a function f and the sequence $\psi(f_n)$ is bounded, then $f \in D(W, R^+)$ and $\psi(f) = lim_{n\to\infty}\psi(f_n)$.*

Proof. Let f_n be an increasing sequence of functions in $D(W, R^+)$ such that f_n converges pointwise to a function f and the sequence $\psi(f_n)$ is bounded. Then for each $n \in N$ there exists an increasing sequence s_{nm} in $S(W, R^+)$ such that

$$\lim_{m\to\infty} s_{nm}(x) = f_n(x)$$

for every $x \in X$. Let

$$s_n = s_{1n} \vee s_{2n} \vee \cdots \vee s_{nn}$$

for every $n \in N$. Then s_n is an increasing sequence of functions in $S(W, R^+)$. Since $s_{nm} \leq f_n \leq f$ for all n and m, we have $s_n \leq f$ for all $n \in N$. So s_n is bounded, and therefore it is convergent. Let

$$\lim_{n\to\infty} s_n(x) = s(x)$$

for every $x \in X$. It is obvious that $s \leq f$. On the other hand, since $s_{nm} \leq s_m \leq s$ if $n \leq m$, passing to the limit $m \to \infty$ we get $f_n \leq s$ for every $n \in N$, and so $f \leq s$. Hence $s = f$. Therefore

$$\lim_{n\to\infty} s_n(x) = f(x)$$

for every $x \in X$. Since $s_n = s_{1n} \vee \cdots \vee s_{nn} \leq f_1 \vee \cdots \vee f_n = f_n$, we have

$$\sigma(s_n) = \psi(s_n) \leq \psi(f_n) \tag{5.2}$$

for every $n \in N$. So the sequence $\sigma(s_n)$ is bounded, because $\psi(f_n)$ is bounded. Thus $f \in D(W, R^+)$.

From inequality (5.2) it also follows that $\lim_{n\to\infty} \sigma(s_n) = \lim_{n\to\infty} \psi(s_n) \leq \lim_{n\to\infty} \psi(f_n)$. Thus

$$\psi(f) \leq \lim_{n\to\infty} \psi(f_n). \tag{5.3}$$

On the other hand

$$\lim_{n\to\infty} \psi(f_n) \leq \psi(f) \tag{5.4}$$

since $f_n \leq f$ for every $n \in N$. Hence from (5.3) and (5.4) we conclude that $\psi(f) = \lim_{n\to\infty} \psi(f_n)$. ∎

Corollary 5.9 *If f_n is a sequence of functions in $D(W, R^+)$ such that $\sum_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ for every $x \in X$, and the series $\sum_{n=1}^{\infty} \psi(f_n)$ is also convergent, then $f \in D(W, R^+)$ and $\psi(f) = \sum_{n=1}^{\infty} \psi(f_n)$.*

Proof. Let f_n be a sequence of functions in $D(W, R^+)$ satisfying the conditions as stated in the hypothesis of this Corollary. For each $n \in N$ let

$$g_n(x) = \sum_{i=1}^{n} f_i(x)$$

for every $x \in X$. Then g_n is a sequence of functions in $D(W, R^+)$, and $g_n(x) \nearrow f(x)$ for every $x \in X$. Moreover $\psi(g_n) = \psi(\sum_{i=1}^{n} f_i) = \sum_{i=1}^{n} \psi(f_i)$ for every $n \in N$. Thus $\lim_{n\to\infty} \psi(g_n) = \sum_{n=1}^{\infty} \psi(f_n) < \infty$, i.e. the sequence $\psi(g_n)$ is bounded. By Proposition 5.8, we get $f \in D(W, R^+)$ and $\psi(f) = \lim_{n\to\infty} \psi(g_n) = \sum_{n=1}^{\infty} \psi(f_n)$. ∎

Proposition 5.10 *Let s_n be an increasing sequence in $S(W, R^+)$ such that s_n converges pointwise to a function f and the sequence $\sigma(s_n)$ is bounded. Then*

$$\lim_{n\to\infty} \sigma(s_n) \;=\; \sup\,\{\sigma(s)\colon\; s \in S(W, R^+),\; s \leq f\}.$$

Proof. Let s_n be a sequence in $S(W, R^+)$ such that s_n is convergent increasingly to a function f and $\sigma(s_n)$ is bounded. Let

$$\sup\,\{\sigma(s)\colon\; s \in S(W, R^+),\; s \leq f\} = r.$$

Then $s_n \leq f$, and therefore $\sigma(s_n) \leq r$, for every $n \in N$. Hence

$$\lim_{n\to\infty} \sigma(s_n) \leq r. \tag{5.5}$$

Take any function $s \in S(W, R^+)$ such that $s \leq f$, and let $t_n = s_n \wedge s$. Then t_n is a sequence in $S(W, R^+)$, which is convergent increasingly to function $f \wedge s = s$.

By Corollary 5.4, we get $\sigma(t_n) \nearrow \sigma(s)$. But $t_n \leq s_n$, and therefore $\sigma(t_n) \leq \sigma(s_n)$, for every $n \in N$. So we must have $\sigma(s) \leq \lim_{n\to\infty} \sigma(s_n)$, which is true for every $s \in S(W, R^+)$ such that $s \leq f$. From this we conclude that

$$r \leq \lim_{n\to\infty} \sigma(s_n). \tag{5.6}$$

The equality we are proving follows from (5.5) and (5.6). ∎

As defined in Section 3.2 of Chapter 3, the set $M(W, C)$ is the collection of all functions $f \in C^X$ which are measurable with respect to the family W of subsets of X. We know that $S(W, C)$ is a subset of $M(W, C)$, and that $M(W, R^+)$ is also a subset of $M(W, C)$. Now we want to show that $D(W, R^+)$ is a subset of $M(W, R^+)$. First we prove the following Lemma.

Lemma 5.11 *Let E_n be an increasing sequence of sets in W such that the sequence $\mu(E_n)$ is bounded. If $E = \bigcup_{n=1}^{\infty} E_n$, then $E \in W$ and $\lim_{n\to\infty} \mu(E_n) = \mu(E)$.*

Proof. The set E belongs to the family V, since V is a σ-algebra. Let $D_1 = E_1$, and $D_n = E_n \setminus E_{n-1}$ for every positive integer $n \geq 2$. Then D_n is a sequence of disjoint sets in V, and $E = \bigcup_{n=1}^{\infty} D_n$. Thus

$$\begin{aligned} \mu(E) &= \mu(\bigcup_{n=1}^{\infty} D_n) \\ &= \sum_{n=1}^{\infty} \mu(D_n) \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} \mu(D_j) \\ &= \lim_{n\to\infty} \mu(\bigcup_{j=1}^{n} D_j) \\ &= \lim_{n\to\infty} \mu(E_n). \end{aligned}$$

Since the sequence $\mu(E_n)$ is bounded, we must have $\mu(E) = \lim_{n\to\infty} \mu(E_n) < \infty$, which means that $E \in W$. ∎

Proposition 5.12 *The set $D(W, R^+)$ is a subset of $M(W, R^+)$.*

Proof. Take any function $f \in D(W, R^+)$. Then there exists an increasing sequence s_n of functions in $S(W, R^+)$ such that s_n converges pointwise to f and the sequence $\sigma(s_n)$ is bounded. Let a be any positive real number. Then

$$\begin{aligned} f^{-1}(a,\infty) &= \{x \in X \mid f(x) > a\} \\ &= \left\{x \in X \,\middle|\, (\forall_{m>\frac{1}{a}})\;\; f(x) > a - \frac{1}{m}\right\} \\ &= \left\{x \in X \,\middle|\, (\forall_{m>\frac{1}{a}})\; (\exists_{n\in N})\;\; s_n(x) > a - \frac{1}{m}\right\} \\ &= \bigcap_{m>\frac{1}{a}} \bigcup_{n\in N} s_n^{-1}\left(a - \frac{1}{m}, \infty\right). \end{aligned}$$

Let $E_{nm} = s_n^{-1}(a - \frac{1}{m}, \infty)$. Then $E_{nm} \in W$ for every positive integer n and m such that $m > \frac{1}{a}$, since $s_n \in M(W, R^+)$ for every $n \in N$. It is easy to see that E_{nm} is an increasing sequence of sets with respect to n for each $m > \frac{1}{a}$. Furthermore, we observe that $(a - \frac{1}{m})c_{E_{nm}} \le s_n$, which implies that $\sigma((a - \frac{1}{m})c_{E_{nm}}) \le \sigma(s_n)$, and therefore $(a - \frac{1}{m})\mu(E_{nm}) \le \sigma(s_n)$, for every $n \in N$ and $m > \frac{1}{a}$. But the sequence $\sigma(s_n)$ is bounded. So the sequence $\mu(E_{nm})$ must be bounded for each $m > \frac{1}{a}$. Hence, applying Lemma 5.11, we get that the set $E_m = \bigcup_{n\in N} E_{nm}$ belongs to the family W for every $m > \frac{1}{a}$. Hence $f^{-1}(a, \infty) = \bigcap_{m>\frac{1}{a}} E_m \in W$, since W is a δ-ring. So $f \in M(W, R^+)$. ∎

5.3 The Space of Summable Functions

A sequence s_n of simple functions in $S(W, C)$ will be called a *fundamental sequence* if it satisfies the following conditions:

1. The series $\sum_{n=1}^{\infty} |s_n(x) - s_{n+1}(x)|$ converges for every $x \in X$.

2. The series $\sum_{n=1}^{\infty} \|s_n - s_{n+1}\|$ converges.

Now let L denote the set of all functions $f \in C^X$ with the property that there exists a fundamental sequence s_n such that $s_n(x) \longrightarrow f(x)$ for every $x \in X$. The members of L are called *summable functions.* Clearly the space $S(W, C)$ is a subset of L.

Proposition 5.13 *The set $D(W, R^+)$ is a subset of L .*

Proof. Take any function $f \in D(W, R^+)$. Then there exists a sequence $s_n \in S(W, R^+)$ such that $\sigma(s_n)$ is bounded and $s_n(x) \nearrow f(x)$ for every $x \in X$. To prove that $f \in L$, it is sufficient to show that the sequence s_n is fundamental. To do so we observe that

$$\begin{aligned}
\sum_{n=1}^{\infty} |s_n(x) - s_{n+1}(x)| &= \lim_{n\to\infty} \sum_{i=1}^{n} |s_i(x) - s_{i+1}(x)| \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} (s_{i+1}(x) - s_i(x)) \\
&= \lim_{n\to\infty} (s_{n+1}(x) - s_1(x)) \\
&= f(x) - s_1(x) < \infty
\end{aligned}$$

for every $x \in X$; and the series

$$\begin{aligned}
\sum_{n=1}^{\infty} \|s_n - s_{n+1}\| &= \lim_{n\to\infty} \sum_{i=1}^{n} \sigma(|s_i - s_{i+1}|) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \sigma(s_{i+1} - s_i) \\
&= \lim_{n\to\infty} (\sigma(s_{n+1}) - \sigma(s_1)) \\
&= \psi(f) - \psi(s_1) < \infty.
\end{aligned}$$

So s_n is a fundamental sequence and therefore $f \in L$. ∎

A function $u: C^k \to C$ is called a *Lipschitzian function* if there exists some positive number λ such that

$$|u(x) - u(y)| \leq \lambda |x - y|$$

for every x and y in C^k. The function u is said to *vanish at zero* if $u(0,\ldots,0) = 0$.

Lemma 5.14 *Let $u: C^k \to C$ be a Lipschitzian function vanishing at zero, and s_n^j a fundamental sequence for each $j \in \{1,2,\ldots,k\}$. Then the sequence s_n, defined by $s_n(x) = u(s_n^1(x), s_n^2(x), \ldots, s_n^k(x))$ for every $x \in X$, is a fundamental sequence.*

Proof. Proposition 3.2 guarantees that $s_n \in S(W,C)$ for every $n \in N$. Since the sequence s_n^j is fundamental, for each $j \in \{1,2,\ldots,k\}$ we have

$$\sum_{n=1}^{\infty} |s_n^j(x) - s_{n+1}^j(x)| < \infty$$

for every $x \in X$, and

$$\sum_{n=1}^{\infty} \|s_n^j - s_{n+1}^j\| < \infty.$$

From the fact that u is Lipschitzian, we get

$$\begin{aligned}
|s_n(x) - s_{n+1}(x)| &= |u(s_n^1(x),\ldots,s_n^k(x)) - u(s_{n+1}^1(x),\ldots,s_{n+1}^k(x))| \\
&\le \lambda|(s_n^1(x),\ldots,s_n^k(x)) - (s_{n+1}^1(x),\ldots,s_{n+1}^k(x))| \\
&= \lambda|(s_n^1(x) - s_{n+1}^1(x),\ldots,s_n^k(x) - s_{n+1}^k(x))| \\
&\le \lambda|s_n^1(x) - s_{n+1}^1(x)| + \cdots + \lambda|s_n^k(x) - s_{n+1}^k(x)|
\end{aligned}$$

for every $n \in N$ and some positive number λ. And so

$$\sum_{n=1}^{m} |s_n(x) - s_{n+1}(x)| \le \lambda \sum_{n=1}^{m} |s_n^1(x) - s_{n+1}^1(x)| + \cdots + \lambda \sum_{n=1}^{m} |s_n^k(x) - s_{n+1}^k(x)|$$

which shows that

$$\sum_{n=1}^{\infty} |s_n(x) - s_{n+1}(x)| < \infty.$$

From the above inequality we also have

$$\begin{aligned}
\|s_n - s_{n+1}\| &= \sigma(|s_n - s_{n+1}|) \\
&\le \lambda\sigma(|s_n^1 - s_{n+1}^1|) + \cdots + \lambda\sigma(|s_n^k - s_{n+1}^k|) \\
&= \lambda\|s_n^1 - s_{n+1}^1\| + \cdots + \lambda\|s_n^k - s_{n+1}^k\|
\end{aligned}$$

which implies that

$$\sum_{n=1}^{\infty} \|s_n - s_{n+1}\| < \infty.$$

Thus the sequence s_n is fundamental. ∎

Corollary 5.15 *If s_n is a fundamental sequence, then the sequence $|s_n|$ is also fundamental.*

Proof. Let s_n be a fundamental sequence. Function $u: C \to C$, defined by $u(z) = |z|$ for every $z \in C$, is a Lipschitzian function vanishing at zero. Thus, by Lemma 5.14, the sequence $|s_n|$ is fundamental. ∎

Corollary 5.16 *The set L is closed under composition with any Lipschitzian function vanishing at zero, in the sense that if $u: C^k \to C$ is a Lipschitzian function such that $u(0, \ldots, 0) = 0$, and $\{f_1, f_2, \ldots, f_k\}$ is any finite subset of L, then $u \circ (f_1, f_2, \ldots, f_k) \in L$.*

Proof. Let $\{f_1, f_2, \ldots, f_k\}$ be any finite subset of L. Then there exist fundamental sequences $s_n^1, s_n^2, \ldots, s_n^k$ such that $\lim_{n\to\infty} s_n^j(x) = f_j(x)$ for each $j \in \{1, 2, \ldots, k\}$ and every $x \in X$. Let $s_n(x) = u(s_n^1(x), \ldots, s_n^k(x))$ for every $x \in X$. By Lemma 5.14, the sequence s_n is fundamental. Moreover

$$\begin{aligned} \lim_{n\to\infty} s_n(x) &= \lim_{n\to\infty} u(s_n^1(x), \ldots, s_n^k(x)) \\ &= u(f_1(x), \ldots, f_k(x)) \end{aligned}$$

for every $x \in X$. Hence $u \circ (f_1, f_2, \ldots, f_k) \in L$. ∎

Proposition 5.17 *The set L is a linear space with respect to scalar multiplication and addition of functions.*

Proof. Consider function $u: C^2 \to C$ defined by $u(z_1, z_2) = z_1 + z_2$ for every $z_1, z_2 \in C$. Function u is Lipschitzian and vanishes at zero. Hence, by Corollary 5.16, we get $u \circ (f_1, f_2) = f_1 + f_2 \in L$ for every f_1 and f_2 in L, i.e. L is closed under addition of functions.

Similarly, by considering function $u: C \to C$ defined by $u(z) = \lambda z$ for every $z \in C$ and any fixed complex number λ, we prove that L is closed under scalar multiplication. ∎

To verify that L is a solid linear subspace of $M(V, C)$, we need several Lemmas. We first prove that the space $M(W, C)$ is closed under dominated convergence.

Let W_1 and W_2 be collections of subsets of X such that $W_1 \subset W_2$. We shall say that W_1 is an *ideal* of W_2 if whenever $E \in W_1$ and $D \in W_2$ then $(E \cap D) \in W_1$. It is easy to see that the collection $W = \{E \in V: \mu(E) < \infty\}$ is an ideal subset of V, and also of W^σ.

Lemma 5.18 *Let W_1 and W_2 be δ-rings of subset of X. Then $M(W_1, C)$ is a solid subset of $M(W_2, C)$ if and only if W_1 is an ideal of W_2.*

Proof. Let $M(W_1, C)$ be a solid subset of $M(W_2, C)$. Take any set $E \in W_1$ and $D \in W_2$. Then $c_E \in M(W_1, C)$ and $c_D \in M(W_2, C)$. And so $c_{E\cap D} = c_E \cdot c_D \in M(W_2, C)$. Since $c_{E\cap D} \leq c_E$ and $M(W_1, C)$ is solid in $M(W_2, C)$, we get $c_{E\cap D} \in M(W_1, C)$. Hence $c_{E\cap D}^{-1}(C \setminus \overline{B(0, 1/2)}) = E \cap D \in W_1$. So W_1 is an ideal in W_2.

Conversely, let W_1 be an ideal of W_2. It is clear $M(W_1, C) \subset M(W_2, C)$. Let $f \in M(W_1, C)$ and $g \in M(W_2, C)$ such that $|g| \leq |f|$. Take any open set G in C such that $0 \notin \overline{G}$. Then $g^{-1}(G) \in W_2$. Let $H = (\overline{G})^c$. Then H is an open set and $0 \in H$. So there exists a positive number r such that the closed ball

$$B = \{z \in C: |z| \leq r\} \subset H.$$

From this it follows that $G \subset H^c \subset B^c$, where B^c is open and its closure does not contain 0. Hence

$$\begin{aligned} g^{-1}(G) &= \{x \in X:\ g(x) \in G\} \\ &\subset \{x \in X:\ g(x) \in B^c\} \\ &= \{x \in X:\ |g(x)| > r\} \\ &\subset \{x \in X:\ |f(x)| > r\} \\ &= \{x \in X:\ f(x) \in B^c\} \\ &= f^{-1}(B^c). \end{aligned}$$

Therefore $g^{-1}(G) \cap f^{-1}(B^c) = g^{-1}(G)$. But $f^{-1}(B^c) \in W_1$, since $f \in M(W_1, C)$. So $g^{-1}(G) \in W_1$, and hence $g \in M(W_1, C)$. ∎

Proposition 5.19 *If W is a δ-ring of subsets of X, then the space $M(W, C)$ is closed under dominated convergence, i.e. if $g \in M(W, C)$ and f_n is a sequence of functions in $M(W, C)$ such that $|f_n(x)| \leq |g(x)|$ for every $n \in N$ and $x \in X$ and f_n converges pointwise to a function f, then $f \in M(W, C)$.*

Proof. Let f_n be a sequence of functions in $M(W, C)$ as described in the hypothesis of this Proposition. Then $f_n \in M(W^\sigma, C)$ for every $n \in N$. Since $M(W^\sigma, C)$ is closed under pointwise convergence, we must have $f \in M(W^\sigma, C)$. But $M(W, C)$ is a solid subset of $M(W^\sigma, C)$, since W is an ideal subset of W^σ. And $|f(x)| \leq |g(x)|$ for every $x \in X$. Hence $f \in M(W, C)$. ∎

Lemma 5.20 *The space $D(W, R^+)$ is a solid subset of $M(W, R^+)$.*

Proof. In Proposition 5.12 we have proved that the space $D(W, R^+)$ is a subset of $M(W, R^+)$. Now take any function $f \in D(W, R^+)$ and $g \in M(W, R^+)$ such that $g \leq f$. Consider a sequence e_n of functions defined as follows

$$e_n(r) = c_{[0,n)}(r)\frac{[2^n r]}{2^n}$$

for every non-negative real number r, where $[r]$ denotes the greatest integer which is less than or equal to r. It is easy to see that $e_n(r) \nearrow e(r) = r$ for every $r \in [0, \infty)$. Let $s_n = e_n \circ g$ for every $n \in N$. Then s_n is an increasing sequence of functions in $S(W, R^+)$ which converges pointwise to function g. So $s_n \leq g \leq f$, and therefore $\sigma(s_n) = \psi(s_n) \leq \psi(f) < \infty$, for every $n \in N$. Thus the sequence $\sigma(s_n)$ is bounded. Hence $g \in D(W, R^+)$. ∎

Lemma 5.21 *The space L is a subset of $M(W, C)$. Moreover, if $f \in L$, then $|f| \in D(W, R^+)$.*

Proof. Let $f \in L$, and s_n be a fundamental sequence such that $s_n(x) \longrightarrow f(x)$ for every $x \in X$. By Corollary 5.15, the sequence $t_n = |s_n|$ is fundamental, and $t_n(x) \longrightarrow |f(x)|$ for every $x \in X$. Let

$$u_n(x) = |t_1(x)| + \sum_{i=1}^{n} |t_{i+1}(x) - t_i(x)|$$

for every $x \in X$. Then $u_n \in S(W, R^+)$, and therefore $u_n \in D(W, R^+)$ for every $n \in N$. Since t_n is fundamental, the sequence u_n converges (increasingly), say to a function g. Moreover

$$\begin{aligned} \psi(u_n) &= \sigma(u_n) \\ &= \sigma(|t_1|) + \sum_{i=1}^{n} \sigma(|t_{i+1} - t_i|) \\ &\leq \|t_1\| + \sum_{i=1}^{\infty} \|t_{i+1} - t_i\| < \infty. \end{aligned}$$

So by Proposition 5.8 we get $g \in D(W, R^+)$, and therefore $g \in M(W, C)$.

We observe that for every $x \in X$ and every $n \in N$ we have

$$\begin{aligned} |s_n(x)| &= t_n(x) \\ &= |t_n(x)| \end{aligned}$$

$$\begin{aligned}
&= |t_1(x) + (t_2(x) - t_1(x)) + \cdots + (t_n(x) - t_{n-1}(x))| \\
&\leq |t_1(x)| + |t_2(x) - t_1(x)| + \cdots + |t_n(x) - t_{n-1}(x)| \\
&= u_{n-1}(x) \\
&\leq g(x).
\end{aligned}$$

Since the space $M(W, C)$ is closed under dominated convergence, we must have $f \in M(W, C)$. This proves that $L \subset M(W, C)$.

The above inequality also implies that $|f(x)| \leq g(x)$ for every $x \in X$. Since $f \in M(W, C)$, we must have $|f| \in M(W, R^+)$. But $D(W, R^+)$ is solid in $M(W, R^+)$. So $|f| \in D(W, R^+)$. ∎

Before we prove the next Lemma, let us recall that if f is a real-valued function defined on X, then the positive part f^+ of f is the function $f^+ = f \vee 0$, and the negative part f^- of f is the function $f^- = (-f) \vee 0$. It is easy to see that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. If $f \in M(V, R)$, then both the positive and the negative parts of f also belong to $M(V, R)$, since $M(V, R)$ is a lattice.

Lemma 5.22 *The space L of summable functions is a solid subset of $M(W, C)$.*

Proof. In Lemma 5.21 we have proved that the space L is a subset of $M(W, C)$. Now take any function $f \in L$ and $g \in M(W, C)$ such that $|g| \leq |f|$. By Lemma 5.21, the function $|f| \in D(W, R^+)$. Let $g_1 = \mathrm{Re} \circ g$ and $g_2 = \mathrm{Im} \circ g$. Then g_1 and g_2 belong to $M(W, R)$, and therefore g_1^+, g_1^-, g_2^+, and g_2^- are all in $M(W, R^+)$. Since $g_1^+ \leq |g_1| \leq |g|$, we have $g_1^+ \leq |f|$. But $D(W, R^+)$ is solid in $M(W, R^+)$, thus $g_1^+ \in D(W, R^+)$, and so $g_1^+ \in L$. Similarly, we can easily show that g_1^-, g_2^+, and g_2^- are also in L. Hence $g_1, g_2 \in L$, and consequently $g \in L$. ∎

Theorem 5.23 *The space L is a solid linear subspace of $M(V, C)$.*

Proof. We have proved that L is a linear space and that L is a subset of $M(W, C)$. Thus L is a linear subspace of $M(V, C)$. Take any function $f \in L$ and $g \in M(V, C)$ such that $|g| \leq |f|$. Then $f \in M(W, C)$. Since W is an ideal of V, the space $M(W, C)$ must be a solid subset of $M(V, C)$. So $g \in M(W, C)$. But L is solid in $M(W, C)$. Thus $g \in L$. ∎

5.4 Construction of the Dirac Integral

Our next step is to extend the functional ψ to a functional $\int$ with domain L and prove that it is a Dirac Integral. To guarantee that this extension is well-defined we first prove the following Lemmas.

Lemma 5.24 *If s_n is a fundamental sequence, then the sequence $\sigma(s_n)$ is convergent.*

Proof. Let s_n be a fundamental sequence. Then the series $\sum_{n=1}^{\infty} \|s_n - s_{n+1}\|$ converges. Given any $\varepsilon > 0$, there exists a positive number n_0 such that

$$\left| \sum_{i=1}^{\infty} \|s_i - s_{i+1}\| - \sum_{i=1}^{n} \|s_i - s_{i+1}\| \right| < \varepsilon$$

for all $n > n_0$. Thus

$$\sum_{i=n+1}^{\infty} \|s_i - s_{i+1}\| < \varepsilon$$

for all $n > n_0$. Let m be a positive integer greater than n. Then

$$\begin{aligned} \|s_n - s_m\| &\leq \sum_{i=n}^{m-1} \|s_i - s_{i+1}\| \\ &\leq \sum_{i=n}^{\infty} \|s_i - s_{i+1}\| < \varepsilon \end{aligned}$$

for all $m > n > n_0 + 1$. This shows that s_n is a Cauchy sequence. But

$$\begin{aligned} |\sigma(s_n) - \sigma(s_m)| &= |\sigma(s_n - s_m)| \\ &\leq \|s_n - s_m\|. \end{aligned}$$

Thus $\sigma(s_n)$ is also a Cauchy sequence (of complex numbers). By the completeness of C we conclude that the sequence $\sigma(s_n)$ is convergent. ∎

Lemma 5.25 *If s_n is a fundamental sequence of functions in $S(W, R^+)$ such that $s_n(x) \longrightarrow 0$ for every $x \in X$, then $\sigma(s_n) \longrightarrow 0$ and $\|s_n\| \longrightarrow 0$.*

Proof. Let s_n be a fundamental sequence of functions in $S(W, R^+)$ such that $s_n(x) \longrightarrow 0$ for every $x \in X$. Let $t_0 = s_1$, and $t_n = s_{n+1} - s_n$ for every positive integer n. Then t_n is a sequence of functions in $S(W, R^+)$, and

$$\begin{aligned}\sum_{n=0}^{\infty} t_n(x) &= \lim_{n\to\infty} \sum_{i=0}^{n} t_i(x) \\ &= \lim_{n\to\infty} s_{n+1}(x) = 0\end{aligned}$$

for every $x \in X$. But

$$\begin{aligned}\sum_{n=0}^{\infty} t_n(x) &= \sum_{n=0}^{\infty} (t_n^+(x) - t_n^-(x)) \\ &= \sum_{n=0}^{\infty} t_n^+(x) - \sum_{n=0}^{\infty} t_n^-(x)\end{aligned}$$

and hence

$$\sum_{n=0}^{\infty} t_n^+(x) = \sum_{n=0}^{\infty} t_n^-(x)$$

for every $x \in X$. We also have $t_n^+(x) \le |t_n(x)|$ and $t_n^-(x) \le |t_n(x)|$ for every $x \in X$. This implies that $\sigma(t_n^+) \le \|t_n\|$ and $\sigma(t_n^-) \le \|t_n\|$ for every non-negative integer n. And therefore

$$\begin{aligned}\sum_{n=0}^{\infty} \sigma(t_n^+) &\le \sum_{n=0}^{\infty} \|t_n\| \\ &= \|s_1\| + \sum_{n=1}^{\infty} \|s_{n+1} - s_n\| < \infty\end{aligned}$$

i.e. the series $\sum_{n=0}^{\infty} \sigma(t_n^+)$ is convergent. Similarly, the series $\sum_{n=0}^{\infty} \sigma(t_n^-)$ is also convergent. Thus, by Corollary 5.6, we get

$$\sum_{n=0}^{\infty} \sigma(t_n^+) = \sum_{n=0}^{\infty} \sigma(t_n^-).$$

And so

$$\begin{aligned}\sum_{n=0}^{\infty}\sigma(t_n) &= \sum_{n=0}^{\infty}\sigma(t_n^+ - t_n^-)\\ &= \sum_{n=0}^{\infty}\sigma(t_n^+) - \sum_{n=0}^{\infty}\sigma(t_n^-) = 0.\end{aligned}$$

But

$$\begin{aligned}\sum_{n=0}^{\infty}\sigma(t_n) &= \lim_{n\to\infty}\sum_{i=0}^{n}\sigma(t_i)\\ &= \lim_{n\to\infty}\sigma(\sum_{i=0}^{n}t_i)\\ &= \lim_{n\to\infty}\sigma(s_{n+1}).\end{aligned}$$

Hence $\lim_{n\to\infty}\sigma(s_n) = 0$.

By Corollary 5.15, the sequence $|s_n|$ is also fundamental; and $|s_n(x)| \longrightarrow 0$ for every $x \in X$. Hence $\|s_n\| = \sigma(|s_n|) \longrightarrow 0$. ∎

Now let us define a functional $\int$ on L. For any function $f \in L$, let

$$\int f = \lim_{n\to\infty}\sigma(s_n)$$

where s_n is a fundamental sequence converging pointwise to f. To show that the functional $\int$ is well defined, we first note that since s_n is a fundamental sequence, by Lemma 5.24, the $\lim_{n\to\infty}\sigma(s_n)$ exists. Now let us suppose that there exist fundamental sequences s_n and t_n such that $s_n(x) \longrightarrow f(x)$ and $t_n(x) \longrightarrow f(x)$ for every $x \in X$. By Lemma 5.14, the sequence $u_n = s_n - t_n$ is fundamental. And $\lim_n u_n = \lim_n s_n - \lim_n t_n = 0$. Thus, by Lemma 5.25, we get $\lim_n \sigma(u_n) = 0$. Consequently $\lim_n \sigma(s_n) = \lim_n \sigma(t_n)$. This shows that the value of $\int f$ does not depend on the fundamental sequence converging to f.

Notice that the functional $\int$ is an extension of ψ from $D(W, R^+)$ onto L, since $D(W, R^+) \subset L$ and for every $f \in D(W, R^+)$ we have $\int f = \lim_{n\to\infty}\sigma(s_n) = \psi(f)$.

Theorem 5.26 *The functional $\int$ is a positive linear functional from L into C.*

Proof. Take any two functions f and g in L, and any complex numbers α and β. There exist fundamental sequences s_n and t_n such that $s_n(x) \longrightarrow f(x)$ and $t_n(x) \longrightarrow g(x)$ for every $x \in X$. By Lemma 5.14, the sequence $u_n = \alpha s_n + \beta t_n$ is fundamental. And $\lim_{n\to\infty} u_n(x) = \alpha f(x) + \beta g(x)$ for every $x \in X$. Thus

$$\begin{aligned}
\int(\alpha f + \beta g) &= \lim_{n\to\infty} \sigma(u_n) \\
&= \lim_{n\to\infty} \sigma(\alpha s_n + \beta t_n) \\
&= \alpha \lim_{n\to\infty} \sigma(s_n) + \beta \lim_{n\to\infty} \sigma(t_n) \\
&= \alpha \int f \; + \; \beta \int g
\end{aligned}$$

which shows that $\int$ is a linear functional.

Let $f \in L$ such that $f(x) \geq 0$ for every $x \in X$, and let s_n be a fundamental sequence converging pointwise to f. Then the sequence $t_n = |s_n|$ is fundamental, and $t_n(x) \longrightarrow |f(x)| = f(x)$ for every $x \in X$, and $t_n(x) \geq 0$ for every $x \in X$. Thus $\int f = \lim_{n\to\infty} \sigma(t_n) \geq 0$, since σ is a positive linear functional. This shows that $\int$ is a positive linear functional. ∎

Theorem 5.27 *If f_n is a sequence of functions in L such that $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ for every $x \in X$, the series $\sum_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ for every $x \in X$, and the series $\sum_{n=1}^{\infty} \int |f_n|$ is also convergent, then $f \in L$ and $\int f = \sum_{n=1}^{\infty} \int f_n$.*

Proof. Let $g_n = \mathrm{Re} \circ f_n$ and $h_n = \mathrm{Im} \circ f_n$ for every $n \in N$. Then both g_n and h_n are sequences in $M(W, C)$ and

$$\begin{aligned}
f_n &= g_n + i h_n \\
&= (g_n^+ - g_n^-) + i(h_n^+ - h_n^-).
\end{aligned}$$

Notice that $g_n^+, g_n^-, h_n^+, h_n^- \in M(W, R^+)$ and

$$g_n^+ \leq |g_n| \leq |f_n| \tag{5.7}$$

for every $n \in N$. But $|f_n| \in D(W, R^+)$, and $D(W, R^+)$ is a solid subset of $M(W, R^+)$. So $g_n^+ \in D(W, R^+)$ for every $n \in N$. From inequality (5.7) we also have

$$\sum_{n=1}^{\infty} g_n^+(x) \leq \sum_{n=1}^{\infty} |f_n(x)| < \infty$$

for every $x \in X$, and

$$\sum_{n=1}^{\infty} \psi(g_n^+) \leq \sum_{n=1}^{\infty} \psi(|f_n|) = \sum_{n=1}^{\infty} \int |f_n| < \infty.$$

Thus, by Corollary 5.9, we get

$$\sum_{n=1}^{\infty} g_n^+ \in D(W, R^+) \subset L$$

and

$$\psi(\sum_{n=1}^{\infty} g_n^+) = \sum_{n=1}^{\infty} \psi(g_n^+).$$

We also have the same results for g_n^-, h_n^+, and h_n^-.

Thus, since

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} f_n(x) \\ &= \sum_{n=1}^{\infty} g_n^+(x) - \sum_{n=1}^{\infty} g_n^-(x) + i \sum_{n=1}^{\infty} h_n^+(x) - i \sum_{n=1}^{\infty} h_n^-(x) \end{aligned}$$

for every $x \in X$, and L is a linear space, we must have $f \in L$. Moreover

$$\begin{aligned} \int f &= \int \sum_{n=1}^{\infty} g_n^+ - \int \sum_{n=1}^{\infty} g_n^- + i \int \sum_{n=1}^{\infty} h_n^+ - i \int \sum_{n=1}^{\infty} h_n^- \\ &= \psi(\sum_{n=1}^{\infty} g_n^+) - \psi(\sum_{n=1}^{\infty} g_n^-) + i\psi(\sum_{n=1}^{\infty} h_n^+) - i\psi(\sum_{n=1}^{\infty} h^{\bar{n}}) \\ &= \sum_{n=1}^{\infty} \psi(g_n^+) - \sum_{n=1}^{\infty} \psi(g_n^-) + i \sum_{n=1}^{\infty} \psi(h_n^+) - i \sum_{n=1}^{\infty} \psi(h_n^-) \end{aligned}$$

$$= \sum_{n=1}^{\infty} \int ((g_n^+ - g_n^-) + i(h_n^+ - h_n^-))$$
$$= \sum_{n=1}^{\infty} \int f_n$$

which completes the proof. ∎

We close this chapter by concluding that if (X, V, μ) is a Lebesgue Measure Space, then the quintuple $(X, C, M(V, C), L, \int)$, as constructed in this chapter, forms a Dirac Integral Space. This Dirac Integral Space is said to be generated by the Lebesgue Measure Space.

Chapter 6

Isomorphism between the Categories DIS and LMS

A *category* C consists of a class O, whose elements are called *objects*, and a class M, whose elements are called *morphisms*, such that

1. For every morphism $f \in M$, there exist objects X and Y in O which are called the *domain* and *codomain* of f respectively. In this case we write $f: X \to Y$. Sometimes we will use the notations dom (f) and cod (f) for the domain and codomain of f, respectively.

2. For every object $X \in O$, there exists a morphism $1_X: X \to X$ which is called the *identity morphism* of the object X.

If the domain of a morphism g coincides with the codomain of another morphism f, then we define the composition $g \circ f$ as a morphism with domain dom (f) and codomain cod (g). Composition of morphisms must satisfy the following conditions:

1. For every composable morphisms f, g and h, the associative law holds, that is $(g \circ f) \circ h = g \circ (f \circ h)$.

2. For every morphism $f: X \to Y$, the identity law is satisfied, that is

$$f \circ 1_X = f = 1_Y \circ f.$$

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *functor* $F:\mathcal{C} \to \mathcal{D}$ is a map from category $\mathcal{C}$ to category $\mathcal{D}$, which consists of two related functions, namely an object function and a morphism function. The object function assigns to each object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{D}$ and the morphism function maps each morphism $f: X \to Y$ in $\mathcal{C}$ to a morphism $F(f): F(X) \to F(Y)$ in $\mathcal{D}$ in such a way that

1. $F(1_X) = 1_{F(X)}$ for every identity morphism 1_X in $\mathcal{C}$.

2. $F(g \circ f) = F(g) \circ F(f)$ for each pair of composable morphisms f and g in $\mathcal{C}$.

It is easy to see that if $\mathcal{C}$ is any category, the identity map $I_{\mathcal{C}}:\mathcal{C} \to \mathcal{C}$ defined by the formula

$$I_{\mathcal{C}}(X) = X \quad \text{and} \quad I_{\mathcal{C}}(f) = f$$

for every object $X \in \mathcal{C}$ and every morphism $f \in \mathcal{C}$, is a functor. It is called the *identity functor* of the category $\mathcal{C}$. If $F:\mathcal{C} \to \mathcal{D}$ and $G:\mathcal{D} \to \mathcal{E}$ are functors, then the composition $G \circ F:\mathcal{C} \to \mathcal{E}$ defined by the formula

$$(G \circ F)(X) = G(F(X)) \quad \text{and} \quad (G \circ F)(f) = G(F(f))$$

for every object $X \in \mathcal{C}$ and every morphism $f \in \mathcal{C}$, is also a functor. From these remarks it follows that categories (as objects) together with functors (as morphisms) form a category.

A functor $F:\mathcal{C} \to \mathcal{D}$ is called an *isomorphism* if and only if there exists a functor $G:\mathcal{D} \to \mathcal{C}$ such that $F \circ G = I_{\mathcal{D}}$ and $G \circ F = I_{\mathcal{C}}$. Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be *isomorphic* if there exists an isomorphism $F:\mathcal{C} \to \mathcal{D}$.

6.1 The Categories DIS and LMS

A morphism of a Dirac Integral Space $(X, C, A, L, \int)$ to a Dirac Integral Space $(X', C, A', L', \int')$ is a Baire algebra morphism $\varphi: A \to A'$ (as defined in Section 2.4) which preserves summable functions and the value of the Dirac integral. In other words, the morphism φ must satisfy the following conditions:

1. $\varphi(\alpha f + \beta g) = \alpha\varphi(f) + \beta\varphi(g)$ for every functions f and g in A and any complex numbers α and β

2. $\varphi(fg) = \varphi(f)\varphi(g)$ for every functions f and g in A

3. If f_n is a sequence of functions in A converging pointwise to a function f, then $\varphi(f_n)$ is a sequence of functions in A' converging pointwise to the function $\varphi(f)$

4. $\varphi(\overline{f}) = \overline{\varphi(f)}$ for every function $f \in A$

5. $\varphi(c_X) = c_{X'}$

6. $f \in L \;\Leftrightarrow\; \varphi(f) \in L'$

7. $\int' \varphi(f) = \int f$ for every function $f \in L$

Dirac Integral Spaces together with such morphisms form a category which will be denoted by DIS. A morphism in DIS will be called a DIS-*morphism.* The identity morphism of a Dirac Integral Space $(X, C, A, L, \int)$ is the operator $1_A: A \to A$ defined by $1_A(f) = f$ for every $f \in A$.

A morphism of a Lebesgue Measure Space (X, V, μ) to a Lebegue Measure Space (X', V', μ') is a set-function $\phi: V \to V'$ which preserves the σ-algebra structure and the value of the Lebesgue measure, i.e. for every sets E and D in V and every sequence E_n of disjoint sets in V, we have

1. $\phi(E \setminus D) = \phi(E) \setminus \phi(D)$.

2. $\phi(\cup_n E_n) = \cup_n \phi(E_n)$.

3. $\phi(X) = X'$.

4. $\mu(E) = \mu'(\phi(E))$.

Lebesgue Measure Spaces together with such morphisms also form a category which will be denoted by LMS. A morphism in LMS will be called an LMS-*morphism*. The identity morphism of a Lebesgue Measure Space (X, V, μ) is the set-function $1_V: V \to V$ defined by $1_V(E) = E$ for every set $E \in V$.

6.2 Functor F from DIS to LMS

In this Section we construct a functor F: DIS $\to$ LMS by defining its object function and morphism function. To do so we need the following Proposition.

Proposition 6.1 *If $\varphi: A \to A'$ is a Baire algebra morphism and $c_E \in A$, then $\varphi(c_E) = c_{E'}$ for some $E' \in V' = \text{trace}(A')$.*

Proof. Since $c_E \cdot c_E = c_E$ and φ is a Baire algebra morphism, we have $\varphi(c_E)\varphi(c_E) = \varphi(c_E)$ for every $E \in V = \text{trace}(A)$. From this it follows that

$$\varphi(c_E)(x)(\varphi(c_E)(x) - 1) = 0$$

for every $x \in X'$. Thus $\varphi(c_E)(x) = 1$ or $\varphi(c_E)(x) = 0$ for every $x \in X'$. This means that there exists a subset E' of X' such that $\varphi(c_E) = c_{E'}$. But $\varphi(c_E) \in A'$. Hence $E' \in V'$. ∎

Let $(X, C, A, L, \int)$ and $(X', C, A', L', \int')$ be Dirac Integral Spaces, and (X, V, μ) and (X', V', μ') the Lebesgue Measure Spaces generated by these Dirac Integral Spaces, respectively (as discussed in Section 4.3 of Chapter 4). Let φ be a DIS-morphism from the Dirac Integral Space $(X, C, A, L, \int)$ to the Dirac Integral Space

$(X', C, A', L', \int')$. We now define a set function $\phi: V \to V'$. Take any set $E \in V$ and let

$$\phi(E) = E' \iff \varphi(c_E) = c_{E'}.$$

In other words, for every $E \in V$ we have

$$\varphi(c_E) = c_{\phi(E)}.$$

Proposition 6.1 guarantees that this set function ϕ is well-defined.

Before we prove that the set function ϕ is an LMS-morphism, let us make the following remark on the convergence of characteristic functions. The sequence c_{E_n} of characteristic functions converges pointwise to c_E if and only if $c_E(x) = \limsup_n c_{E_n}(x) = \liminf_n c_{E_n}(x)$ for every $x \in X$. From the fact that

$$\begin{aligned} \limsup_n c_{E_n}(x) &= \inf_n \sup \{c_{E_m}(x):\ m \geq n\} \\ &= c_{\bigcap_n \bigcup_{m \geq n} E_m}(x) \end{aligned}$$

for every $x \in X$, we derive that the sequence c_{E_n} converges pointwise to c_E if and only if $E = \bigcap_n \bigcup_{m \geq n} E_m = \bigcup_n \bigcap_{m \geq n} E_m$.

Proposition 6.2 *The set-function ϕ as define above is a* LMS-*morphism from the Lebesgue Measure Space (X, V, μ) to the Lebesgue Measure Space (X', V', μ').*

Proof. Take any two sets E and D from V. Then

$$\begin{aligned} c_{\phi(E \setminus D)} &= \varphi(c_{E \setminus D}) \\ &= \varphi(c_E - c_E c_D) \\ &= \varphi(c_E) - \varphi(c_E)\varphi(c_D) \\ &= c_{\phi(E)} - c_{\phi(E)} c_{\phi(D)} \\ &= c_{\phi(E) \setminus \phi(D)}. \end{aligned}$$

Hence $\phi(E \setminus D) = \phi(E) \setminus \phi(D)$.

Let E_n be a sequence of disjoint sets in V. Then $E = \cup_n E_n \in V$. Let $D_n = \cup_{i=1}^n E_i$ for every $n \in N$. The sequence c_{D_n} converges pointwise to the characteristic function c_E. Therefore the sequence $\varphi(c_{D_n})$ converges pointwise to the function $\varphi(c_E) = c_{\phi(E)}$. Consider a function $u: C^n \to C$ defined by

$$u(z_1, \ldots, z_n) = \max \{|z_1|, \ldots, |z_n|\}$$

for every $z_1, \ldots, z_n \in C$. The function u is continuous. Thus by Theorem 2.18 we get

$$\varphi(u \circ (c_{E_1}, \ldots, c_{E_n})) = u \circ (\varphi(c_{E_1}), \ldots, \varphi(c_{E_n})).$$

But

$$\begin{aligned}(u \circ (c_{E_1}, \ldots, c_{E_n}))(x) &= \max \{c_{E_1}(x), \ldots, c_{E_n}(x)\} \\ &= c_{E_1 \cup \cdots \cup E_n}(x) \\ &= c_{D_n}(x)\end{aligned}$$

for every $x \in X$. So

$$\varphi(u \circ (c_{E_1}, \ldots, c_{E_n})) = \varphi(c_{D_n})$$

which converges pointwise to the function $c_{\phi(E)}$. And

$$\begin{aligned}u \circ (\varphi(c_{E_1}), \ldots, \varphi(c_{E_n})) &= u \circ (c_{\phi(E_1)}, \ldots, c_{\phi(E_n)}) \\ &= c_{\phi(E_1) \cup \cdots \cup \phi(E_n)}.\end{aligned}$$

Thus the sequence $c_{\phi(E_1) \cup \cdots \cup \phi(E_n)}$ must converge pointwise to the function $c_{\phi(E)}$. By the remark on the convergence of characteristic functions we conclude that $\phi(E) = \cup_n \phi(E_n)$.

From the fact that $\varphi(c_X) = c_{X'}$, we immediately get $\phi(X) = X'$. Now take any set $E \in V$. If $E \in \text{trace}(L)$, then $c_E \in L$, and therefore $\mu(E) = \int c_E = \int' \varphi(c_E) =$

$\int' c_{\phi(E)} = \mu'(\phi(E))$. If $E \notin \text{trace}(L)$, then $c_E \notin L$, and therefore $\mu(E) = \infty$. Since $c_E \notin L$, we have $\varphi(c_E) = c_{\phi(E)} \notin L'$, and therefore $\mu'(\phi(E)) = \infty$. Thus $\mu(E) = \mu'(\phi(E))$. ∎

We shall say that the LMS-morphism ϕ as defined above is induced by the DIS-morphism φ. Now let us consider a function F from the category DIS to the category LMS defined as follows

1. Function F maps each Dirac Integral Space in DIS to the Lebesgue Measure Space generated by it.

2. Function F maps each DIS-morphism in DIS to the LMS-morphism induced by it. Thus for every DIS-morphism φ we have

$$F(\varphi) = \phi \iff \varphi(c_E) = c_{\phi(E)}$$

   for every $E \in V$.

Theorem 6.3 *The function F: DIS $\to$ LMS as defined above is a functor.*

Proof. Let $(X, C, A, L, \int)$ be a Dirac Integral Space, with the identity morphism 1_A, and (X, V, μ) the Lebesgue Measure Space generated by it, with the identity morphism 1_V. We need to prove that $F(1_A) = 1_V$. Let $F(1_A) = \phi$ and E be any set in V. Then $c_E \in A$ and $1_A(c_E) = c_E$. But $1_A(c_E) = c_{\phi(E)}$. Hence $\phi(E) = E = 1_V(E)$ for every set $E \in V$, which shows that $F(1_A) = 1_V$.

Now let $\varphi: A \to A'$ and $\psi: A' \to A''$ be two composable DIS-morphisms, and $F(\varphi) = \phi: V \to V'$ and $F(\psi) = \tau: V' \to V''$ their images under the function F. We need to verify that $F(\psi \circ \varphi) = F(\psi) \circ F(\varphi)$. Take any set $E \in V$. Then $c_E \in A$ and

$$(\psi \circ \varphi)(c_E) = \psi(\varphi(c_E))$$

$$\begin{aligned} &= \psi(c_{\phi(E)}) \\ &= c_{\tau(\phi(E))} \\ &= c_{(\tau\circ\phi)(E)}. \end{aligned}$$

Hence $F(\psi \circ \varphi) = \tau \circ \phi = F(\psi) \circ F(\varphi)$. ∎

6.3 Functor G from LMS to DIS

To establish the isomorphism between the two categories we need to show that there exists a functor G: LMS → DIS such that the composition of this functor with the functor F yields the identity functor. To prove the existence of such a functor we need several Propositions.

First we observe that the set $C_0 = \{0, 1\}$ is a field with respect to addition $(\oplus)$ and multiplication $(\cdot)$ defined as follows

$$\begin{array}{ll} 0 \oplus 0 = 0 & 0 \cdot 0 = 0 \\ 0 \oplus 1 = 1 & 0 \cdot 1 = 0 \\ 1 \oplus 0 = 1 & 1 \cdot 0 = 0 \\ 1 \oplus 1 = 0 & 1 \cdot 1 = 1 \end{array}$$

Notice that $x \oplus y = |x - y|$ for every x and y in C_0, if we consider C_0 as a subset of C. Using this field C_0 we can define Baire algebra of functions in C_0^X and the space of compositors of order T which will be denoted by Comp(C_0, T). It is easy to see that all the Propositions and Theorems in Chapter 2 are also valid if we replace the field C of complex numbers by the field C_0.

If V is a family of subsets of a space X, then let

$$F(V) = \{c_E \colon E \in V\}.$$

It is clear that $F(V)$ is a subset of the space C_0^X. If c_E and c_D are in $F(V)$, then for every $x \in X$ we have

$$(c_E \oplus c_D)(x) \;=\; c_E(x) \oplus c_D(x)$$

$$
\begin{aligned}
&= |c_E(x) - c_D(x)| \\
&= |c_{E\setminus D}(x) + c_{E\cap D}(x) - c_{D\setminus E}(x) - c_{D\cap E}(x)| \\
&= |c_{E\setminus D}(x) - c_{D\setminus E}(x)| \\
&= c_{E\setminus D}(x) + c_{D\setminus E}(x) \\
&= c_{E\triangle D}(x)
\end{aligned}
$$

where $E \triangle D = (E \setminus D) \cup (D \setminus E)$. Notice also that

$$
\begin{aligned}
(c_E \cdot c_D)(x) &= c_E(x) \cdot c_D(x) \\
&= c_{E\cap D}(x)
\end{aligned}
$$

for every $x \in X$. Thus

$$c_E \oplus c_D = c_{E\triangle D}$$

$$c_E \cdot c_D = c_{E\cap D}$$

for every $c_E, c_D \in F(V)$.

Proposition 6.4 *A family V of subsets of X is a σ-algebra if and only if $F(V)$ is a Baire algebra of functions in C_0^X.*

Proof. $(\Rightarrow)$: Suppose that V is a σ algebra of subsets of X. Take any two characteristic functions c_E and c_D from $F(V)$. Then $E \in V$ and $D \in V$, and $c_E \oplus c_D = c_{E\triangle D} \in F(V)$, since $(E \triangle D) \in V$. Also $c_E \cdot c_D = c_{E\cap D} \in F(V)$ since $(E \cap D) \in V$.

It is clear that $F(V)$ is closed under scalar multiplication (where the scalars are elements of C_0), and under involution (where the conjugate of an element in C_0 is the element itself), and is containing all constant functions (since V is a σ-algebra).

To show that $F(V)$ is closed under pointwise convergence, we take any sequence c_{E_n} of characteristic functions from $F(V)$ such that $c_{E_n}(x) \longrightarrow c_E(x)$ for every $x \in X$. Then $E_n \in V$ for every $n \in N$. By the remark on the convergence of characteristic functions and the fact that every σ-ring is closed under countable unions and intersections we get $E = \lim \bigcup_{n \geq m} E_n \in V$. Hence $c_E \in F(V)$.

$(\Leftarrow)$: Assume that $F(V)$ is a Baire algebra of functions in C_0^X. The space $X \in V$ since $c_X \in F(V)$. Take any two sets E and D from V. Then both c_E and c_D belong to $F(V)$. Notice that

$$\begin{aligned} c_{E \setminus D} &= c_{E \triangle (E \cap D)} \\ &= c_E \oplus (c_E \cdot c_D) \in F(V). \end{aligned}$$

Thus $(E \setminus D) \in V$.

We also have

$$\begin{aligned} c_{E \cup D} &= c_{(E \triangle D) \triangle (E \cap D)} \\ &= (c_E \oplus c_D) \oplus (c_E \cdot c_D) \in F(V) \end{aligned}$$

which proves that $(E \cup D) \in V$.

Take any countable collection E_n of sets from V and let $D_n = E_1 \cup \cdots \cup E_n$ and $E = \bigcup_{n \in N} E_n$. Then $D_n \in V$ for every $n \in N$, and so c_{D_n} is a sequence of functions in $F(V)$. As shown in the proof of Theorem 3.12, the sequence c_{D_n} converges pointwise to the function c_E. Since $F(V)$ is closed under pointwise convergence, we get $c_E \in F(V)$. Thus $E \in V$. This completes the proof that V is a σ-algebra. ∎

Let V and V' be σ-algebras of subsets of spaces X and X' respectively, and $\phi: V \rightarrow V'$ be a *σ-algebra morphism*, i.e. a set-function which preserves the σ-algebra structure. From Proposition 6.4 we know that $F(V)$ and $F(V')$ are

Baire algebras in C_0^X. Now let us define an operator $\psi\colon F(V) \to F(V')$ by

$$\psi(c_E) = c_{\phi(E)}$$

for every $c_E \in F(V)$. Notice that $\psi(c_\emptyset) = c_{\phi(\emptyset)} = c_\emptyset$, since $\phi(\emptyset) = \phi(E \setminus E) = \phi(E) \setminus \phi(E) = \emptyset$.

Proposition 6.5 *The operator ψ as defined above is a Baire algebra morphism.*

Proof. Take any two characteristic functions c_E and c_D from $F(V)$. Then E and D are in V and

$$\begin{aligned}
\psi(c_E \oplus c_D) &= \psi(c_{E\triangle D}) \\
&= c_{\phi(E\triangle D)} \\
&= c_{\phi(E)\triangle\phi(D)} \\
&= c_{\phi(E)} \oplus c_{\phi(D)} \\
&= \psi(c_E) \oplus \psi(c_D).
\end{aligned}$$

Also

$$\begin{aligned}
\psi(c_E \cdot c_D) &= \psi(c_{E\cap D}) \\
&= c_{\phi(E\cap D)} \\
&= c_{\phi(E)\cap\phi(D)} \\
&= c_{\phi(E)} \cdot c_{\phi(D)} \\
&= \psi(c_E) \cdot \psi(c_D).
\end{aligned}$$

It is easy to see that ψ preserves the scalar multiplication, constant functions and involution.

Let c_{E_n} be a sequence of characteristic functions in $F(V)$ converging pointwise to c_E. Then $E = \bigcap_n \bigcup_{m\geq n} E_m = \bigcup_n \bigcap_{m\geq n} E_m$. Thus $\phi(E) = \bigcap_n \bigcup_{m\geq n} \phi(E_m) =$

$\bigcup_n \bigcap_{m\geq n} \phi(E_m)$. This implies that $c_{\phi(E_n)}(x) \longrightarrow c_{\phi(E)}(x)$ for every $x \in X$, or equivalently the sequence $\psi(c_{E_n})$ converges pointwise to the function $\psi(c_E)$. So the operator ψ preserves pointwise convergence. This completes the proof that ψ is a Baire algebra morphism. ∎

Let (X, V, μ) and (X', V', μ') be Lebesgue Measure Spaces, which generate Dirac Integral Spaces $(X, C, M(V, C), L, \int)$ and $(X', C, M(V', C), L', \int')$, respectively, as discussed in Chapter 5. Suppose that ϕ is a LMS-morphism from (X, V, μ) to (X', V', μ'). From Theorem 2.17 and Proposition 3.11, we know that

$$f \in M(V, C) \iff f = u \circ (c_E)_{E\in V}$$

for some $u \in \text{Comp}(V)$. Let us now define an operator $\varphi: M(V, C) \to M(V', C)$ by the formula

$$\varphi(f) = u \circ (c_{\phi(E)})_{E\in V}$$

if $f = u \circ (c_E)_{E\in V}$ for some $u \in \text{Comp}(V)$.

To show that this operator φ is well-defined, let $f \in M(V, C)$ such that $f = u\circ(c_E)_{E\in V} = v\circ(c_E)_{E\in V}$, where u and v are compositors of order V. Let $w = u - v$. Then $w \in \text{Comp}(V)$ (and therefore w is a Baire function), and

$$w \circ (c_E)_{E\in V} = 0 = c_\emptyset. \tag{6.1}$$

Consider the injective function $i: C_0^V \to C^V$ defined by $i(x) = x$ for every $x \in C_0^V$, and a constant function $r: C \to C_0$ defined by $r(z) = 0$ for every $z \in C$. Both functions are continuous, and therefore they are Baire functions. So by Proposition 2.6 the function $w_0 = r \circ w \circ i$ is a Baire function from C_0^V into C_0. Thus $w_0 \in \text{Comp}(C_0, V)$ and therefore $w_0 \circ (c_E)_{E\in V} \in F(V)$ since $c_E \in F(V)$ for every $E \in V$ and $F(V)$ is a Baire algebra in C_0^X. We also note that w_0 is the restriction of w to C_0^V. Thus

$$w_0 \circ (c_E)_{E\in V} = w \circ (c_E)_{E\in V} = c_\emptyset$$

by equation (6.1). Let $\psi: F(V) \to F(V')$ be the operator defined on page 107. Then

$$\psi(w_0 \circ (c_E)_{E\in V}) = \psi(c_\emptyset).$$

Since ψ is a Baire algebra morphism, by Theorem 2.18 we get

$$w_0 \circ (\psi(c_E))_{E\in V} = c_\emptyset.$$

Thus $w_0 \circ (c_{\phi(E)})_{E\in V} = 0$ or equivalently

$$w \circ (c_{\phi(E)})_{E\in V} = 0.$$

From this it follows that

$$u \circ (c_{\phi(E)})_{E\in V} = v \circ (c_{\phi(E)})_{E\in V}$$

which shows that the operator φ is well-defined.

Proposition 6.6 *The operator $\varphi: M(V,C) \to M(V',C)$ as defined above is a Baire algebra morphism.*

Proof. Take any two functions f and g from the space $M(V,C)$. Then $f = u \circ (c_E)_{E\in V}$ and $g = v \circ (c_E)_{E\in V}$ for some compositors $u, v \in \mathrm{Comp}(V)$, and so

$$\begin{aligned} \alpha f + \beta g &= \alpha(u \circ (c_E)_{E\in V}) + \beta(v \circ (c_E)_{E\in V}) \\ &= (\alpha u) \circ (c_E)_{E\in V} + (\beta v) \circ (c_E)_{E\in V} \\ &= (\alpha u + \beta v) \circ (c_E)_{E\in V} \end{aligned}$$

or any complex numbers α and β. Thus

$$\begin{aligned} \varphi(\alpha f + \beta g) &= (\alpha u + \beta v) \circ (c_{\phi(E)})_{E\in V} \\ &= \alpha(u \circ (c_{\phi(E)})_{E\in V}) + \beta(v \circ (c_{\phi(E)})_{E\in V}) \\ &= \alpha\varphi(f) + \beta\varphi(g). \end{aligned}$$

We can prove analogously that $\varphi(fg) = \varphi(f)\varphi(g)$.

Let f_n be a sequence of functions in $M(V, C)$ converging pointwise to some function $f \in M(V, C)$. Then there exists a sequence u_n of compositors in Comp(V) such that $f_n = u_n \circ (c_E)_{E \in V}$. Let $v: C^N \to C$ be the function defined by

$$v((z_n)_{n \in N}) = \begin{cases} \lim_n z_n & \text{if } z_n \text{ converges} \\ 0 & \text{if } z_n \text{ does not converge.} \end{cases}$$

From Lemma 2.16, we know that $v \in$ Comp(N). Since the sequence f_n converges pointwise to function f, we have $v \circ (f_n)_{n \in N} = f$. But

$$\begin{aligned} v \circ (f_n)_{n \in N} &= v \circ (u_n \circ (c_E)_{E \in V})_{n \in N} \\ &= v \circ (u_n)_{n \in N} \circ (c_E)_{E \in V} \\ &= w \circ (c_E)_{E \in V} \end{aligned}$$

where $w = v \circ (u_n)_{n \in N}$. Notice that $w \in$ Comp(V) since $v \in$ Comp(N) and Comp(V) is a Baire algebra. Thus $f = w \circ (c_E)_{E \in V}$, and so

$$\begin{aligned} \varphi(f) &= w \circ (c_{\phi(E)})_{E \in V} \\ &= (v \circ (u_n)_{n \in N}) \circ (c_{\phi(E)})_{E \in V} \\ &= v \circ (u_n \circ (c_{\phi(E)})_{E \in V})_{n \in N} \\ &= v \circ (\varphi(f_n))_{n \in N} \\ &= \lim_n \varphi(f_n) \end{aligned}$$

which shows that φ preserves pointwise convergence.

If $f = u \circ (c_E)_{E \in V} \in M(V, C)$ for some $u \in$ Comp(V), then $\bar{f} = \bar{u} \circ (c_E)_{E \in V}$, and so

$$\begin{aligned} \varphi(\bar{f}) &= \bar{u} \circ (c_{\phi(E)})_{E \in V} \\ &= \overline{u \circ (c_{\phi(E)})_{E \in V}} \\ &= \overline{\varphi(f)}. \end{aligned}$$

We notice that $c_X = p_X \circ (c_E)_{E\in V}$ where the projection $p_D((c_E)_{E\in V}) \doteq c_D$ is a compositor of order V for every $D \in V$. Hence

$$\begin{aligned}\varphi(c_X) &= p_X \circ (c_{\phi(E)})_{E\in V} \\ &= c_{\phi(X)} \\ &= c_{X'}.\end{aligned}$$

This completes the proof of the Proposition. ∎

We recall that if (X, V, μ) is a Lebesgue Measure Space then $S(W, C)$ is the set of all simple functions with respect to the family $W = \{E \in V\colon \mu(E) < \infty\}$. For the operator $\varphi\colon M(V, C) \to M(V', C)$ as defined above we prove the following Lemmas.

Lemma 6.7 *A function s belongs to the space $S(W, C)$ if and only if $\varphi(s) \in S(W', C)$. Moreover, for every $s \in S(W, C)$ we have $\int s = \int' \varphi(s)$.*

Proof. We first observe that $c_E = p_E \circ (c_D)_{D\in V}$, where the projection $p_E \in \mathrm{Comp}(V)$ for every $E \in V$. Thus

$$\begin{aligned}\varphi(c_E) &= p_E \circ (c_{\phi(D)})_{D\in V} \\ &= c_{\phi(E)}.\end{aligned}$$

From this it follows that

$$\begin{aligned}c_E \in S(W, C) &\iff E \in W \\ &\iff \mu(E) < \infty \\ &\iff \mu'(\phi(E)) < \infty \\ &\iff \phi(E) \in W' \\ &\iff c_{\phi(E)} \in S(W', C) \\ &\iff \varphi(c_E) \in S(W', C).\end{aligned}$$

Hence, by the linearity of the spaces $S(W, C)$ and $S(W', C)$, and of the operator φ, we get the equivalence

$$s \in S(W, C) \iff \varphi(s) \in S(W', C).$$

Moreover

$$\begin{aligned} \int c_E &= \mu(E) \\ &= \mu'(\phi(E)) \\ &= \int' c_{\phi(E)} \\ &= \int' \varphi(c_E) \end{aligned}$$

for every $c_E \in S(W, C)$. Thus by the linearity of the integrals and the operator φ we get

$$\int s = \int' \varphi(s)$$

for every $s \in S(W, C)$. ∎

Let X be a topological Hausdorff space. Consider the Baire space $B(X, C)$ of functions from X into C. Let $V = \text{trace}(B(X, C))$. The members of V are called *Baire sets.* In other words, a subset E of X is a Baire set if and only if the characteristic function c_E of the set E is a Baire function. Since $B(X, C)$ is a Baire algebra, from Corollary 3.13 we know that the collection V of all Baire sets forms a σ-algebra of subsets of X. Let us prove the following Proposition.

Proposition 6.8 *If X is a metric space, then every open or closed subset of X is a Baire set.*

Proof. Let G be any open subset of a metric space X with metric d. Consider a function $f: X \to R$ defined by

$$f(x) = d(x, G^c)$$

for every $n \in N$. The function f is continuous on X, and $f(x) > 0$ if and only if $x \in G$. So the function $g_n = nf \wedge c_X$ is also continuous on X, and therefore $g_n \in B(X,C)$, for every $n \in N$. Moreover the sequence g_n converges pointwise to the characteristic function c_G. Since the Baire space $B(X,C)$ is closed under pointwise convergence, we have $c_G \in B(X,C)$. Hence G is a Baire set.

If F is a closed subset of X, then $F = G^c$ for some open set G. Thus F is also a Baire set. ∎

For each $m \in N$ consider a function $v_m: C^N \to R$ defined by the following formula

$$v_m((z_n)_{n\in N}) = \sum_{n=1}^{m} |z_n - z_{n+1}|$$

for every $((z_n)_{n\in N}) \in C^N$. This function v_m is continuous on C^N for every $m \in N$. Let B be the set of all $((z_n)_{n\in N}) \in C^N$ such that the series $\sum_{n=1}^{\infty} |z_n - z_{n+1}|$ converges. Then

$$\begin{aligned} B &= \left\{ (z_n)_{n\in N} \;\middle|\; \sum_{n=1}^{\infty} |z_n - z_{n+1}| < \infty \right\} \\ &= \left\{ (z_n)_{n\in N} \;\middle|\; (\exists_{k\in N})\ (\forall_{m\in N})\ \sum_{n=1}^{m} |z_n - z_{n+1}| < k \right\} \\ &= \bigcup_{k\in N} \bigcap_{m\in N} \left\{ (z_n)_{n\in N} \;\middle|\; \sum_{n=1}^{m} |z_n - z_{n+1}| < k \right\} \\ &= \bigcup_{k\in N} \bigcap_{m\in N} v_m^{-1}((-\infty, k)) \end{aligned}$$

where $v_m^{-1}((-\infty,k))$ is an open set in C^N for every $m \in N$ and $k \in N$. Since C^N is metrizable, the set $v_m^{-1}((-\infty,k))$ is a Baire set for each $m \in N$ and $k \in N$. Hence the set B is a Baire set, and consequently c_B is a Baire function. So $c_B \in \mathrm{Comp}(N)$.

Lemma 6.9 *Let s_n be any sequence of functions from a Baire algebra A in C^X, and the set B as defined above. Then the following two conditions are equivalent:*

1. *$\sum_{n=1}^{\infty} |s_n(x) - s_{n+1}(x)| < \infty$ for all $x \in X$*

2. $c_B \circ (s_n)_{n \in N} = c_X$

Proof. Take any $x \in X$. Then

$$\sum_{n=1}^{\infty} |s_n(x) - s_{n+1}(x)| < \infty \iff (s_n(x))_{n \in N} \in B$$
$$\iff c_B((s_n(x))_{n \in N}) = 1$$
$$\iff (c_B \circ (s_n)_{n \in N})(x) = c_X(x) \qquad (6.2)$$

which implies that the conditions stated in the Lemma are equivalent. ∎

Lemma 6.10 *If a sequence $s_n \in S(W, C)$ is fundamental, then the sequence $t_n = \varphi(s_n) \in S(W', C)$ is fundamental.*

Proof. Take any fundamental sequence s_n from $S(W, C)$. Then

$$\sum_{n=1}^{\infty} |s_n(x) - s_{n+1}(x)| < \infty \quad \forall x \in X,$$

which is equivalent to

$$c_B \circ (s_n)_{n \in N} = c_X$$

by Lemma 6.9. Applying the Baire algebra morphism φ to both sides of the above equality, we get

$$\varphi(c_B \circ (s_n)_{n \in N}) = \varphi(c_X).$$

Using the fact that $c_B \in \text{Comp}(N)$ we have

$$c_B \circ (\varphi(s_n))_{n \in N} = c_{X'}$$

or equivalently

$$c_B \circ (t_n)_{n \in N} = c_{X'}.$$

By Lemma 6.9 the last condition is equivalent to

$$\sum_{n=1}^{\infty} |t_n(x') - t_{n+1}(x')| < \infty \quad \forall x' \in X'.$$

By Lemma 6.7 we get

$$\begin{aligned}\int |s_n - s_{n+1}| &= \int' \varphi(|s_n - s_{n+1}|) \\ &= \int' |\varphi(s_n - s_{n+1})| \\ &= \int' |\varphi(s_n) - \varphi(s_{n+1})| \\ &= \int' |t_n - t_{n+1}|\end{aligned}$$

which implies that

$$\sum_{n=1}^{\infty} \|s_n - s_{n+1}\| < \infty \iff \sum_{n=1}^{\infty} \|t_n - t_{n+1}\|' < \infty.$$

Thus the proof of the Lemma is complete. ∎

Lemma 6.11 *A function s belongs to the space $S(W,C)$ if and only if $s \in S(V,C)$ and $s \in L$.*

Proof. If $s \in S(W,C)$ then it is clear that $s \in S(V,C)$ and also $s \in L$. So take any simple function $s \in S(V,C)$ such that s also belongs to the space L. Then

$$s = \sum_{j=1}^{n} z_j c_{E_j}$$

where $\{E_j\colon\ j = 1,2,\ldots,n\}$ is collection of disjoint sets from V. Thus $c_{E_j} \in M(V,C)$, and consequently $c_{E_j}|s| \in M(V,C)$, for $j = 1,2,\ldots,n$. But $c_{E_j}|s| \leq |s|$. Since L is solid in $M(V,C)$, we must have $c_{E_j}|s| = c_{E_j}|z_j| \in L$. Without loss of generality we may assume that $z_j \neq 0$ for $j = 1,2,\ldots,n$. Thus $c_{E_j} = \frac{1}{|z_j|}c_{E_j}|s| \in L$ and therefore $E_j \in W$ for $j = 1,2,\ldots,n$. Hence $s \in S(W,C)$. ∎

Proposition 6.12 *The operator $\varphi\colon M(V,C) \to M(V',C)$ as defined above is a* DIS*-morphism from the Dirac Integral Space $(X,C,M(V,C),L,\int)$ to the Dirac Integral Space $(X',C,M(V',C),L',\int')$.*

Proof. In Proposition 6.6 we have proved that φ is a Baire algebra morphism. So we need only to show that φ preserves summable functions and the value of the Dirac integral.

Let $f \in L$ and s_n be a fundamental sequence in $S(W, C)$ converging pointwise to f. Then $\varphi(s_n)$ is a fundamental sequence in $S(W', C)$ converging pointwise to the function $\varphi(f) \in M(V', C)$. Thus $\varphi(f) \in L'$.

Conversely, take any function $\varphi(f) \in L'$ where $f \in M(V, C)$. Let u_n be the sequence of functions in $S(K, C)$ such that $u_n(z) \longrightarrow e(z) = z$ and $|u_n(z)| \nearrow |z|$ for every $z \in C$ (cfr. Lemma 3.7). Then by Lemma 3.8 we get $(u_n \circ f) \in S(V, C)$ for every $n \in N$. Therefore $\varphi(u_n \circ f) = u_n \circ \varphi(f) \in S(V', C) \subset M(V', C)$ for every $n \in N$. Since $\varphi(f) \in L'$ and $|u_n \circ \varphi(f)| \leq |\varphi(f)|$ and L' is solid in $M(V', C)$, we have $(u_n \circ \varphi(f)) \in L'$ for evey $n \in N$. And so by Lemma 6.11 we get $(u_n \circ \varphi(f)) \in S(W', C)$ for every $n \in N$. We also see that $|u_n \circ \varphi(f)| \nearrow |\varphi(f)|$; and therefore $\int' |u_n \circ \varphi(f)| \nearrow \int' |\varphi(f)|$, which means that the sequence $\int' |u_n \circ \varphi(f)|$ is bounded. Notice that $|u_n \circ f| \nearrow |f|$ and $\int |u_n \circ f| = \int' |u_n \circ \varphi(f)|$ which is bounded. By the Monotone Convergence Theorem we get $|f| \in L$. But L is solid in $M(V, C)$, so we must have $f \in L$. Thus we have proved that $f \in L \Leftrightarrow \varphi(f) \in L'$.

Take any function $f \in L$ and let $s_n \in S(W, C)$ be some fundamental sequence converging pointwise to f. Then $\varphi(s_n) \in S(W', C)$ is a fundamental sequence converging pointwise to the function $\varphi(f) \in L'$. Thus

$$\begin{aligned} \int f &= \lim_n \int s_n \\ &= \lim_n \int' \varphi(s_n) \\ &= \int' \varphi(f) \end{aligned}$$

for every $f \in L$. ∎

The morphism φ introduced by means of the above construction will be called the DIS-morphism induced by the LMS-morphism ϕ. Now let us consider a function G from the category LMS to the category DIS defined as follows

1. Function G maps each Lebesgue Measure Space in LMS to the Dirac Integral Space generated by it.

2. Function G maps each LMS-morphism in LMS to the DIS-morphism induced by it.

Theorem 6.13 *The function* $G\colon \mathrm{LMS} \to \mathrm{DIS}$ *as defined above is a functor.*

Proof. Let (X, V, μ) be any Lebesgue Measure Space, with identity morphism 1_V, and let $(X, C, M(V, C), L, \int)$ be the Dirac Integral Space generated by it, with the identity morphism 1_M. We have to show that $G(1_V) = 1_M$. Take any function $f \in M(V, C)$. Then $f = u \circ (c_E)_{E \in V}$ for some function $u \in \mathrm{Comp}(V)$. Thus $G(1_V)(f) = u \circ (c_{1_V(E)})_{E \in V} = u \circ (c_E)_{E \in V} = f = 1_M(f)$. Hence $G(1_V) = 1_M$.

Now let $\phi\colon V \to V'$ and $\tau\colon V' \to V''$ be any two composable LMS-morphisms. Notice that for every $E \in V$ we have

$$\begin{aligned} G(\tau)(c_{\phi(E)}) &= G(\tau)(p_{\phi(E)} \circ (c_D)_{D \in V}) \\ &= p_{\phi(E)} \circ (G(\tau)(c_D))_{D \in V} \\ &= p_{\phi(E)} \circ (c_{\tau(D)})_{D \in V} \\ &= c_{\tau(\phi(E))}. \end{aligned}$$

Take any function $f \in M(V, C)$. Then $f = u \circ (c_E)_{E \in V}$ for some function $u \in \mathrm{Comp}(V)$. Thus

$$\begin{aligned} G(\tau \circ \phi)(f) &= u \circ (c_{(\tau \circ \phi)(E)})_{E \in V} \\ &= u \circ (c_{\tau(\phi(E))})_{E \in V} \\ &= u \circ (G(\tau)(c_{\phi(E)}))_{E \in V} \end{aligned}$$

$$= G(\tau)(u \circ (c_{\phi(E)})_{E \in V})$$

$$= G(\tau)(G(\phi)(f))$$

$$= (G(\tau) \circ G(\phi))(f)$$

for every $f \in M(V, C)$, which proves that $G(\tau \circ \phi) = G(\tau) \circ G(\phi)$. ∎

6.4 Isomorphism between the two Categories

In this Section we are going to show that the functor F: DIS $\rightarrow$ LMS and the functor G: LMS $\rightarrow$ DIS are inverse to each other, i.e. to prove that $F \circ G = I_{\text{LMS}}$ and $G \circ F = I_{\text{DIS}}$ where I_{LMS} and I_{DIS} are the identity functors of the categories LMS and DIS, respectively.

Proposition 6.14 *The composite functor $F \circ G$ is the identity functor of the category* LMS.

Proof. We have to show that $(F \circ G)(X, V, \mu) = (X, V, \mu)$ and $(F \circ G)(\phi) = \phi$ for every Lebesgue Measure Space (X, V, μ) and every LMS-morphism ϕ in the category LMS.

$$\begin{array}{ccccc}
(X,V,\mu) & \xrightarrow{\;G\;} & (X,C,M(V,C),L,\int) & \xrightarrow{\;F\;} & (X,\tilde{V},\tilde{\mu}) \\
\Big\downarrow \phi & & \Big\downarrow G(\phi)=\varphi & & \Big\downarrow F(\varphi)=\tilde{\phi} \\
(X',V',\mu') & \xrightarrow{\;G\;} & (X',C,M(V',C),L',\int') & \xrightarrow{\;F\;} & (X',\widetilde{V'},\widetilde{\mu'})
\end{array}$$

Take any Lebesgue Measure Space (X, V, μ) and let $(X, C, M(V, C), L, \int)$ be the Dirac Integral Space generated by it. Suppose that $(X, \tilde{V}, \tilde{\mu})$ is the Lebesgue

Measure Space generated by the Dirac Integral Space $(X, C, M(V,C), L, \int)$. We need to verify that $\tilde{V} = V$ and $\tilde{\mu} = \mu$.

Take any set $E \in V$. Then $c_E \in M(V,C)$, and so $E \in \text{trace}(M(V,C)) = \tilde{V}$. Thus $V \subset \tilde{V}$. Conversely, take any set $E \in \tilde{V} = \text{trace}(M(V,C))$. Then $c_E \in M(V,C)$. Let $G = \{z \in C: |z - 1| < 1/2\}$. Then G is an open set in C and $0 \notin \overline{G}$. So $c_E^{-1}(G) \in V$. But $x \in c_E^{-1}(G) \Leftrightarrow c_E(x) \in G \Leftrightarrow c_E(x) = 1 \Leftrightarrow x \in E$, which means that $c_E^{-1}(G) = E$. Thus $E \in V$, and so $\tilde{V} \subset V$. Hence $\tilde{V} = V$.

Take any set $E \in V$. If $E \in W$, then $c_E \in L$, and so $\tilde{\mu}(E) = \int c_E = \mu(E)$. If $E \notin W$, then $c_E \notin L$, and so $\tilde{\mu}(E) = \infty = \mu(E)$. Thus $\tilde{\mu} = \mu$.

Now let ϕ be any LMS-morphism and φ be the DIS-morphism induced by it. Suppose that $\tilde{\phi}$ is the LMS-morphism induced by φ. We need to show that $\tilde{\phi} = \phi$. In the above paragraph we have proved that the domains of the morphisms ϕ and $\tilde{\phi}$ are the the same, say V. Take any set $E \in V$. Then

$$
\begin{aligned}
c_{\tilde{\phi}(E)} &= \varphi(c_E) \\
&= \varphi(p_E \circ (c_D)_{D \in V}) \\
&= p_E \circ (c_{\phi(D)})_{D \in V} \\
&= c_{\phi(E)}.
\end{aligned}
$$

Hence $\tilde{\phi}(E) = \phi(E)$ for every $E \in V$, which means that $\tilde{\phi} = \phi$. ∎

To prove that the composite functor $G \circ F$ is the identity functor of the category DIS we need the following Lemmas. Let $(X, C, A, L, \int)$ be any Dirac Integral Space and (X, V, μ) be the Lebesgue Measure Space generated by it. We recall that the set $S(W, C)$ is the collection of all simple functions with respect to the family $W = \{E \in V\colon \mu(E) < \infty\}$. Let $(X, C, M(V,C), \tilde{L}, \tilde{\int})$ be the Dirac Integral Space generated by the Lebesgue Measure Space (X, V, μ). In Theorem 3.16 we have proved that $M(V,C) = A$.

$$(X, C, A, L, \int) \xrightarrow{F} (X, V, \mu) \xrightarrow{G} (X, C, M(V, C), \tilde{L}, \tilde{\int})$$

Lemma 6.15 *If $s \in S(W, C)$, then s belongs to both L and $\tilde{L}$, and $\int s = \tilde{\int} s$.*

Proof. Take any simple function $s \in S(W, C)$. It is obvious that $s \in \tilde{L}$. By definition $s = \sum_{i=1}^{n} z_i c_{E_i}$ for some complex numbers $z_1, \ldots, z_n$ and some collection $\{E_i\}$ of disjoint sets from W. Since $E_i \in W$, we have $\mu(E_i) < \infty$, which means that $c_{E_i} \in L$. Thus by the linearity of the spaces L we get $s \in L$. Moreover

$$\begin{aligned} \tilde{\int} s &= \sum_{i=1}^{n} z_i \mu(E_i) \\ &= \sum_{i=1}^{n} z_i \int c_{E_i} \\ &= \int \sum_{i=1}^{n} z_i c_{E_i} \\ &= \int s \end{aligned}$$

since both integrals are linear. ∎

Lemma 6.16 *If $f \in \tilde{L}$, then $f \in L$ and $\int f = \tilde{\int} f$.*

Proof. Take any function $f \in \tilde{L}$ and let u_n be the sequence of functions in $S(K, C)$ such that $u_n(z) \to e(z) = z$ and $|u_n(z)| \nearrow |z|$ for every $z \in C$ (cfr. Lemma 3.7). Let $v_n = u_n \circ f$. Then $v_n \in M(V, C)$ for every $n \in N$. Since $|v_n| = |u_n \circ f| \leq |f|$ and $\tilde{L}$ is solid in $M(V, C)$, we have $v_n \in \tilde{L}$ for every $n \in N$. By Lemma 6.11, we get $v_n \in S(W, C)$ for every $n \in N$. Notice that the sequence v_n converges pointwise to the function $e \circ f = f$. Thus, by the Dominated Convergence Theorem, we get

$$\tilde{\int} |v_n - f| = \|v_n - f\| \longrightarrow 0.$$

This implies that $\|v_n - v_m\| \longrightarrow 0$ when n and m tend to infinity. So we can find an increasing sequence k_n of positive integers such that

$$\|v_{k_n} - v_m\| < 2^{-n}$$

for all $m > k_n$. Let $s_n = v_{k_n}$ for every $n \in N$. It is obvious that s_n is a sequence of functions in $S(W, C)$. Let

$$h_n = \sum_{j=1}^{n} |s_j - s_{j+1}|.$$

Then h_n is an increasing sequence of non-negative real-valued functions in L and

$$\begin{aligned} \int h_n &= \sum_{j=1}^{n} \int |s_j - s_{j+1}| \\ &= \sum_{j=1}^{n} \|s_j - s_{j+1}\| \\ &= \sum_{j=1}^{n} \|v_{k_j} - v_{k_{j+1}}\| \\ &< \sum_{j=1}^{n} 2^{-j} \leq 1. \end{aligned}$$

Let E be set of all $x \in X$ at which the sequence $h_n(x)$ is not convergent. From Lemma 4.18 we know that the set E is a null-set. Let $D = X \setminus E$ and $t_n = c_D s_n$ and $g = c_D f$. Then $t_n \in S(W, C)$ for every $n \in N$. Since

$$t_n(x) = \begin{cases} s_n(x) & \text{if } x \notin E \\ 0 & \text{if } x \in E, \end{cases}$$

we must have

$$\sum_{n=1}^{\infty} |t_n(x) - t_{n+1}(x)| < \infty$$

for every $x \in X$. We also observe that

$$\begin{aligned} \sum_{n=1}^{\infty} \|t_n - t_{n+1}\| &= \sum_{n=1}^{\infty} \|s_n - s_{n+1}\| \\ &< \sum_{n=1}^{\infty} 2^{-n} = 1 \end{aligned}$$

since $t_n - t_{n+1} = s_n - s_{n+1}$ almost everywhere, and $t_n(x) \longrightarrow c_D(x)f(x) = g(x)$ for every $x \in X$. Thus t_n is a fundamental sequence in $S(W,C)$ converging pointwise to the function g. Hence $g \in \tilde{L}$, and $\tilde{\int} g = \lim_n \tilde{\int} t_n = \lim_n \int t_n$. But, since $|t_n| \leq |f|$ a.e. for every $n \in N$, by the Strong Dominated Convergence Theorem we have $g \in L$ and $\int t_n \longrightarrow \int g$. So $\tilde{\int} g = \int g$. Notice that $f \in L \subset A = M(V,C)$ and $f = g$ a.e. Therefore $f \in L$. Moreover $\int f = \int g$ and $\tilde{\int} f = \tilde{\int} g$. Hence $\int f = \tilde{\int} f$. ∎

Proposition 6.17 *The composite functor $G \circ F$ is the identity functor of the category* DIS.

Proof. We need to verify that $(G \circ F)(X,C,A,L,\int) = (X,C,A,L,\int)$ and $(G \circ F)(\varphi) = \varphi$ for every Dirac Integral Space $(X,C,A,L,\int)$ and every DIS-morphism φ in the category DIS.

$$\begin{array}{ccccc}
(X,C,A,L,\int) & \xrightarrow{F} & (X,V,\mu) & \xrightarrow{G} & (X,C,M(V,C),\tilde{L},\tilde{\int}) \\
\Big\downarrow \varphi & & \Big\downarrow F(\varphi)=\phi & & \Big\downarrow G(\phi)=\tilde{\varphi} \\
(X',C,A',L',\int') & \xrightarrow{F} & (X',V',\mu') & \xrightarrow{G} & (X',C,M(V',C),\widetilde{L'},\tilde{\int}')
\end{array}$$

Let $(X,C,A,L,\int)$ be any Dirac Integral Space and (X,V,μ) be the Lebesgue Measure Space generated by it. Let $(X,C,M(V,C),\tilde{L},\tilde{\int})$ be the Dirac Integral Space generated by the Lebesgue Measure Space (X,V,μ). In Theorem 3.16 we have proved that $M(V,C) = A$. We still have to verify that $\tilde{L} = L$ and $\tilde{\int} = \int$. By Lemma 6.16 we have $L \subset \tilde{L}$ and $\int f = \tilde{\int} f$ for every $f \in L$. So it suffices to show that $\tilde{L} \subset L$. Take any function $f \in \tilde{L}$. Then there exists a fundamental sequence

s_n in $S(W, C)$ such that s_n converges pointwise to f. From Lemma 6.15 we know that $s_n \in L$ for every $n \in N$. Let

$$g_n(x) = \sum_{i=1}^{n} |s_i(x) - s_{i+1}(x)|$$

and

$$g(x) = \sum_{i=1}^{\infty} |s_i(x) - s_{i+1}(x)|$$

for every $x \in X$. Then g_n is an increasing sequence of real-valued functions in L converging pointwise to the function g. Moreover, the relations

$$\begin{aligned} \int g_n &= \sum_{i=1}^{n} \int |s_i - s_{i+1}| \\ &= \sum_{i=1}^{n} \tilde{\int} |s_i - s_{i+1}| \\ &= \sum_{i=1}^{n} \|s_i - s_{i+1}\|^{\sim} \\ &\leq \sum_{i=1}^{\infty} \|s_i - s_{i+1}\|^{\sim} < \infty \end{aligned}$$

show that the sequence $\int g_n$ is bounded, where $\|s\|^{\sim} = \tilde{\int} |s|$. Thus, by the Monotone Convergence Theorem, we get $g \in L$. Notice that

$$\begin{aligned} |s_n| &= |s_1 + (s_2 - s_1) + \cdots + (s_n - s_{n-1})| \\ &\leq |s_1| + |s_2 - s_1| + \cdots + |s_n - s_{n-1}| \\ &= |s_1| + g_{n-1} \\ &\leq |s_1| + g \end{aligned}$$

for every $n \in N$, and $|s_1| + g \in L$. So by the Dominated Convergence Theorem we get $f \in L$. Thus $\tilde{L} \subset L$. Hence $\tilde{L} = L$ and consequently $\tilde{\int} = \int$.

Let φ be any DIS-morphism and $\phi: V \to V'$ be the LMS-morphism induced by it. Let $\tilde{\varphi}$ be the DIS-morphism induced by the LMS-morphism ϕ. We need to show that $\tilde{\varphi} = \varphi$. It has been shown in the above paragraph that the domains of the morphisms $\tilde{\varphi}$ and φ are the same, i.e. $M(V, C) = A$. Now take any function $f \in A$. Then $f = u \circ (c_E)_{E \in V}$ for some $u \in \text{Comp}(V)$. So

$$\begin{aligned}
\tilde{\varphi}(f) &= u \circ (c_{\phi(E)})_{E \in V} \\
&= u \circ (\varphi(c_E))_{E \in V} \\
&= \varphi(u \circ (c_E)_{E \in V}) \\
&= \varphi(f),
\end{aligned}$$

which shows that $\tilde{\varphi} = \varphi$. ∎

We end this paper by stating the concluding Theorem which follows directly from Propositions 6.14 and 6.17.

Theorem 6.18 *The functor F establishes the isomorphism of the category of Dirac Integral Spaces* DIS *with the category of Lebesgue Measure Spaces* LMS.

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List of Symbols

$B(X,Y)$	Baire space of functions from X into Y, 11
C^X	Space of functions from X into C, 6
$C(X,Y)$	Space of continuous functions from X into Y, 11
$\mathrm{Comp}(T)$	Space of compositors of order T, 7
c_E	Characteristic function of a set E, 8
f^+, f^-	Positive and negative parts of a function f, 90
$f_n \to f$	Sequence f_n converges pointwise to f, 6
$f_n \nearrow f$	Sequence f_n converges increasingly to f, 57, 77
$(f_t)_{t\in T}$	Function generated by a set $\{f_t\colon\ t \in T\}$, 7
$\bar{f}$	Involution of a function f, 6
$\bar{G}$	Closure of a set G, 32
$\mathrm{Im}(z)$	Imaginary part of a complex number z, 6
I_C	Identity functor of a category C, 98
1_X	Identity morphism of an object X, 97
L/L_0	Quotient space generated by L_0, 68
$M(V,C)$	Space of complex-valued measurable functions generated by a family V of sets, 34
p_α	Projection onto the α-th coordinate, 8
$\mathrm{Re}(z)$	Real part of a complex number, 6
$S(V)$	Family of simple sets generated by a family V, 26
$S(V,Y)$	Space of simple functions with values in Y generated by a family V of sets, 27
$\mathrm{supp}(f)$	Support of a function f, 58
$\mathrm{trace}(L)$	Trace of a space L of functions, 41
V^σ	Collection of all countable unions of sets from V, 31
$(X,C,A,L,\int)$	Dirac Integral Space, 46
(X,V,μ)	Lebesgue Measure Space, 62
$\bigvee_{n\in N} x_n$; $\bigwedge_{n\in N} x_n$	$\sup\{x_n\colon\ n \in N\}$; $\inf\{x_n\colon\ n \in N\}$, 20
$\\|\cdot\\|$, $\#\cdot\#$	(Semi-)norm, 51, 65, 69

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