A study of well-balanced finite volume methods and refinement indicators for the shallow water equations

Sudi Mungkasi

September 2012

A thesis submitted for the degree of
Doctor of Philosophy
of The Australian National University

Dedicated to my wife, Asti,
our daughter, Daniella,
and my parents, Wagimin and Sulasmi.

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Declaration

The work in this thesis is my own except where otherwise stated.

Sudi Mungkasi

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Contents

Acknowledgements **xi**

Publications, Talks, and Awards **xiii**

Abstract **xvii**

Abbreviations **xix**

1 Thesis overview **1**

1.1 Motivation 1

1.2 Purpose 2

1.3 Outline and contributions 4

PART I: WELL-BALANCED METHODS **7**

2 Well-balanced methods **9**

2.1 Introduction 9

2.2 Governing equations and numerical methods 10

2.3 Quantity reconstructions 12

2.4 Numerical tests 15

2.4.1 Steady state: a lake at rest 16

2.4.2 Unsteady state: oscillation on a parabolic bed 18

2.5 Concluding remarks 19

PART II: SOME EXACT SOLUTIONS **23**

3 Avalanche involving a dry area **25**

3.1 Introduction 25

3.2 Saint-Venant models 26

3.2.1 Equations in the standard Cartesian coordinate 28
3.2.2 Equations in the topography-linked coordinate 30
3.3 Existing solutions 32
3.4 A new solution 33
3.5 Numerical tests 36
3.6 Concluding remarks 41

4 Avalanche involving a shock 43
4.1 Introduction 43
4.2 A new solution in the standard Cartesian system 45
4.2.1 Derivation of the analytical solution 45
4.2.2 Properties of the analytical solution 51
4.3 A new solution in the topography-linked system 52
4.4 Numerical tests 54
4.5 Concluding remarks 57

5 The Carrier-Greenspan periodic solution 59
5.1 Introduction 59
5.2 Governing equations 61
5.3 Periodic waves on a sloping beach 62
5.3.1 Carrier-Greenspan solution 63
5.3.2 The formulation of Johns 64
5.3.3 Calculating the stage and velocity 65
5.3.4 The Johns approximate solution 67
5.3.5 More accurate approximations 68
5.4 Computational experiments 70
5.4.1 First test case: the Johns prescription is successful 72
5.4.2 Second test case: the Johns prescription fails 77
5.5 Rate of convergence 82
5.6 Concluding remarks 83

PART III: INDICATORS FOR ADAPTIVE METHODS 85

6 Numerical entropy production 87
6.1 Introduction 87
6.2 Numerical entropy production in 1D 88
6.2.1 An existing numerical entropy scheme 88

6.2.2 An alternative numerical entropy scheme 91
6.2.3 Numerical tests 93
6.3 Numerical entropy production in 1.5D 104
6.3.1 Governing equations and numerical schemes 104
6.3.2 Numerical tests 105
6.4 Numerical entropy production in 2D 109
6.4.1 Numerical entropy scheme 109
6.4.2 Numerical tests 111
6.5 Concluding remarks 115

7 Weak local residuals **117**
7.1 Introduction 117
7.2 Weak local residuals of balance laws 118
7.3 Weak local residuals of shallow water equations 120
7.3.1 Well-balancing the weak local residual 121
7.3.2 Wet/dry interface treatment for weak local residuals 124
7.3.3 Numerical tests 125
7.4 Weak local residuals in adaptive methods 131
7.4.1 A one-dimensional flow with topography 134
7.4.2 A one-dimensional dam break with passive tracer 135
7.4.3 A planar dam break 136
7.4.4 A radial dam break 138
7.5 Concluding remarks 141

8 Conclusions and future work **143**
8.1 Conclusions 143
8.2 Future work 144

A Equations and methods in two dimensions **145**
A.1 Integral forms of shallow water equations 145
A.2 Finite volume methods 146

B Data structure for triangulations in *i*FEM **149**
B.1 A newest vertex bisection for refinement 150
B.2 A nodal-wise algorithm for coarsening 150
B.3 Conserved quantites in finite volume methods 151

References **153**

Acknowledgements

The work for this thesis was undertaken at the Department of Mathematics, Mathematical Sciences Institute, The Australian National University (ANU). Here I acknowledge the contribution and support of a number of people, institutions and committees over the years of my candidature.

First and foremost, I sincerely thank my primary supervisor, Stephen Roberts, for his guidance, support, and encouragement. I have learnt a lot from him academically and non-academically. He has been very helpful in supporting me to do research in the directions of my own interests, which make me enjoy my PhD research. He has also been supportive to me in getting funds and travel grants during my PhD studies.

I am very grateful to my co-supervisors, Markus Hegland and Linda Stals. I acknowledge some discussions with Markus on weak local residuals and with Linda on adaptive grid methods.

I am indebted to some lecturers, postdoctoral fellows, staff, colleagues, and friends at the ANU. For some discussions, I thank Christopher Zoppou, Vikram Sunkara, Padarn Wilson, Brendan Harding, Mathew Langford, Shi Bai, Bin Zhou, Jiakun Liu, Srinivasa Subramanya Rao, Kowshik Bettadapura, Yuan Fang, and Chin Foon Khoo. I also thank Bishnu Lamichhane, John Urbas and Paul Leopardi also for some discussions. Kind thanks are due to Jodi Tutty and Annie Bartlett at the ANU Academic Skills and Learning Centre for some suggestions on the writing of this thesis. Thanks also go to Nick Guoth, Warren Yang and Kelly Wicks for technical and administrative support.

I thank my friends from Indonesia in Australia for the friendship, which has made me feel at home. Thanks are due to Robertus Anugerah Purwoko Putro and family, Hendra Gunawan Harno, Teddy Kurniawan, Jack Matsay and family, Supomo Suryo Hudoyo and family, Br Budi Hernawan OFM, and other Indonesian friends during my studies in Australia.

In my PhD period, I was visiting the Department of Applied Mathematics, the University of Washington, U.S.A. and Institut für Geometrie und Praktische Mathematik, RWTH Aachen University, Germany. I thank Professor Randall John LeVeque for his assistance and for some discussions, and Jihwan Kim, Jonathan Varkovitzky and Grady Lemoine at the University of Washington for some discussions. I thank Professor Sebastian Noelle for his assistance and some supervision on part of my work, and K. R. Arun, Guoxian Chen, and Marcel Makowski at RWTH Aachen University for some discussions.

During my PhD candidature, I was supported financially by The Australian National University. I acknowledge and thank the following institutions, organisations and committees for funding supports during my PhD candidature.

- The Australian National University for ANU PhD and Tuition Scholarships, ANU Vice Chancellor's travel grants, ANU Miscellaneous (top-up) Scholarships, and funding support for conferences,
- The Australian Mathematical Society (AustMS) for partial funding for my presentations in the AustMS conference meetings,
- ANZIAM and CSIRO for partial funding through ANZIAM-CSIRO Student Support Schemes for my presentations in the Engineering Mathematics and Applications Conference as well as Computational Techniques and Applications Conferences,
- Sanata Dharma University for covering the premium of my Overseas Health Cover Insurance,
- RWTH Aachen University for partial accommodation support while I visited Professor Sebastian Noelle,
- The University of Washington for one time in-city travel assistance while I visited Professor Randall John LeVeque,
- The Committe of the 2012 Gene Golub SIAM Summer School (G2S3) for the G2S3 travel grant funded by the U.S.A. Naval Postgraduate School for my participation in the summer school.

Last but not least, I thank my parents for their continuous encouragement and my wife, Asti, for her constant love.

Publications, Talks, and Awards

The following publications and preprints correspond to the work in this thesis.

- S. Mungkasi and S. G. Roberts, 2010, "On the best quantity reconstructions for a well balanced finite volume method used to solve the shallow water wave equations with a wet/dry interface", *ANZIAM Journal (E)*, 51: C48–C65, Australian Mathematical Society. (Based on the work in Chapter 2.)
  http://journal.austms.org.au/ojs/index.php/ANZIAMJ/article/view/2576/1289

- S. Mungkasi and S. G. Roberts, 2011, "A new analytical solution for testing debris avalanche numerical models", *ANZIAM Journal (E)*, 52: C349–C363, Australian Mathematical Society. (Based on the work in Chapter 3.)
  http://journal.austms.org.au/ojs/index.php/ANZIAMJ/article/view/3785/1465

- S. Mungkasi and S. G. Roberts, 2012, "Analytical solutions involving shock waves for testing debris avalanche numerical models", *Pure and Applied Geophysics*, 169 (10): 1847–1858, Springer. (Based on the work in Chapter 4.)
  http://dx.doi.org/10.1007/s00024-011-0449-1

- S. Mungkasi and S. G. Roberts, 2012, "Approximations of the Carrier-Greenspan periodic solution to the shallow water wave equations for flows on a sloping beach", *International Journal for Numerical Methods in Fluids*, 69 (4): 763–780, John Wiley & Sons. (Based on the work in Chapter 5.)
  http://dx.doi.org/10.1002/fld.2607

- S. Mungkasi and S. G. Roberts, 2011, "Numerical entropy production for shallow water flows", *ANZIAM Journal (E)*, 52: C1–C17, Australian Mathematical Society. (Based on part of the work in Chapter 6.)
  http://journal.austms.org.au/ojs/index.php/ANZIAMJ/article/view/3786/1410

- S. Mungkasi, S. G. Roberts, and S. Noelle, 2012, "Numerical entropy production as a refinement indicator for shallow water equations", *preprint*. (Based on part of the work in Chapter 6.)

- S. Mungkasi and S. G. Roberts, 2012, "Well-balanced computations of weak local residuals for shallow water equations", *preprint*. (Based on part of the work in Chapter 7.)

- S. Mungkasi and S. G. Roberts, 2012, "Weak local residuals as refinement indicators for shallow water equations", *preprint*. (Based on part of the work in Chapter 7.)

- S. Mungkasi and S. G. Roberts, 2011, "A finite volume method for shallow water flows on triangular computational grids", *Proceedings of IEEE International Conference on Advanced Computer Science and Information Systems (ICACSIS) 2011*, pages 79–84. (Based on the work in Appendix A.) `http://ieeexplore.ieee.org/xpl/freeabs_all.jsp?arnumber=6140781`

I have presented some parts of this thesis in talks during my PhD candidature. The talks are as follows.

- "Well-balanced finite volume methods for shallow water flows involving wet/dry interface" in the 53rd Australian Mathematical Society Annual Meeting, The University of South Australia, Adelaide, Australia, 28 Sep – 1 Oct 2009

- "Well-balanced finite volume methods for shallow water flows involving wet/dry interface" in Engineering Mathematics and Applications Conference 2009, The University of Adelaide, Adelaide, Australia, 6 – 9 Dec 2009

- "A new analytical solution for testing debris avalanche numerical models in the standard Cartesian coordinate system" in the 54th Australian Mathematical Society Annual Meeting, The University of Queensland, Brisbane, Australia, 27 – 30 Sep 2010

- "A new analytical solution for testing debris avalanche numerical models" in Computational Techniques and Applications Conference 2010, The University of New South Wales, Sydney, Australia, 28 Nov – 1 Dec 2010

- "Numerical entropy production for shallow water flows" in Computational Techniques and Applications Conference 2010, The University of New South Wales, Sydney, Australia, 28 Nov – 1 Dec 2010

- "Approximations of the Carrier-Greenspan periodic solution to the shallow water wave equations for flows on a sloping beach" in the 55th Australian Mathematical Society Annual Meeting, The University of Wollongong, Wollongong, Australia, 26 – 29 Sep 2011

- "A finite volume method for shallow water flows on triangular computational grids" in the IEEE International Conference on Advanced Computer Science and Information Systems (ICACSIS) 2011, Jakarta, Indonesia, 17 – 18 Dec 2011

- "Some exact solutions to shallow-water-type models and their use as numerical test problems" in a Seminar at IGPM, RWTH Aachen University, Aachen, Germany, 9 February 2012

I received the following awards during my PhD candidature.

- Best paper award in the IEEE International Conference on Advanced Computer Science and Information Systems (ICACSIS) 2011, Jakarta, Indonesia, 17 – 18 Dec 2011,

- Best presentation award, again in the IEEE International Conference on Advanced Computer Science and Information Systems (ICACSIS) 2011, Jakarta, Indonesia, 17 – 18 Dec 2011,

- ANU Vice Chancellor's travel grants to visit Professor Randall John LeVeque at the University of Washington, Seattle, U.S.A., 16 Nov–14 Dec 2011 and Professor Sebastian Noelle at RWTH Aachen University, Aachen, Germany, 6 Jan – 17 Feb 2012, and

- Travel grant from the U.S.A. Naval Postgraduate School to participate in the Gene Golub SIAM Summer School "Simulation and Supercomputing in the Geosciences", Monterey, California, U.S.A., 29 Jul – 10 Aug 2012.

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Abstract

This thesis studies solutions to the shallow water equations analytically and numerically. The study is separated into three parts.

The first part is about well-balanced finite volume methods to solve steady and unsteady state problems. A method is said to be well-balanced if it preserves an unperturbed steady state at the discrete level. We implement hydrostatic reconstructions for the well-balanced methods with respect to the steady state of a lake at rest. Four combinations of quantity reconstructions are tested. Our results indicate an appropriate combination of quantity reconstructions for dealing with steady and unsteady state problems.

The second part presents some new analytical solutions to debris avalanche problems and reviews the implicit Carrier-Greenspan periodic solution for flows on a sloping beach. The analytical solutions to debris avalanche problems are derived using characteristics and a variable transformation technique. The analytical solutions are used as benchmarks to test the performance of numerical solutions. For the Carrier-Greenspan periodic solution, we show that the linear approximation of the Carrier-Greenspan periodic solution may result in large errors in some cases. If an explicit approximation of the Carrier-Greenspan periodic solution is needed, higher order approximations should be considered. We propose second order approximations of the Carrier-Greenspan periodic solution and present a way to get higher order approximations.

The third part discusses refinement indicators used in adaptive finite volume methods to detect smooth and nonsmooth regions. In the adaptive finite volume methods, smooth regions are coarsened to reduce the computational costs and nonsmooth regions are refined to get more accurate solutions. We consider the numerical entropy production and weak local residuals as refinement indicators. Regarding the numerical entropy production, our work is the first to implement the numerical entropy production as a refinement indicator into adaptive finite

volume methods used to solve the shallow water equations. Regarding weak local residuals, we propose formulations to compute weak local residuals on nonuniform meshes. Our numerical experiments show that both the numerical entropy production and weak local residuals are successful as refinement indicators.

Abbreviations

1D, 1.5D, 2D	one dimension, one-and-a-half dimensions, two dimensions
CFL	Courant-Friedrichs-Lewy
CG	Carrier-Greenspan
CK	Constantin-Kurganov
FVM	Finite Volume Method(s)
*i*FEM	innovative Finite Element Method
KKP	Karni-Kurganov-Petrova
MHR	Mangeney-Heinrich-Roche
NEP	Numerical Entropy Production
nNEP	normalised Numerical Entropy Production
SI	Systéme International (International System)
WLR	Weak Local Residual(s)