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Steady flow over an arbitrary obstruction based on the gravity wave equations

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Abstract. We derive an analytical solution to a steady state problem of the gravity wave equations. An arbitrary bottom topography is considered. The problem is assumed to be one dimensional. The depth, discharge and topography elevation at the left-end of the space domain are assumed to have the same values as those at the right-end. We obtain that the fluid surface on the whole interior space domain remains horizontal and is not influenced by the topography shape when we use the gravity wave equations. Furthermore, the analytical solution that we derive is used to test the performance of a finite volume method. We find that the gravity wave equations give some advantages in comparison to the shallow water equations.

1. Introduction

The shallow water equations have been widely accepted as a model for shallow water flows on open channels, such as urban floods, tsunamis, river flows, etc. These equations are often simplified in order to reduce the computational costs and to increase the numerical stability leading to some simpler models, for example, kinematic wave equations, diffusive wave equations and gravity wave equations [1].

Martins et al. [1] presented the analytical solution to the classical dam break problem based on the gravity wave equations. Then they [2] reported the effectiveness of the gravity wave equations for urban flood simulations. Their work [1, 2] was on unsteady state problems. Steady state problems based on the gravity wave equations have, to our knowledge, not been investigated.

Our contributions are as follows. We derive the analytical solution to a steady state problem of the gravity wave equations with an arbitrary bottom topography. In addition, we verify the performance of the finite volume method used by Hidayat et al. [3] in solving steady state problems of the gravity wave equations.

This paper is organized as follows. Mathematical models are recalled in Section 2. We derive the analytical solution to the steady state problem in Section 3. Numerical results for a scenario of the gravity wave equations and the shallow water equations are presented in Section 4. Finally concluding remarks are drawn in Section 5.

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2. Mathematical models

The gravity wave equations are [1]

$$\frac{\partial}{\partial t}h+\frac{\partial}{\partial x}(uh)=0\,, \tag{1}$$

$$\frac{\partial}{\partial t}(uh)+\frac{\partial}{\partial x}\left(\frac{1}{2}gh^2\right)=-gh\frac{d}{dx}z\,. \tag{2}$$

Here x is one dimensional space variable, t is time variable, $h=h(x,t)$ represents the fluid depth (height), $u=u(x,t)$ denotes the fluid velocity, $z=z(x)$ is the bed (bottom) topography, and g is the acceleration due to gravity. Note that the gravity wave equations (1)-(2) are simplifications of the standard shallow water equations

$$\frac{\partial}{\partial t}h+\frac{\partial}{\partial x}(uh)=0\,, \tag{3}$$

$$\frac{\partial}{\partial t}(uh)+\frac{\partial}{\partial x}\left(u^2h+\frac{1}{2}gh^2\right)=-gh\frac{d}{dx}z\,, \tag{4}$$

where the convective term $\frac{\partial}{\partial x}(u^2h)$ is neglected [1, 2]. We assume that friction is not involved in the models. The expression uh is the fluid discharge. Equations (1)-(2) and (3)-(4) are one dimensional. Readers interested in two dimensional equations may consult the shallow water literature, such as the work of Aronica et al. [4].

3. Analytical results

Now we suppose that a space domain is given where the fluid depth $h(x,t)$ is positive everywhere on the domain at any time. A fixed bottom obstruction $z(x)$ is also known.

A steady state is achieved when the water depth $h(x,t)$ and the velocity $u(x,t)$ are independent with respect to time t. This leads to

$$\frac{\partial}{\partial t}h=0 \qquad \text{and} \qquad \frac{\partial}{\partial t}(uh)=0\,. \tag{5}$$

As a result we have

$$\frac{\partial}{\partial x}(uh)=0 \qquad \text{and} \qquad \frac{\partial}{\partial x}\left(\frac{1}{2}gh^2\right)=-gh\frac{d}{dx}z\,. \tag{6}$$

That is,

$$uh=q \qquad \text{and} \qquad h+z=c\,, \tag{7}$$

where q and c are constants. Far from the obstacle, the bottom elevation, the water depth and the velocity are assumed to be $z(x)=0$, $h(x,t)=h_0$ and $u(x,t)=u_0$ respectively. Thus for the whole domain, we obtain

$$uh=u_0h_0 \qquad \text{and} \qquad h+z=h_0 . \tag{8}$$

The analytical solution to the steady state problems of the gravity wave equations is as follows. The analytical solution for the fluid depth is

$$h(x,t)=h_0-z(x), \tag{9}$$

and the analytical solution for the fluid velocity is

$$u(x,t)=\frac{u_0h_0}{h_0-z(x)}, \tag{10}$$

at any space point x and any time $t>0$.

4. Numerical results

In this section we present some numerical results related to the analytical solution that we obtain in the previous section for a scenario of the steady state problem based on the gravity wave equations. All measured quantities are assumed to have SI units. The acceleration due to gravity is 9.81.

Consider the gravity wave equations (1)-(2). We refer to the finite volume method described by Hidayat et al. [5] with first order accuracy. Following Hidayat et al. [5] we take the channel space domain $0\le x\le 25$. The bottom topography is

$$z(x)=\begin{cases} 0.2-0.05(x-10)^2 & \text{if} \quad 8\le x\le 12, \\ \\ 0 & \text{if} \quad \text{otherwise.} \end{cases} \tag{11}$$

The boundary condition is

$$q(0,t)=4.42 \qquad \text{and} \qquad u(25,t)=2, \tag{12}$$

at the ends of the space domain for $t>0$. The initial condition is

$$q(x,0)=4.42 \qquad \text{and} \qquad h(x,0)+z(x)=2.21, \tag{13}$$

for $t>0$.

At time $t=30$ the results of the gravity wave equations are shown in Figure 1. We use 100 cells with the time step is 0.05 times the cell width. The numerical solution matches with the exact analytical solution with the depth error is 7.6983×10^{-15} and the velocity error is 2.9088×10^{-14}. This means that the finite volume method is able to solve the problem very accurately up to the machine precision.

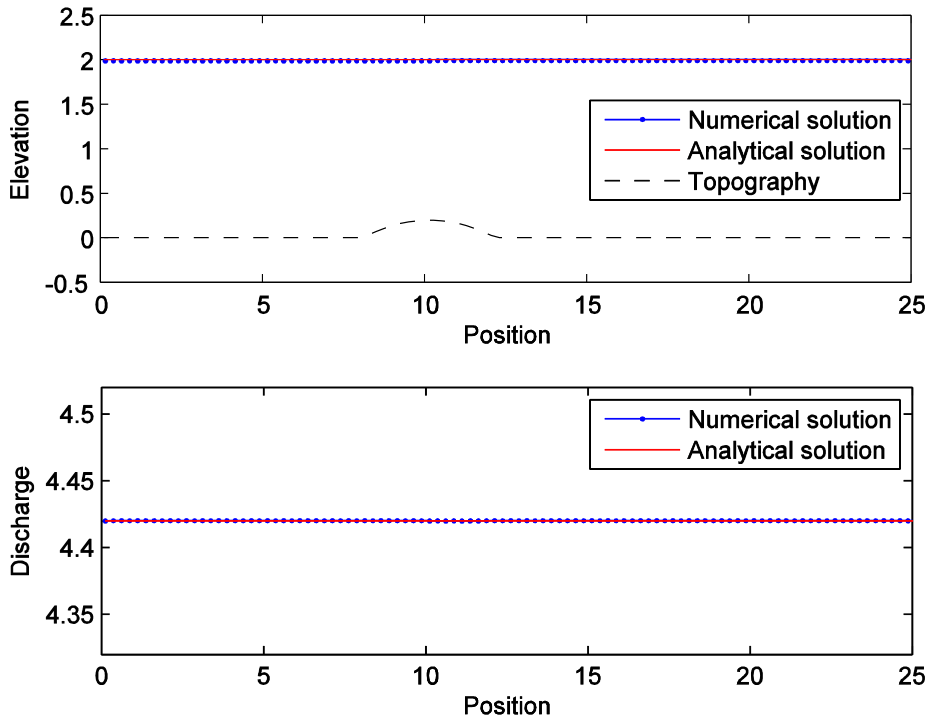

Figure 1. Simulation results of the gravity wave equations. The numerical solution matches the analytical solution up to the machine precision.

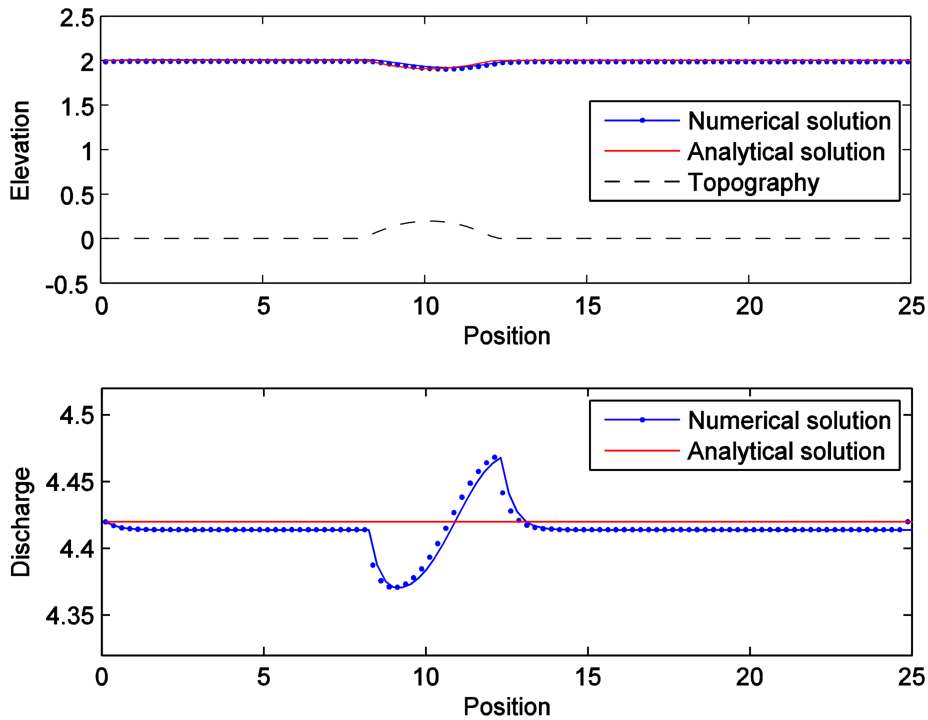

Figure 2. Simulation results of the shallow water equations. The finite volume method solves the shallow water equations not as accurately as the gravity wave equations.

In comparison to the gravity wave equations, we also plot here the results of the same problem, but based on the shallow water equations (3)-(4), as shown in Figure 2. We also take 100 cells with the time step is 0.05 times the cell width. The numerical solution matches with the exact solution with the depth error is 0.0053 and the velocity error is 0.0102. We obtain from Figure 2 that the source of errors is mainly at the region where the topographical obstruction exists. However, these errors can be made small by taking more number of computational cells. More results on numerical solutions to the shallow water equations can be found in the work of Mungkasi et al. [5-8]. Note that the analytical work on the steady state problems of shallow water equations was presented in some literature, for example that done by Houghton and Kasahara [9].

We observe that, for the steady state problem, the finite volume method performs better in solving the gravity wave equations than in solving the shallow water equations. This is another advantage of using the gravity wave equations for water flow simulation in addition to their simpler forms, as long as the gravity effect is much more significant than the convective effect. That is, as long as the convective term of the shallow water equations is negligible for the particular problem under consideration.

5. Conclusion

We have derived an analytical solution to the steady state problem of flows over an arbitrary topography based on the gravity wave equations. The analytical solution can be applied to test the performance of numerical methods used to solve the gravity wave equations. Numerical results show that the finite volume method solves the gravity wave equations more accurately than the shallow water equations. This means that the gravity wave equations indeed have great advantages in saving computational costs and in giving opportunity to be solved more accurately.

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