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Stochastic currents of fractional Brownian motion: Existence and regularity

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ABSTRACT

By using white noise analysis, we study the integral kernel $\xi(x)$, $x \in \mathbb{R}^d$, $d \in \mathbb{N}$, of stochastic currents corresponding to fractional Brownian motion with Hurst parameter $0 < H < 1$. For all $x \in \mathbb{R}^d \setminus \{0\}$ and all $d \geq 1$ we prove that the integral kernel $\xi(x)$ is well-defined as a Hida distribution for all $H \in (0, 1)$. For $dH < 1$ and $0 < H < 1$, we furthermore show that the integral kernel $\xi(x)$, when evaluated at $x = 0$, is a Hida distribution as well. In the case $d = 1$, $x \neq 0$, and $0 < H < 1$, we show that $\xi(x) \in \mathcal{G}'$, the space of regular generalized functions. Elements of the space $\mathcal{G}'$ and elements from the negative Sobolev–Watanabe distribution spaces share the property that the partial sums of their chaos decomposition are square integrable functions. More precisely, we show that $\xi(x) \in \mathcal{G}_{-s} \subset \mathcal{G}'$ for all $x \neq 0$, $0 < H < 1$, and all $s > 0$.

1. Introduction

The concept of currents has its origins in geometric measure theory. A typical 1-current is given by

$$\varphi \mapsto \int_0^T (\varphi(\gamma(t)), \gamma'(t))_{\mathbb{R}^d} \, \mathrm{d}t, \qquad 0 < T < \infty, \quad d \in \mathbb{N},$$

where $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ and $[0, T] \ni t \mapsto \gamma(t) \in \mathbb{R}^d$ is a rectifiable curve. The interested reader may find definitions, results, and applications on the subject in the books [1,2].

To obtain its integral kernel, one can propose the ansatz

$$\zeta(x) := \int_0^T \delta(x - \gamma(t))\gamma'(t) \, \mathrm{d}t, \quad x \in \mathbb{R}^d,$$

where δ is the Dirac delta function, and try to give a mathematically rigorous meaning in an appropriate space of generalized functions.

The stochastic analog of the integral kernel $\zeta(x)$ arises if we substitute the deterministic curve γ by the sample path of a stochastic process X taking values in $\mathbb{R}^d$. Hence, we obtain the following kernel

$$\xi(x) := \int_0^T \delta(x - X(t)) \, \mathrm{d}X(t), \quad x \in \mathbb{R}^d. \tag{1}$$

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The stochastic integral (1) must be properly defined.

The object $\xi(x)$ appears naturally in the study of fluid-dynamical models. Based on evidence that turbulent fluids are composed of vortex structures, in particular filament-like structures, Chorin [3] proposed describing such vortex filaments as paths of stochastic processes and relating the statistical properties of these processes to those of turbulent fluids. The intuitive idea of a vortex filament is a curve, a vortex line (an integral curve of the vorticity field) over which the vorticity field is supported. This directly leads to the concept of a 1-current, a distributional vector field, concentrated along curves. If X is the stochastic process whose paths represent the vortex filaments, then the random distributional vorticity field is given by $\xi(x)$. The two main questions concern the statistical properties and the realism of these random vortex structures. For a more comprehensive discussion, including advanced applications and open problems, we refer the reader to [4, Section 4] and the references cited therein.

In this paper, we choose X to be a d-dimensional fractional Brownian motion (fBm) B_H, with Hurst parameter $H \in (0,1)$. Therefore, the main objective of our study is

$$\xi(x) := \int_0^T \delta(x - B_H(t))\,\mathrm{d}B_H(t). \tag{2}$$

The stochastic integral is interpreted as a fractional Itô integral developed in [5]. Other approaches, such as Malliavin calculus and stochastic integrals through regularization, were investigated in [6–8] to study ξ. In [6–8], pathwise with probability one, ξ was constructed as a random variable taking values in a negative Sobolev space. I.e., for a fixed path, ξ is a generalized function and is therefore not pointwise defined in $x \in \mathbb{R}^d$. Moreover, in [8], for all $x \in \mathbb{R}$, the kernel $\xi(x)$ was constructed in a negative Sobolev–Watanabe distribution space using Malliavin calculus for $H \in [1/2, 1)$.

In this work, we show that, if $x \in \mathbb{R}^d \backslash \{0\}$, $\xi(x)$ is a Hida distribution for any $H \in (0,1)$ and $d \geq 1$ while for $x = 0 \in \mathbb{R}^d$, $\xi(x)$ is a Hida distribution whenever $dH < 1$; see Theorem 3.1 and Corollary 3.2. For $d = 1$, $x \neq 0$, and $H \in (0,1)$, we show that $\xi(x) \in \mathcal{G}'$, the space of regular generalized functions. The new idea to prove this type of regularity can be found in the key Lemma 4.1. Elements of the space $\mathcal{G}'$ and elements from the negative Sobolev–Watanabe distribution spaces share the property that the partial sums of their chaos decomposition are square integrable functions. More precisely, we show that $\xi(x) \in \mathcal{G}_{-s} \subset \mathcal{G}'$ for $x \neq 0$, $H \in (0,1)$, and all $s > 0$; see Theorem 4.4. The question of whether $\xi(x) \in \mathcal{G}_0 = L^2(\mu)$ remains open. Should this be the case, the integral kernel of stochastic currents would even exist as a real-valued random variable.

The paper is organized as follows. In Section 2, we recall the background from the white noise analysis needed later. In Section 3, we prove the existence result, and in Section 4, we prove the regularity result. The appendix contains the proofs for certain technical lemmas.

2. Gaussian white noise calculus

In this section, we briefly recall the concepts and results of white noise analysis used throughout this work. For a detailed explanation, see, e.g., [9–14].

2.1. Hida test and generalized functions

The starting point of the white noise analysis is the real Gelfand triple

$$S_d \subset L^2_d \subset S'_d,$$

where $L^2_d := L^2(\mathbb{R}, \mathbb{R}^d)$, $d \geq 1$, is the real Hilbert space of all vector-valued square-integrable functions with respect to the Lebesgue measure on $\mathbb{R}$, and S_d and S'_d are the Schwartz space of vector-valued test functions and tempered distributions, respectively. We denote the L^2_d-norm by $\|\cdot\|_2$ and the dual pairing between S'_d and S_d by $\langle\cdot,\cdot\rangle$, which is a bilinear extension of the inner product on L^2_d, that is,

$$\langle \varphi, f \rangle = \sum_{i=1}^{d} \int_{\mathbb{R}} \varphi_i(x) f_i(x)\,\mathrm{d}x,$$

for all $f = (f_1, \dots, f_d) \in L^2_d$ and all $\varphi = (\varphi_1, \dots, \varphi_d) \in S_d$. By the Minlos theorem, there is a unique probability measure μ on the σ-algebra $\mathcal{B}$ generated by the cylinder sets on S'_d with characteristic function C given by

$$C(\varphi) := e^{-\frac{1}{2}\|\varphi\|_2^2} = \int_{S'_d} e^{i\langle \varphi, \omega\rangle}\,\mathrm{d}\mu(\omega), \quad \varphi \in S_d.$$

In this way, we have defined the white noise measure space $(S'_d, \mathcal{B}, \mu)$. Within this formalism, one can show that

$$\big(\langle \mathbb{1}_{[0,t)}, w_1\rangle, \dots, \langle \mathbb{1}_{[0,t)}, w_d\rangle\big), \quad w = (w_1, \dots, w_d) \in S'_d,\ t \geq 0,$$

has a continuous modification $B(t,w)$, which is a d-dimensional Brownian motion. Here, $\mathbb{1}_A$ denotes the indicator function of the Borel set $A \subset \mathbb{R}$ and $\langle \mathbb{1}_A, w_i\rangle$, $i = 1, \dots, d$ is defined as an $L^2(\mu)$-limit. For an arbitrary Hurst parameter $0 < H < 1$,

$$\big(\langle \eta_t, w_1\rangle, \dots, \langle \eta_t, w_d\rangle\big), \quad w = (w_1, \dots, w_d) \in S'_d,\ \eta_t := M_-^H \mathbb{1}_{[0,t)},$$

has a continuous modification $B_H(t,w)$ that is a d-dimensional fBm. On the one hand, for a real-valued function f and $\frac{1}{2} < H < 1$, the operator M_-^H is defined by

$$(M_-^H f)(x) := K_H \cdot (I_-^H f)(x) := \frac{K_H}{\Gamma\left(H-\frac{1}{2}\right)} \int_0^\infty f(x+t) t^{H-\frac{3}{2}}\, \mathrm{d}t, \tag{3}$$

provided that the integral exists for almost all $x \in \mathbb{R}$ and the normalization constant is given by

$$K_H := \Gamma\left(H+\frac{1}{2}\right)\left(\frac{1}{2H} + \int_0^\infty \left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right)^2 \mathrm{d}s\right)^{-\frac{1}{2}}.$$

On the other hand, for a real-valued function f and $0 < H < \frac{1}{2}$, the operator M_-^H has the form

$$(M_-^H f)(x) := K_H \cdot (D_-^H f)(x) := \frac{(\frac{1}{2}-H)K_H}{\Gamma\left(H+\frac{1}{2}\right)} \lim_{\epsilon \to 0^+} \int_\epsilon^\infty \frac{f(x)-f(x+y)}{y^{\frac{3}{2}-H}}\, \mathrm{d}y, \tag{4}$$

if the limit exists for almost all $x \in \mathbb{R}$. For $H = 1/2$, $M_-^{1/2}$ is the identity operator. For more details, see, e.g., [5,15], and the references therein.

To introduce the corresponding fractional noise W_H, we first need to define the operator M_+^H, which is dual to the operator M_-^H. Therefore, for $\frac{1}{2} < H < 1$ we define

$$(M_+^H f)(x) := K_H \cdot (I_+^H f)(x) := K_H \cdot (f * g_H)(x),$$

where $g_H(t) := \frac{1}{\Gamma(H-1/2)} t^{H-3/2}$, $t > 0$, whenever the convolution integral exists for almost all $x \in \mathbb{R}$. For $0 < H < \frac{1}{2}$ the operator M_+^H is defined by

$$(M_+^H f)(x) := K_H \cdot (D_+^H f)(x) := \frac{(\frac{1}{2}-H)K_H}{\Gamma\left(H+\frac{1}{2}\right)} \lim_{\epsilon \to 0^+} \int_\epsilon^\infty \frac{f(x)-f(x-y)}{y^{\frac{3}{2}-H}}\, \mathrm{d}y,$$

if the limit exists for almost all $x \in \mathbb{R}$.

There are several examples of functions f for which $M_\pm^H f$ exists for any $H \in (0,1)$. For example, $f = \mathbb{1}_{[0,t)}$, $t \geq 0$, or $f \in S_1(\mathbb{R})$. For functions f_1, f_2 being either of these two types, it is easy to prove the following equality

$$\int_{\mathbb{R}} f_1(x)(M_-^H f_2)(x)\, \mathrm{d}x = \int_{\mathbb{R}} (M_+^H f_1)(x) f_2(x)\, \mathrm{d}x,$$

showing that M_-^H and M_+^H are dual operators, cf. Eq. (12) in [5].

The corresponding d-dimensional fractional noise $W_H(t)$ in the sense of Hida distributions is given by

$$W_H(t) := (W_{H,1}(t), \dots, W_{H,d}(t)) := (\langle M_-^H \delta_t, P_1\rangle, \dots, \langle M_-^H \delta_t, P_d\rangle), \tag{5}$$

where $P_i : S_d' \to S_1'$, $i = 1, \dots, d$, denotes the projection on the ith component, see Definition 2.18 in [5] for $d = 1$. The generalized function $M_-^H \delta_t \in S_1'$ is defined for any test function $\varphi \in S_1$ by

$$\langle \varphi, M_-^H \delta_t\rangle := (M_+^H \varphi)(t). \tag{6}$$

For $H = 1/2$, the operator $M_+^{1/2}$ is defined as the identity, and $W_{1/2}(t) = (\langle \delta_t, P_1\rangle, \dots, \langle \delta_t, P_d\rangle)$ coincides with the vector-valued white noise.

Let us now consider the complex Hilbert space $L^2(\mu) := L^2(S_d', \mathcal{B}, \mu; \mathbb{C})$. This space is canonically isomorphic to the symmetric Fock space of symmetric square-integrable functions,

$$L^2(\mu) \simeq \left(\bigoplus_{k=0}^\infty \mathrm{Sym}\, L^2(\mathbb{R}^k, k!\mathrm{d}^k x)\right)^{\otimes d},$$

leading to the chaos expansion of the elements in $L^2(\mu)$,

$$F(w_1, \dots, w_d) = \sum_{(n_1,\dots,n_d) \in \mathbb{N}_0^d} \langle F_{(n_1,\dots,n_d)}, :w_1^{\otimes n_1}: \otimes \cdots \otimes :w_d^{\otimes n_d}:\rangle, \tag{7}$$

with kernel functions $F_{(n_1,\dots,n_d)} \in (L^2_{d,\mathbb{C}})^{\hat{\otimes} n}$ and $w = (w_1, \dots, w_d) \in S_d'$. For simplicity, we use the notation

$$\boldsymbol{n} = (n_1, \cdots, n_d) \in \mathbb{N}_0^d, \quad n = \sum_{i=1}^d n_i, \quad \boldsymbol{n}! = \prod_{i=1}^d n_i!,$$

$$:w^{\otimes \boldsymbol{n}}: := :w_1^{\otimes n_1}: \otimes \cdots \otimes :w_d^{\otimes n_d}:,$$

that reduces the chaos expansion in (7) to

$$F(w) = \sum_{\boldsymbol{n} \in \mathbb{N}_0^d} \langle F_{\boldsymbol{n}}, :w^{\otimes \boldsymbol{n}}:\rangle, \quad w \in S_d'.$$

To proceed further, we have to consider a Gelfand triple around the space $L^2(\mu)$. We use the space $(S_d)'$ of Hida distributions and the corresponding Gelfand triple

$$(S_d) \subset L^2(\mu) \subset (S_d)'. \tag{8}$$

Here (S_d) is the space of the white noise test functions such that its dual space (with respect to $L^2(\mu)$) is the space $(S_d)'$. Instead of reproducing the explicit construction of $\left(S_d\right)'$ (see e.g., [11]), we characterize this space by its S-transform in Theorem 2.3. We recall that given a $\varphi \in S_d$, the Wick exponential is defined by

$$: \exp(\langle \varphi, w\rangle) : := \sum_{\boldsymbol{n}\in\mathbb{N}_0^d} \frac{1}{\boldsymbol{n}!}\langle \varphi^{\otimes \boldsymbol{n}}, : w^{\otimes \boldsymbol{n}} :\rangle = C(\varphi)e^{\langle \varphi, w\rangle},$$

where $\varphi^{\otimes \boldsymbol{n}} := \varphi_1^{\otimes n_1} \otimes \cdots \otimes \varphi_d^{\otimes n_d}$ and the S-transform of $\Phi \in (S_d)'$ is defined by

$$S\Phi(\varphi) := \langle\!\langle : \exp(\langle \varphi, \cdot\rangle) :, \Phi\rangle\!\rangle, \quad \varphi \in S_d. \tag{9}$$

Here $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes the dual pairing between $\left(S_d\right)$ and $\left(S_d\right)'$, which is a bilinear extension of $(\!(\cdot, \bar{\cdot})\!)$, where $(\!(\cdot, \cdot)\!)$ is the sesquilinear inner product on $L^2(\mu)$. We observe that the multilinear expansion of (9),

$$S\Phi(\varphi) := \sum_{\boldsymbol{n}\in\mathbb{N}_0^d} \langle \varphi^{\otimes \boldsymbol{n}}, \Phi_{\boldsymbol{n}}\rangle,$$

extends the chaos expansion to $\Phi \in \left(S_d\right)'$ with distribution valued kernels $\Phi_{\boldsymbol{n}} \in (S_d')^{\otimes \boldsymbol{n}}$ such that

$$\langle\!\langle \varphi, \Phi\rangle\!\rangle = \sum_{\boldsymbol{n}\in\mathbb{N}_0^d} \boldsymbol{n}!\langle \varphi_{\boldsymbol{n}}, \Phi_{\boldsymbol{n}}\rangle,$$

for every test function $\varphi \in (S_d)$ with kernel functions $\varphi_{\boldsymbol{n}} \in S_d^{\otimes \boldsymbol{n}}$. This allows us to represent Φ by its generalized chaos expansion

$$\Phi = \sum_{\boldsymbol{n}\in\mathbb{N}_0^d} I_{\boldsymbol{n}}(\Phi_{\boldsymbol{n}}), \quad \Phi_{\boldsymbol{n}} \in (S_d')^{\otimes \boldsymbol{n}},$$

where

$$\langle\!\langle \varphi, I_{\boldsymbol{n}}(\Phi_{\boldsymbol{n}})\rangle\!\rangle := \boldsymbol{n}!\langle \varphi_{\boldsymbol{n}}, \Phi_{\boldsymbol{n}}\rangle, \quad \varphi \in (S_d).$$

Example 2.1. Let $d = 1$ and $W_H(t)$ be the fractional noise introduced in (5). Then its S-transform is given by (cf. [5])

$$SW_H(t)(\varphi) = (M_+^H \varphi)(t), \quad \varphi \in S_1.$$

To characterize the space $(S_d)'$ through its S-transform, we first need the following definition.

Definition 2.2 (U-functional). A function $F : S_d \to \mathbb{C}$ is called a U-functional whenever

1. for every $\varphi_1, \varphi_2 \in S_d$ the mapping $\mathbb{R} \ni \lambda \longmapsto F(\lambda\varphi_1 + \varphi_2)$ has an entire extension to $\lambda \in \mathbb{C}$,
2. there are constants $K_1, K_2 < \infty$ such that

$$|F(z\varphi)| \le K_1 \exp\left(K_2|z|^2\|\varphi\|^2\right), \quad z \in \mathbb{C}, \varphi \in S_d$$

for some continuous norm $\|\cdot\|$ on S_d.

We are now ready to state the characterization theorem mentioned above.

Theorem 2.3 (cf. [16,17]). *The S-transform defines a bijection between the space $(S_d)'$ and the space of the U-functionals.*

As a consequence of Theorem 2.3, one may derive the following statement, which concerns the Bochner integration of a family of the same type of distributions. For more details and proofs; see, e.g., [11,16,17].

Corollary 2.4. *Let $(\Omega, \mathcal{F}, m)$ be a measure space and $\lambda \mapsto \Phi_\lambda$ be a mapping from Ω to $(S_d)'$. We assume that the S-transform of Φ_λ fulfills the following two properties:*

1. *The mapping $\lambda \mapsto S\Phi_\lambda(\varphi)$ is measurable for every $\varphi \in S_d$.*
2. *The function $S\Phi_\lambda$ obeys the estimate*

$$|S\Phi_\lambda(z\varphi)| \le C_1(\lambda)e^{C_2(\lambda)|z|^2\|\varphi\|^2}, \quad z \in \mathbb{C}, \varphi \in S_d,$$

for some continuous norm $\|\cdot\|$ on S_d and $C_1 \in L^1(\Omega, m)$, $C_2 \in L^\infty(\Omega, m)$.

Then

$$\int_\Omega \Phi_\lambda \, \mathrm{d}m(\lambda) \in (S_d)',$$

and

$$S\left(\int_\Omega \Phi_\lambda \, \mathrm{d}m(\lambda)\right)(\varphi) = \int_\Omega S\Phi_\lambda(\varphi) \, \mathrm{d}m(\lambda), \qquad \varphi \in S_d.$$

Example 2.5 (Donsker's delta function)**.** As a typical example of a Hida distribution, we have the Donsker delta function needed later. More precisely, the following Bochner integral is a well-defined element in $(S_d)'$:

$$\Phi_{x,H} := \delta(x - B_H(t)) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(\lambda, x - B_H(t))_{\mathbb{R}^d}} \, \mathrm{d}\lambda, \quad x \in \mathbb{R}^d.$$

In fact, the S-transform of $\Phi_{x,H}$ for any $z \in \mathbb{C}$ and $\varphi \in S_d$ is given by

$$S\Phi_{x,H}(z\varphi) = \frac{1}{(2\pi t^{2H})^{d/2}} \exp\left(-\frac{1}{2t^{2H}} \sum_{j=1}^{d} (x_j - \langle z\varphi_j, \eta_t \rangle)^2 \right). \tag{10}$$

It is well known that the Wick product is a well-defined operation in Gaussian analysis; see, for example, [12,18], and [19].

Definition 2.6. For any $\Phi, \Psi \in (S_d)'$, the Wick product $\Phi \diamondsuit \Psi$ is defined by

$$S(\Phi \diamondsuit \Psi) = S\Phi \cdot S\Psi. \tag{11}$$

Since the space of U-functionals is an algebra, by Theorem 2.3 there exists a unique element $\Phi \diamondsuit \Psi \in (S_d)'$ such that (11) holds.

2.2. Characterization of regular generalized functions

In this subsection, we introduce the space of regular generalized functions, which is used in Section 4. We restrict the construction to $d = 1$ and adapt the notation from Section 2.1.

Consider the Wiener-Itô chaos decomposition of $F \in L^2(\mu)$

$$F = \sum_{n=0}^{\infty} \left\langle f^{(n)}, :\cdot^{\otimes n}: \right\rangle, \quad f^{(n)} \in (L^2_{1,\mathbb{C}})^{\hat{\otimes} n},$$

with

$$\|F\|^2_{L^2(\mu)} = \sum_{n=0}^{\infty} n! |f^{(n)}|^2_{(L^2_{1,\mathbb{C}})^{\otimes n}}, \tag{12}$$

where $:\omega^{\otimes n}: \in (S'_{1,\mathbb{C}})^{\hat{\otimes} n}$ denotes the Wick power of nth order of $\omega \in S'_{1,\mathbb{C}}$ and $\langle \cdot, \cdot \rangle$ denotes the bilinear dual pairing on $S_{\mathbb{C}}(\mathbb{R})^{\otimes n} \times S'_{\mathbb{C}}(\mathbb{R})^{\otimes n}$. We remark that $\langle \cdot, :\cdot^{\otimes n}: \rangle$ extends in the first variable to $(L^2_{1,\mathbb{C}})^{\hat{\otimes} n}$ in the sense of an $L^2(\mu)$-limit.

Now, let $s \in [0, \infty)$ be given. If we require

$$\|F\|^2_{\mathcal{G}_s} := \sum_{n=0}^{\infty} n! 2^{ns} |f^{(n)}|^2_{(L^2_{1,\mathbb{C}})^{\otimes n}}$$

to be finite, in addition to (12), we obtain

$$\mathcal{G}_s := \left\{ F = \sum_{n=0}^{\infty} \left\langle f^{(n)}, :\cdot^{\otimes n}: \right\rangle \in L^2(\mu) \mid f^{(n)} \in (L^2_{1,\mathbb{C}})^{\hat{\otimes} n}, \|F\|_{\mathcal{G}_s} < \infty \right\}.$$

Thus, $(\mathcal{G}_s)_{s \in [0,\infty)}$ is a nested family of Hilbert subspaces of $L^2(\mu)$ with $\mathcal{G}_0 = L^2(\mu)$. We write $\mathcal{G} := \bigcap_{s \in [0,\infty)} \mathcal{G}_s$ for the projective limit of this family. This yields the topological dual space $\mathcal{G}' = \bigcup_{s \in [0,\infty)} \mathcal{G}_{-s}$, where $\mathcal{G}_{-s} := \mathcal{G}'_s$. The elements of $\mathcal{G}'$ are referred to as regular generalized functions because the partial sums within their chaos decomposition are square integrable functions.

Note that the construction of $\mathcal{G}$ is almost identical to that of (S_1). The difference lies, in addition to the slightly weaker summability assumption, in the fact that we do not require the kernels $f^{(n)}$ to possess additional regularity. Hence, we have the following refinement of the Gelfand triple (8) with $d = 1$

$$(S_1) \subset \mathcal{G} \subset \mathcal{G}_s \subset L^2(\mu) \subset \mathcal{G}_{-s} \subset \mathcal{G}' \subset (S_1)', \qquad s \in [0, \infty). \tag{13}$$

Example 2.7. Let $t \geq 0$ be fixed and $H = \frac{1}{2}$. Then the Brownian motion $B_{\frac{1}{2}}(t) = \langle \mathbb{1}_{[0,t)}, \cdot \rangle$ and white noise $W_{\frac{1}{2}}(t) = \langle \delta_t, \cdot \rangle$ described above are such that $B_{\frac{1}{2}}(t) \in \mathcal{G} \setminus (S_1)$ and $W_{\frac{1}{2}}(t) \in (S_1)' \setminus \mathcal{G}'$.

In the following, we provide the characterization of regular generalized functions, see [20] for details. More precisely, given a $\Phi \in (S_1)'$, we have a necessary and sufficient condition to check whether $\Phi \in \mathcal{G}_s$ for some $s \in \mathbb{Z}$. Before we state this characterization theorem (cf. Theorem 2.10 below), we need some preparations. Recall the Gaussian measure μ_{σ^2} on $(S'_1, \mathcal{B})$ with covariance $\sigma^2 > 0$ is given by its characteristic functional

$$C_{\sigma^2}(\varphi) := \exp\left(-\frac{\sigma^2}{2} \|\varphi\|^2_2 \right), \quad \varphi \in S_1. \tag{14}$$

Definition 2.8. We identify $S'_{1,\mathbb{C}}$ with $S'_1 \times S'_1$ and define the product measure $\nu := \mu_{\frac{1}{2}} \otimes \mu_{\frac{1}{2}}$ on $(S'_{1,\mathbb{C}}, \mathcal{B} \otimes \mathcal{B})$.

To characterize elements $\Phi \in \mathcal{G}_s$, it suffices to consider their S-transform $S\Phi$, a function defined on S_1. Since it is also a U-functional by Theorem 2.3, there is a unique extension of $S\Phi$ to an entire function on $S_{1,\mathbb{C}}$, see Lemma 10 in [17].

Starting with an element $z \in S'_{1,\mathbb{C}}$, we want to project it into $S_{1,\mathbb{C}}$ to which we may apply $S\Phi$.

Definition 2.9. Let $n \in \mathbb{N}$ be given and $(\varphi_i)_{i=1}^n \subset S_1$ be orthonormal in L^2_1. We call

$$P : S'_{1,\mathbb{C}} \longrightarrow S_{1,\mathbb{C}},\ z \mapsto Pz := \sum_{i=1}^n \langle \varphi_i, z\rangle\, \varphi_i,$$

an *orthogonal projection* from $S'_{1,\mathbb{C}}$ into $S_{1,\mathbb{C}}$. Here, $\langle\cdot,\cdot\rangle$ denotes the dual pairing on $S_{1,\mathbb{C}} \times S'_{1,\mathbb{C}}$ which is a bilinear extension of $(\cdot,\bar{\cdot})_{L^2_{1,\mathbb{C}}}$. The set of all orthogonal projections from $S'_{1,\mathbb{C}}$ into $S_{1,\mathbb{C}}$ is denoted by $\mathbb{P}$.

Like orthogonal projections in the classical sense, any $P \in \mathbb{P}$ has the following property

$$\langle Pz, z'\rangle = \left\langle \sum_{i=1}^n \langle\varphi_i, z\rangle\varphi_i, z' \right\rangle = \sum_{i=1}^n \langle\varphi_i, z\rangle\langle\varphi_i, z'\rangle = \langle Pz', z\rangle, \tag{15}$$

for every $z, z' \in S'_{1,\mathbb{C}}$. We are now able to state the characterization theorem and refer to Theorem 2.11 in [20] for the proof and more details.

Theorem 2.10. *Let $\Phi \in (S_1)'$ be given. Then for $s \in \mathbb{R}$ it holds*

$$\Phi \in \mathcal{G}_s \iff \sup_{P\in\mathbb{P}} \int_{S'_{1,\mathbb{C}}} \left| S\Phi\left(2^{\frac{s}{2}} Pz\right)\right|^2 \mathrm{d}\nu(z) < \infty.$$

In particular, the spaces $\mathcal{G}$ and $\mathcal{G}'$ are characterized by

(i) $\Phi \in \mathcal{G} \iff \forall \lambda > 0,\ \sup_{P\in\mathbb{P}} \int_{S'_{1,\mathbb{C}}} |S\Phi(\lambda Pz)|^2 \mathrm{d}\nu(z) < \infty$,

(ii) $\Phi \in \mathcal{G}' \iff \exists \varepsilon > 0,\ \sup_{P\in\mathbb{P}} \int_{S'_{1,\mathbb{C}}} |S\Phi(\varepsilon Pz)|^2 \mathrm{d}\nu(z) < \infty$.

3. Stochastic current of fractional Brownian motion

As motivated in the introduction, using white noise analysis, we investigate for $x \in \mathbb{R}^d$ the following (generalized) function

$$\begin{aligned}\xi(x) &:= \int_0^T \delta(x - B_H(t))\,\mathrm{d}B_H(t)\\ &:= \left(\int_0^T \delta(x - B_H(t))\diamondsuit W_{H,1}(t)\,\mathrm{d}t, \dots, \int_0^T \delta(x - B_H(t))\diamondsuit W_{H,d}(t)\,\mathrm{d}t\right)\\ &=: \big(\xi_1(x), \dots, \xi_d(x)\big),\end{aligned}$$

where $W_H := (W_{H,1}, \dots, W_{H,d})$ is the vector-valued fractional noise defined in (5). The above stochastic integral was introduced in [5, Equation (26)] and is called the fractional Itô integral. If $H = 1/2$ and the integrand is an adapted, square integrable function, then this stochastic integral coincides with the classical Itô integral.

From this point onward, C will denote a positive finite constant whose value may change from line to line.

Theorem 3.1. *For $x \in \mathbb{R}^d\backslash\{0\}$, $0 < T < \infty$, $H \in (0,1)$, $d \geq 1$, and for each $i = 1, \dots, d$, the Bochner integral*

$$\xi_i(x) = \int_0^T \delta(x - B_H(t))\diamondsuit W_{H,i}(t)\,\mathrm{d}t \tag{16}$$

is a Hida distribution, and its S-transform is given, for any $\varphi \in S_d$, by

$$S\big(\xi_i(x)\big)(\varphi) = \frac{1}{(2\pi)^{d/2}} \int_0^T \frac{1}{t^{Hd}} \mathrm{e}^{-\sum_{j=1}^d \frac{(x_j - \langle\varphi_j, \eta_t\rangle)^2}{2t^{2H}}} (M^H_+\varphi_i)(t)\,\mathrm{d}t. \tag{17}$$

Proof. First, we compute the S-transform of the integrand Φ_t, $t \in (0,T]$, in (16), that is,

$$\Phi_t := \delta(x - B_H(t))\diamondsuit W_{H,i}(t).$$

It follows from Definition 2.6, Examples 2.1 and 2.5 that, for any $\varphi \in S_d$, we obtain

$$\begin{aligned}S\Phi_t(\varphi) &= S\big(\delta(x - B_H(t))\big)(\varphi) S\big(W_{H,i}(t)\big)(\varphi)\\ &= \frac{1}{(2\pi t^{2H})^{d/2}} \exp\left(-\frac{1}{2t^{2H}} \sum_{j=1}^d (x_j - \langle\varphi_j, \eta_t\rangle)^2\right)(M^H_+\varphi_i)(t).\end{aligned} \tag{18}$$

It is clear that $(0,T] \ni t \mapsto S\Phi_t(\varphi) \in \mathbb{C}$ is Borel measurable for every $\varphi \in S_d$. Furthermore, for any $z \in \mathbb{C}$ and all $\varphi \in S_d$, we estimate $|S\Phi_t(z\varphi)|$ as follows

$$\begin{aligned}&|S\Phi_t(z\varphi)|\\ &\leq \frac{1}{(2\pi t^{2H})^{d/2}} \exp\left(-\frac{1}{2t^{2H}}|x|^2_{\mathbb{R}^d}\right) \exp\left(\frac{1}{t^{2H}} \sum_{j=1}^d |x_j||z||\langle\varphi_j, \eta_t\rangle|\right)\end{aligned}$$

$$\times \exp\left(\frac{1}{2t^{2H}}|z|^2\sum_{j=1}^{d}|\langle\varphi_j,\eta_t\rangle|^2\right)|z|\left|M_+^H\varphi_i(t)\right|$$
$$\leq \frac{1}{(2\pi t^{2H})^{d/2}}\exp\left(-\frac{1}{2t^{2H}}|x|^2_{\mathbb{R}^d}\right)\exp\left(\frac{1}{t^{2H}}\sum_{j=1}^{d}\left(\frac{1}{4}|x_j|^2+|z|^2|\langle\varphi_j,\eta_t\rangle|^2\right)\right)$$
$$\times \exp\left(\frac{1}{2t^{2H}}|z|^2t^{2H}\|\varphi\|_2^2\right)|z|\|M_+^H\varphi\|_\infty$$
$$\leq \frac{1}{(2\pi t^{2H})^{d/2}}\exp\left(-\frac{1}{4t^{2H}}|x|^2_{\mathbb{R}^d}\right)\exp\left(C|z|^2\|\varphi\|_2^2\right)\exp\left(|z|^2\|M_+^H\varphi\|_\infty^2\right)$$
$$\leq \frac{1}{(2\pi t^{2H})^{d/2}}\exp\left(-\frac{1}{4t^{2H}}|x|^2_{\mathbb{R}^d}\right)\exp\left(C|z|^2\|\varphi\|^2\right)$$

for some continuous norm $\|\cdot\|$ on S_d. The third inequality is a consequence of

$$|\langle\varphi,\eta_t\rangle|^2 \leq \|\varphi\|_2^2\|M_-^H\mathbb{1}_{[0,t)}\|_2^2 = t^{2H}\|\varphi\|_2^2.$$

The last inequality follows from $\|M_+^H\varphi\|_\infty^2 \leq \|\varphi\|^2$, which is due to Theorem 2.3 in [5]. By using a gamma function argument, the function $(0,T]\ni t\mapsto \frac{1}{(2\pi t^{2H})^{d/2}}\exp\left(-\frac{1}{4t^{2H}}|x|^2_{\mathbb{R}^d}\right)$ is integrable with respect to the Lebesgue measure on $[0,T]$.

As the second factor $\exp\left(C|z|^2\|\varphi\|^2\right)$ is independent of $t\in[0,T]$, this shows that the conditions of Corollary 2.4 are satisfied and

$$\int_0^T \delta(x-B_H(t))\diamondsuit W_{H,i}(t)\,\mathrm{d}t \in (S_d)'.$$

□

Analyzing the proof of Theorem 3.1, we see that it is also possible to include $x=0\in\mathbb{R}^d$.

Corollary 3.2. *For all $dH<1$, $d\in\mathbb{N}$, and $0<H<1$ we have $\xi(0)\in(S_d)'$.*

Remark 3.3. To cover the case $H\in[1/d,1)$, we may truncate $\xi(0)$ and use techniques similar to those in the analysis of local times for fBm; see, for example, [21,22] and other sources cited therein.

4. Regularity of stochastic currents of fractional Brownian motion

Consider the S-transform of the stochastic current of 1-dimensional fBm with Hurst parameter $H\in(0,1)$:

$$S(\xi(x))(\varphi)=\frac{1}{(2\pi)^{1/2}}\int_0^T\frac{1}{t^H}\mathrm{e}^{-\frac{(x-\langle\varphi,\eta_t\rangle)^2}{2t^{2H}}}(M_+^H\varphi)(t)\,\mathrm{d}t,\qquad \varphi\in S_1. \tag{19}$$

To show the integrability condition of Theorem 2.10 of the S-transform (19), a more suitable representation of $S(\xi(x))$ is needed. Point evaluations are too singular. The test function should appear only in integrated form. The new idea for obtaining such a representation can be found in the following key lemma:

Lemma 4.1. *For every $x\in\mathbb{R}$, $H\in(0,1)$, the S-transform of the stochastic current $\xi(x)$ of fBm has a representation given by*

$$\begin{aligned}S(\xi(x))(\varphi)&=\frac{H}{\sqrt{2\pi}}\int_0^T\int_0^1\frac{1}{t^{H+1}}\mathrm{e}^{-\frac{(x-\tau\langle\varphi,\eta_t\rangle)^2}{2t^{2H}}}\langle\varphi,\eta_t\rangle\,\mathrm{d}\tau\,\mathrm{d}t\\&+\frac{1}{\sqrt{2\pi}T^H}\int_0^1\mathrm{e}^{-\frac{(x-\tau\langle\varphi,\eta_T\rangle)^2}{2T^{2H}}}\langle\varphi,\eta_T\rangle\,\mathrm{d}\tau\\&-\frac{H}{\sqrt{2\pi}}\int_0^T\int_0^1\frac{(x-\tau\langle\varphi,\eta_t\rangle)^2}{t^{3H+1}}\mathrm{e}^{-\frac{(x-\tau\langle\varphi,\eta_t\rangle)^2}{2t^{2H}}}\langle\varphi,\eta_t\rangle\,\mathrm{d}\tau\,\mathrm{d}t,\end{aligned} \tag{20}$$

for any $\varphi\in S_1$.

Proof. *Step 1*: For any test function $\varphi\in S_1$ and every $x\in\mathbb{R}$ we define the maps

$$f:(0,\infty)\times\mathbb{R}\longrightarrow\mathbb{R},\ (t,u)\mapsto f(t,u):=\int_0^u\exp\left(-\frac{(x-v)^2}{2t^{2H}}\right)\mathrm{d}v,$$
$$F:(0,\infty)\longrightarrow\mathbb{R},\ t\mapsto F(t):=f(t,\psi(t)),\quad \psi(t):=\langle\varphi,\eta_t\rangle.$$

By Corollary 2.8 in [5], $\psi:(0,\infty)\longrightarrow\mathbb{R}$ is differentiable, and its derivative $t\mapsto\psi'(t)=(M_+^H\varphi)(t)$ is a continuous function. It follows that F is differentiable, and its derivative is given by

$$\begin{aligned}F'(t)&=\frac{\partial f}{\partial t}(t,\psi(t))+\frac{\partial f}{\partial u}(t,\psi(t))\frac{\partial\psi}{\partial t}(t)\\&=H\int_0^{\psi(t)}\frac{(x-v)^2}{t^{2H+1}}\exp\left(-\frac{(x-v)^2}{2t^{2H}}\right)\mathrm{d}v+\exp\left(-\frac{(x-\psi(t))^2}{2t^{2H}}\right)(M_+^H\varphi)(t).\end{aligned} \tag{21}$$

For every $t > 0$, the change of variable $\psi(t)\tau = v$ in the integral above yields

$$\begin{aligned}&\int_0^{\psi(t)} \frac{(x-v)^2}{t^{2H+1}} \exp\left(-\frac{(x-v)^2}{2t^{2H}}\right) \mathrm{d}v \\ &= \int_0^1 \frac{(x-\tau\psi(t))^2}{t^{2H+1}} \exp\left(-\frac{(x-\tau\psi(t))^2}{2t^{2H}}\right) \psi(t)\, \mathrm{d}\tau. \end{aligned} \tag{22}$$

It follows from (21) and (22) that

$$\begin{aligned}\frac{F'(t)}{\sqrt{2\pi}t^H} = \frac{H}{\sqrt{2\pi}} \frac{1}{t^H} \int_0^1 \frac{(x-\tau\psi(t))^2}{t^{2H+1}} \exp\left(-\frac{(x-\tau\psi(t))^2}{2t^{2H}}\right) \psi(t) \mathrm{d}\tau \\ + \frac{1}{\sqrt{2\pi}t^H} \exp\left(-\frac{(x-\psi(t))^2}{2t^{2H}}\right) (M_+^H \varphi)(t). \end{aligned} \tag{23}$$

Step 2: The function

$$(0,T] \ni t \mapsto \int_0^1 \frac{(x-\tau\psi(t))^2}{t^{2H+1}} \exp\left(-\frac{(x-\tau\psi(t))^2}{2t^{2H}}\right) \psi(t)\, \mathrm{d}\tau \tag{24}$$

is $\mathrm{d}t$-integrable on $[0,T]$. The details on the integrability of the function in (24) are given in Lemma A.1, below. If we integrate both sides of (23) from $t = 0$ to $t = T$ we obtain

$$\begin{aligned}S(\xi(x))(\varphi) &= \frac{1}{\sqrt{2\pi}} \int_0^T \frac{F'(t)}{t^H}\, \mathrm{d}t \\ &- \frac{H}{\sqrt{2\pi}} \int_0^T \int_0^1 \frac{(x-\tau\psi(t))^2}{t^{3H+1}} \exp\left(-\frac{(x-\tau\psi(t))^2}{2t^{2H}}\right) \psi(t)\, \mathrm{d}\tau\, \mathrm{d}t. \end{aligned} \tag{25}$$

Step 3: An integration by parts shows that

$$\begin{aligned}\int_0^T \frac{F'(t)}{t^H}\, \mathrm{d}t = \frac{1}{T^H} \int_0^1 \exp\left(-\frac{(x-\tau\psi(T))^2}{2T^{2H}}\right) \psi(T)\, \mathrm{d}\tau \\ + H \int_0^T \frac{1}{t^{H+1}} \int_0^1 \exp\left(-\frac{(x-\tau\psi(t))^2}{2t^{2H}}\right) \psi(t)\, \mathrm{d}\tau\, \mathrm{d}t. \end{aligned} \tag{26}$$

We refer to Lemma A.2, below, for details on the integration by parts in Eq. (26). The statement of Lemma 4.1 now follows from (25) combined with *Step 3*. □

Proposition 4.2. *For any $x \in \mathbb{R}$ and $H \in (0,1)$, the expression of the S-transform of $\xi(x)$ given in (20) is a U-functional.*

Proof. The proof is given in Appendix B. □

As a consequence of the characterization theorem, we now immediately obtain the following corollary.

Corollary 4.3. *For any $x \in \mathbb{R}$ and $H \in (0,1)$, the expression of the S-transform of $\xi(x)$ given in (20) extends uniquely to an entire function on $S_{1,\mathbb{C}}$; see [17].*

We are now ready to present the regularity result.

Theorem 4.4. *Let $x \neq 0$ and $H \in (0,1)$ be given. Then the stochastic current $\xi(x)$ of 1-dimensional fBm is an element of $\mathcal{G}_{-s}$ for all $s > 0$.*

Proof. We have to show the condition of Theorem 2.10, that is,

$$\sup_{P \in \mathbb{P}} \int_{S'_{1,\mathbb{C}}} |S(\xi(x))(\varepsilon P z)|^2\, \mathrm{d}\nu(z) < \infty, \tag{27}$$

for all $0 < \varepsilon < 1$. To this end, we use the representation of $S(\xi(x))$ on $S_{1,\mathbb{C}}$ provided by Corollary 4.3 and show the ν-integrability.

Let $0 < \varepsilon < 1$ be fixed and $P \in \mathbb{P}$ be given. It follows from (20) and the property (15) that for every $z \in S'_{1,\mathbb{C}}$

$$\begin{aligned}&S(\xi(x))(\varepsilon P z) \\ &= \frac{H}{(2\pi)^{1/2}} \int_0^T \int_0^1 \frac{1}{t^{H+1}} \exp\left(-\frac{(x-\tau\varepsilon\langle P\eta_t, z\rangle)^2}{2t^{2H}}\right) \varepsilon\langle P\eta_t, z\rangle\, \mathrm{d}\tau\, \mathrm{d}t \\ &+ \frac{1}{\sqrt{2\pi}T^H} \int_0^1 \exp\left(-\frac{(x-\tau\varepsilon\langle P\eta_T, z\rangle)^2}{2T^{2H}}\right) \varepsilon\langle P\eta_T, z\rangle\, \mathrm{d}\tau \\ &- \frac{H}{\sqrt{2\pi}} \int_0^T \int_0^1 \frac{(x-\tau\varepsilon\langle P\eta_t, z\rangle)^2}{t^{3H+1}} \exp\left(-\frac{(x-\tau\varepsilon\langle P\eta_t, z\rangle)^2}{2t^{2H}}\right) \varepsilon\langle P\eta_t, z\rangle\, \mathrm{d}\tau\, \mathrm{d}t. \end{aligned} \tag{28}$$

We show the ν-integrability of the last summand in (28). All the other summands are particular cases, and the same arguments apply. Define

$$G(x,\varphi,t,\tau) := \big(x - \tau\langle\varphi,\eta_t\rangle\big)^2 \exp\left(-\frac{\big(x-\tau\langle\varphi,\eta_t\rangle\big)^2}{2t^{2H}}\right) \langle\varphi,\eta_t\rangle, \tag{29}$$

where $\varphi \in S_{1,\mathbb{C}}$, $t \in (0,T]$, and $\tau \in [0,1]$. The following estimate follows from Jensen's inequality

$$\begin{aligned}
&\int_{S'_{1,\mathbb{C}}} \left| \int_0^T \int_0^1 \frac{1}{t^{3H+1}} G(x, \varepsilon P z, t, \tau)\, \mathrm{d}\tau\, \mathrm{d}t \right|^2 \mathrm{d}\nu(z) \\
&\leq T \int_{S'_{1,\mathbb{C}}} \int_0^T \left| \int_0^1 \frac{1}{t^{3H+1}} G(x, \varepsilon P z, t, \tau)\, d\tau \right|^2 \mathrm{d}t\, \mathrm{d}\nu(z) \\
&= T \int_{S'_{1,\mathbb{C}}} \int_0^T \int_0^1 \int_0^1 \frac{1}{t^{6H+2}} G(x, \varepsilon P z, t, \tau_1) \overline{G(x, \varepsilon P z, t, \tau_2)}\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2\, \mathrm{d}t\, \mathrm{d}\nu(z).
\end{aligned} \tag{30}$$

From the definition of the function G, it follows that

$$S'_{1,\mathbb{C}} \times (0,T] \times [0,1] \ni (z,t,\tau) \mapsto G(x, \varepsilon P z, t, \tau) \in \mathbb{C}$$

is measurable. Apply Fubini's theorem, and we have to evaluate the ν-integral

$$\int_{S'_{1,\mathbb{C}}} G(x, \varepsilon P z, t, \tau_1) \overline{G(x, \varepsilon P z, t, \tau_2)}\, \mathrm{d}\nu(z) \tag{31}$$

for fixed $t \in (0,T]$, $\tau_1, \tau_2 \in [0,1]$. Our aim is to apply Lemma C.1 to compute the integral in (31). Hence, we split the integrand into an exponential factor and a polynomial factor.

Let $t \in (0,T]$ be such that $P\eta_t \neq 0$, otherwise the integral in (31) is zero. For every $z \in S'_{1,\mathbb{C}}$ the exponential part of the integrand $G(x, \varepsilon P z, t, \tau_1) \cdot \overline{G(x, \varepsilon P z, t, \tau_2)}$ is given by

$$\begin{aligned}
&\exp\left(-\frac{\big(x - \tau_1 \varepsilon \langle P\eta_t, z\rangle\big)^2}{2t^{2H}} \right) \exp\left(-\frac{\big(x - \tau_2 \varepsilon \langle P\eta_t, \overline{z}\rangle\big)^2}{2t^{2H}} \right) \\
&= \exp\left(-\frac{x^2}{t^{2H}} \right) \exp\left(\frac{x(\tau_1+\tau_2)\varepsilon \langle P\eta_t, \Re z\rangle}{t^{2H}} + \frac{ix(\tau_1-\tau_2)\varepsilon \langle P\eta_t, \Im z\rangle}{t^{2H}} \right) \\
&\quad \times \exp\left(-\frac{\tau_1^2 \varepsilon^2 \langle P\eta_t, z\rangle^2}{2t^{2H}} - \frac{\tau_2^2 \varepsilon^2 \langle P\eta_t, \overline{z}\rangle^2}{2t^{2H}} \right).
\end{aligned}$$

The polynomial part of $G(x, \varepsilon P z, t, \tau_1)\overline{G(x, \varepsilon P z, t, \tau_2)}$ gives

$$\begin{aligned}
&\big(x - \tau_1 \varepsilon \langle P\eta_t, z\rangle\big)^2 \varepsilon \langle P\eta_t, z\rangle \big(x - \tau_2 \varepsilon \langle P\eta_t, e\overline{z}\rangle\big)^2 \varepsilon \langle P\eta_t, e\overline{z}\rangle \\
&= \varepsilon^2 \big[\langle P\eta_t, \Re z\rangle^2 - (i\langle P\eta_t, \Im z\rangle)^2 \big] \big[x - \tau_1 \varepsilon (\langle P\eta_t, \Re z\rangle + i\langle P\eta_t, \Im z\rangle) \big]^2 \\
&\quad \times \big[x - \tau_2 \varepsilon (\langle P\eta_t, \Re z\rangle - i\langle P\eta_t, \Im z\rangle) \big]^2.
\end{aligned}$$

Expanding the right side of the above equality in a finite sum of terms, such as

$$C x^{k_1} \varepsilon^{k_2} \tau_1^{k_3} \tau_2^{k_4} \langle P\eta_t, \Re z\rangle^n (i\langle P\eta_t, \Im z\rangle)^m$$

with $C \in \mathbb{Z}$ and $0 \leq k_j, n, m \leq 6$, $j = 1,2,3,4$. Therefore, the integral in (31) simplifies to a finite sum of integrals of the form

$$\begin{aligned}
&C x^{k_1} \varepsilon^{k_2} \tau_1^{k_3} \tau_2^{k_4} \exp\left(-\frac{x^2}{t^{2H}} \right) \\
&\times \int_{S'_{1,\mathbb{C}}} \exp\left(\frac{x(\tau_1+\tau_2)\varepsilon \langle P\eta_t, \Re z\rangle}{t^{2H}} + \frac{ix(\tau_1-\tau_2)\varepsilon \langle P\eta_t, \Im z\rangle}{t^{2H}} \right) \\
&\times \exp\left(-\frac{\tau_1^2 \varepsilon^2 \langle P\eta_t, z\rangle^2}{2t^{2H}} - \frac{\tau_2^2 \varepsilon^2 \langle P\eta_t, \overline{z}\rangle^2}{2t^{2H}} \right) \langle P\eta_t, \Re z\rangle^n (i\langle P\eta_t, \Im z\rangle)^m\, \mathrm{d}\nu(z).
\end{aligned} \tag{32}$$

Note that the image of the measure ν under the map

$$(S'_{1,\mathbb{C}}, \nu) \longrightarrow \mathbb{R}^2,\ z \mapsto (\langle \varphi, \Re z\rangle, \langle \phi, \Im z\rangle),$$

for $L^2(\mathbb{R})$-normalized test functions $\varphi, \phi \in S_1$, is the centered Gaussian measure with covariance matrix $\frac{1}{2}\mathrm{Id}_2$. We introduce the following notation.

$$\begin{aligned} c_P &:= c_P(t) := \frac{\|P\eta_t\|_2}{t^H}, \\ \alpha &:= \alpha_{P,\varepsilon}(t,\tau_1,\tau_2) := \frac{x(\tau_1+\tau_2)\varepsilon c_P}{t^H}, \\ \beta &:= \beta_{P,\varepsilon}(t,\tau_1,\tau_2) := \frac{x(\tau_1-\tau_2)\varepsilon c_P}{t^H}, \\ \gamma &:= \gamma_{P,\varepsilon}(t,\tau_1,\tau_2) := -(\tau_1^2-\tau_2^2)\varepsilon^2 c_P^2, \\ \sigma &:= \sigma_{P,\varepsilon}(t,\tau_1,\tau_2) := \frac{2}{2+(\tau_1^2+\tau_2^2)\varepsilon^2 c_P^2}, \\ \sigma' &:= \sigma'_{P,\varepsilon}(t,\tau_1,\tau_2) := \frac{2}{2-(\tau_1^2+\tau_2^2)\varepsilon^2 c_P^2}. \end{aligned} \tag{33}$$

Evaluating the integral (32) using the above functions yields

$$\begin{aligned} &\frac{1}{\pi} C x^{k_1}\varepsilon^{k_2}\tau_1^{k_3}\tau_2^{k_4} \exp\left(-\frac{x^2}{t^{2H}}\right)\|P\eta_t\|_2^{n+m} \int_{\mathbb{R}^2} \exp\left(\alpha u + \mathrm{i}\beta v + \mathrm{i}\gamma uv\right) \\ &\quad \times \exp\left(-\frac{(\tau_1^2+\tau_2^2)\varepsilon^2 c_P^2(u^2-v^2)}{2} - (u^2+v^2)\right) u^n(\mathrm{i}v)^m \,\mathrm{d}u\,\mathrm{d}v \\ &= \frac{1}{\pi} C x^{k_1}\varepsilon^{k_2}\tau_1^{k_3}\tau_2^{k_4} \exp\left(-\frac{x^2}{t^{2H}}\right)\|P\eta_t\|_2^{n+m} \\ &\quad \times \int_{\mathbb{R}^2} \exp\left(-\frac{1}{\sigma}u^2 + \alpha u\right)\exp\left(-\frac{1}{\sigma'}v^2 + i\beta v\right)\exp(\mathrm{i}\gamma uv) u^n(\mathrm{i}v)^m \,\mathrm{d}u\,\mathrm{d}v \\ &= \frac{1}{\pi} C x^{k_1}\varepsilon^{k_2}\tau_1^{k_3}\tau_2^{k_4} \exp\left(-\frac{x^2}{t^{2H}}\right)\|P\eta_t\|_2^{n+m} \int_{\mathbb{R}^2} h(\alpha,\beta,u,v) u^n(\mathrm{i}v)^m \,\mathrm{d}u\,\mathrm{d}v, \end{aligned}$$

where $h(\alpha,\beta,u,v)$ is given by

$$h(\alpha,\beta,u,v) := \exp\left(-\frac{1}{\sigma}u^2 + \alpha u\right)\exp\left(-\frac{1}{\sigma'}v^2 + i\beta v\right)\exp(\mathrm{i}\gamma uv).$$

Define I_h by

$$I_h : \mathbb{R}^2 \longrightarrow \mathbb{R},\ (\alpha,\beta) \mapsto \int_{\mathbb{R}^2} h(\alpha,\beta,u,v)\,\mathrm{d}u\,\mathrm{d}v.$$

It follows from Lemma C.1 and Remark C.2 that

$$\int_{\mathbb{R}^2} h(\alpha,\beta,u,v) u^n(\mathrm{i}v)^m \,\mathrm{d}u\,\mathrm{d}v = \frac{\partial^{n+m} I_h}{\partial\alpha^n \partial\beta^m}(\alpha,\beta) = I_h(\alpha,\beta) p_{n+m}(\alpha,\beta).$$

Here, $p_{n+m}(\alpha,\beta)$ is a finite sum of terms of the form

$$D c^{l_1} c'^{l_2} (\sigma'\gamma)^{l_3} \alpha^{l_4} \beta^{l_5}$$

with $D \in \mathbb{Q}$ and $0 \le l_1, \dots, l_5 \le n+m$. Here, c and c' are defined in (C.1). Recall that h, I_h, c and c' depend on σ, σ', γ, which are functions of t, τ_1, τ_2. Putting all together, the integral in (31) is given as a finite sum of terms of the form

$$\frac{1}{\pi} C D x^{k_1}\varepsilon^{k_2}\tau_1^{k_3}\tau_2^{k_4} \exp\left(-\frac{x^2}{t^{2H}}\right)\|P\eta_t\|_2^{n+m} c^{l_1} c'^{l_2} (\sigma'\gamma)^{l_3} \alpha^{l_4} \beta^{l_5} I_h(\alpha,\beta). \tag{34}$$

It remains to show that (34) multiplied by $\frac{1}{t^{6H+2}}$ is integrable with respect to $\mathrm{d}t \otimes \mathrm{d}\tau_1 \otimes \mathrm{d}\tau_2$ on $(0,T]\times[0,1]^2$ for any $P \in \mathbb{P}$, and that the supremum of the resulting set of integrals over all $P \in \mathbb{P}$ is finite.

First, we note that $I_h(\alpha,\beta)$ can be estimated by

$$I_h(\alpha,\beta) \le \frac{\pi}{\sqrt{1-\varepsilon^2}} \exp\left(\varepsilon^2 \frac{x^2}{t^{2H}}\right),$$

see Lemma C.4. Hence, using (C.2) and bounding the functions in (33), the expression in (34) multiplied by $\frac{1}{t^{6H+2}}$ is bounded by

$$\frac{2^{l_4}|CD|T^{H(n+m)}}{4^{l_1}\left(1-\varepsilon^2\right)^{l_3+1/2}} |x|^{k_1+l_4+l_5} \left(\frac{1}{4(1-\varepsilon^2)} + \frac{1}{16(1-\varepsilon^2)^2}\right)^{l_2} \frac{\mathrm{e}^{(\varepsilon^2-1)\frac{x^2}{t^{2H}}}}{t^{6H+2+H(l_4+l_5)}}.$$

This bound is independent of τ_1, τ_2 and P. Furthermore, since $\varepsilon^2 - 1 < 0$, the function

$$(0,T] \ni t \mapsto \frac{\mathrm{e}^{(\varepsilon^2-1)\frac{x^2}{t^{2H}}}}{t^{6H+2+H(l_4+l_5)}}$$

is $\mathrm{d}t$-integrable. The result of the theorem follows. □

Remark 4.5. The question of whether $\xi(x) \in \mathcal{G}_0 = L^2(\mu)$ remains open. Should this be the case, the integral kernel of stochastic currents would even exist as a real-valued random variable. This is well known for the local time of fractional Brownian motion for $dH < 1$, which is given by the Bochner integral

$$\int_0^T \delta\big(x - B_H(t)\big)\,\mathrm{d}t.$$

In comparison with the Bochner integral representation for the integral kernel of stochastic currents, see (16), we observe in the latter one an additional Wick product with the fractional noise W_H. At this point, its impact on the square integrability of $\xi(x)$ remains open.

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Appendix A. Details on the proof of Lemma 4.1

Lemma A.1. *The function in Eq. (24) is integrable on* $[0,T]$, $T > 0$.

Proof. Let us estimate the function

$$t \mapsto \frac{(x-\tau\psi(t))^2}{t^{2H+1}} \exp\left(-\frac{(x-\tau\psi(t))^2}{2t^{2H}}\right)\psi(t). \tag{A.1}$$

First, note that $|\psi(t)| = |\langle \varphi, M_-^H \mathbb{1}_{[0,t)}\rangle| \le t\|M_+^H\varphi\|_\infty$ and $|\psi(t)|^2 \le t^{2H}\|\varphi\|_2^2$. The first factor in (A.1) with $\tau \in [0,1]$ can be estimated as

$$\frac{(x-\tau\psi(t))^2}{t^{2H+1}} \le \frac{|x|^2}{t^{2H+1}} + \frac{2|x|\|M_+^H\varphi\|_\infty}{t^{2H}} + \frac{\|M_+^H\varphi\|_\infty^2}{t^{2H-1}}.$$

The exponential factor in (A.1) is estimated by

$$\begin{aligned}
&\exp\left(-\frac{(x-\tau\psi(t))^2}{2t^{2H}}\right) \le \exp\left(-\frac{|x|^2}{2t^{2H}}\right)\exp\left(\frac{\tau^2 t^{2H}\|\varphi\|_2^2}{2t^{2H}} + \frac{|x|\tau|\psi(t)|}{t^{2H}}\right)\\
&\le \exp\left(-\frac{|x|^2}{2t^{2H}}\right)\exp\left(\frac{\|\varphi\|_2^2}{2} + \frac{|x|^2}{4t^{2H}} + \frac{t^{2H}\|\varphi\|_2^2}{t^{2H}}\right).
\end{aligned}$$

Putting everything together, the integrand function (A.1) is bounded by

$$\begin{aligned}
&\left|\frac{(x-\tau\psi(t))^2}{t^{2H+1}} \exp\left(-\frac{(x-\tau\psi(t))^2}{2t^{2H}}\right)\psi(t)\right|\\
&\le \frac{1}{t^H}\left(\frac{|x|^2}{t^H} + \frac{2|x|\|M_+^H\varphi\|_\infty}{t^{H-1}} + \frac{\|M_+^H\varphi\|_\infty^2}{t^{H-2}}\right)\\
&\times \exp\left(-\frac{|x|^2}{2t^{2H}}\right)\exp\left(\frac{3\|\varphi\|_2^2}{2} + \frac{|x|^2}{4t^{2H}}\right)\|M_+^H\varphi\|_\infty.
\end{aligned}$$

Using the fact that $t \in [0,T]$ we further estimate the integrand as

$$\begin{aligned}
&\le \frac{1}{t^{2H}}\left(|x|^2 + 2T|x|\|M_+^H\varphi\|_\infty + T^2\|M_+^H\varphi\|_\infty^2\right)\\
&\times \exp\left(\frac{3\|\varphi\|_2^2}{2}\right)\exp\left(-\frac{|x|^2}{4t^{2H}}\right)\|M_+^H\varphi\|_\infty.
\end{aligned}$$

Note that the map $(0,T] \ni t \mapsto t^{-2H}\exp(-|x|^2/(4t^{2H}))$, is integrable on $[0,T]$. Then, we may conclude that the double integral

$$\frac{H}{\sqrt{2\pi}}\int_0^T\int_0^1 \frac{(x-\tau\langle\varphi,\eta_t\rangle)^2}{t^{2H+1}} \mathrm{e}^{-\frac{(x-\tau\langle\varphi,\eta_t\rangle)^2}{2t^{2H}}}\langle\varphi,\eta_t\rangle\,\mathrm{d}\tau\,\mathrm{d}t$$

is finite. □

Lemma A.2. *Integration by parts in Eq. (26).*

Proof. Given that both functions F and the mapping $(\varepsilon,T) \ni t \mapsto \frac{1}{t^H}$ are continuously differentiable on the interval (ε,T), $\varepsilon > 0$, applying integration by parts results in

$$\int_\varepsilon^T \frac{F'(t)}{t^H}\,\mathrm{d}t = \frac{1}{T^H}\int_0^1 \exp\left(-\frac{(x-\tau\psi(T))^2}{2T^{2H}}\right)\psi(T)\,\mathrm{d}\tau$$

$$- \frac{1}{\varepsilon^H} \int_0^1 \exp\left(-\frac{(x-\tau\psi(\varepsilon))^2}{2\varepsilon^{2H}}\right)\psi(\varepsilon)\,\mathrm{d}\tau$$
$$+ H \int_\varepsilon^T \frac{1}{t^{H+1}} \int_0^1 \exp\left(-\frac{(x-\tau\psi(t))^2}{2t^{2H}}\right)\psi(t)\,\mathrm{d}\tau\,\mathrm{d}t.$$

It is easy to obtain the estimates:

$$\left|-\frac{1}{\varepsilon^H} \int_0^1 \exp\left(-\frac{(x-\tau\psi(\varepsilon))^2}{2\varepsilon^{2H}}\right)\psi(\varepsilon)\,\mathrm{d}\tau\right| \le \|M_+^H\varphi\|_\infty \varepsilon^{1-H}$$

and

$$\left|\frac{1}{t^{H+1}} \int_0^1 \exp\left(-\frac{(x-\tau\psi(t))^2}{2t^{2H}}\right)\psi(t)\,\mathrm{d}\tau\right| \le \|M_+^H\varphi\|_\infty \frac{1}{t^H}.$$

As the function $(0,T] \ni t \mapsto \|M_+^H\varphi\|_\infty \frac{1}{t^H}$ is integrable, then an application of the Lebesgue dominated convergence theorem as $\varepsilon \to 0$ yields

$$\int_0^T \frac{F'(t)}{t^H}\,\mathrm{d}t = \frac{1}{T^H} \int_0^1 \exp\left(-\frac{(x-\tau\psi(T))^2}{2T^{2H}}\right)\psi(T)\,\mathrm{d}\tau$$
$$+ H \int_0^T \frac{1}{t^{H+1}} \int_0^1 \exp\left(-\frac{(x-\tau\psi(t))^2}{2t^{2H}}\right)\psi(t)\,\mathrm{d}\tau\,\mathrm{d}t.$$

This shows the claim. □

Appendix B. Proof of Proposition 4.2

Proof. We have to show that for any $x \in \mathbb{R}$ and $H \in (0,1)$ the expression of the S-transform of $\xi(x)$ given in (20) satisfies the conditions of Definition 2.2. Consider the measure space $\big((0,T]\times[0,1], \mathcal{B}((0,T]) \otimes \mathcal{B}([0,1])), \mathrm{d}t \otimes \mathrm{d}\tau\big)$ and fix $(t,\tau) \in (0,T]\times[0,1]$. We consider only the last summand since it is the most intricate one, and the other two can be estimated similarly. The corresponding functional to be integrated is denoted by

$$S_1 \ni \varphi \mapsto U_{t,\tau}(\varphi) := \frac{(x-\tau\langle\varphi,\eta_t\rangle)^2}{t^{3H+1}} \exp\left(-\frac{(x-\tau\langle\varphi,\eta_t\rangle)^2}{2t^{2H}}\right)\langle\varphi,\eta_t\rangle.$$

Let us check the first condition of Definition 2.2. Given $\varphi,\theta \in S_1$ we have to show that the function

$$\mathbb{R} \ni y \mapsto U_{t,\tau}(\varphi + y\theta) \in \mathbb{C}$$

extends to an entire function on $\mathbb{C}$. This follows easily from the fact that the function $\mathbb{R} \ni y \mapsto \langle\varphi,\eta_t\rangle + y\langle\theta,\eta_t\rangle$ is entire, and the composition of entire functions is again an entire function. Next, we prove the second condition of Definition 2.2. Let $z \in \mathbb{C}$ be given. The following estimate holds

$$\begin{aligned}
&|U_{t,\tau}(z\varphi)| \\
&= \left|\langle z\varphi,\eta_t\rangle \frac{(x-\tau\langle z\varphi,\eta_t\rangle)^2}{t^{3H+1}} \exp\left(-\frac{(x-\tau\langle z\varphi,\eta_t\rangle)^2}{2t^{2H}}\right)\right| \\
&\le \frac{|z|\|\varphi\|_2}{t^{3H+1}}\left(|x| + \tau|z|t\|M_+^H\varphi\|_\infty\right)^2 \exp\left(-\frac{x^2 - 2\tau x\langle z\varphi,\eta_t\rangle + \tau^2\langle z\varphi,\eta_t\rangle^2}{2t^{2H}}\right) \\
&\le \frac{2}{t^{3H+1}}\left(|x|^2 + t^2|z|^2\|M_+^H\varphi\|_\infty^2\right) \exp\left(-\frac{|x|^2}{4t^{2H}} + \frac{3}{2}|z|^2\|\varphi\|_2^2\right) \exp\left(|z|\|\varphi\|_2\right) \\
&= \frac{2}{t^{3H+1}} \exp\left(|x|^2 + t^2|z|^2\|M_+^H\varphi\|_\infty^2 - \frac{|x|^2}{4t^{2H}} + \frac{5}{2}|z|^2\|\varphi\|_2^2\right) \\
&\le \frac{2}{t^{3H+1}} \exp\left(|x|^2 - \frac{|x|^2}{4t^{2H}}\right) \exp\left(T^2|z|^2\|M_+^H\varphi\|_\infty^2 + \frac{5}{2}|z|^2\|\varphi\|_2^2\right) \\
&\le C(t,\tau,x,H) \exp\left(\left(T^2 + \frac{5}{2}\right)|z|^2\|\varphi\|^2\right),
\end{aligned}$$

where $\|\cdot\|$ is a continuous norm on S_1 and $C(t,\tau,x,H)$ is given by

$$C(t,\tau,x,H) := \frac{2}{t^{3H+1}} \exp\left(|x|^2 - \frac{|x|^2}{4t^{2H}}\right).$$

Thus, $U_{t,\tau}$ is a U-functional. Moreover, the function $(0,T]\times[0,1] \ni (t,\tau) \mapsto C(t,\tau,x,H)$ is $\mathrm{d}t \otimes \mathrm{d}\tau$-integrable, which implies that

$$S_1 \ni \varphi \mapsto \int_0^T \int_0^1 U_{t,\tau}(\varphi)\,\mathrm{d}\tau\,\mathrm{d}t$$

is a U-functional by Corollary 2.4. □

Appendix C. Other Lemmas

In this concluding appendix, we present several lemmas. The proofs are based on computations involving Gaussian integrals and estimations of basic polynomials. While not challenging, these proofs require substantial space. As such, we leave them to interested readers. Below, the constants c and c' are defined by

$$c := \frac{1}{\frac{4}{\sigma} + \sigma'\gamma^2}, \qquad c' := -\frac{\sigma'}{4} + c\frac{\sigma'^2\gamma^2}{4}. \tag{C.1}$$

Lemma C.1. *Given $\sigma, \sigma' > 0$ and $\gamma \in \mathbb{R}$, define $h : \mathbb{R}^4 \longrightarrow \mathbb{C}$ by*

$$h(\alpha, \beta, u, v) := \exp\left(-\frac{1}{\sigma}u^2 + \alpha u\right)\exp\left(-\frac{1}{\sigma'}v^2 + \mathrm{i}\beta v\right)\exp(\mathrm{i}\gamma u v).$$

Then, for every $\alpha, \beta \in \mathbb{C}$, $h(\alpha, \beta, \cdot, \cdot) \in L^1_{\mathbb{C}}(\mathrm{d}u \otimes \mathrm{d}v)$ and we have

$$\int_{\mathbb{R}^2} h(\alpha, \beta, u, v)\,\mathrm{d}u\,\mathrm{d}v = 2\pi\sqrt{c\sigma'}\exp\left(-\frac{\sigma'}{4}\beta^2 + c\big(\alpha - \frac{\sigma'\gamma}{2}\beta\big)^2\right) \in \mathbb{C}.$$

Moreover, the function

$$I_h : \mathbb{R}^2 \longrightarrow \mathbb{C},\ (\alpha, \beta) \mapsto \int_{\mathbb{R}^2} h(\alpha, \beta, u, v)\,\mathrm{d}u\,\mathrm{d}v$$

is in $C^\infty(\mathbb{R}^2)$ and the partial derivatives are given by

$$\frac{\partial^{n+m} I_h}{\partial\alpha^n \partial\beta^m}(\alpha, \beta) = \int_{\mathbb{R}^2} h(\alpha, \beta, u, v) u^n (iv)^m\,\mathrm{d}u\,\mathrm{d}v, \qquad n, m \in \mathbb{N}_0.$$

Remark C.2. The partial derivatives of I_h from Lemma C.1 may be computed directly. In fact, if we rewrite I_h as

$$I_h(\alpha, \beta) = 2\pi\sqrt{c\sigma'}\exp\left(c\alpha^2 + c'\beta^2 - c\sigma'\gamma\alpha\beta\right), \quad \alpha, \beta \in \mathbb{R},$$

then it is clear that the partial derivatives of I_h have the form

$$\frac{\partial^{n+m} I_h}{\partial\alpha^n \partial\beta^m}(\alpha, \beta) = I_h(\alpha, \beta)\, p_{n+m}(\alpha, \beta), \quad \alpha, \beta \in \mathbb{R}, \qquad n, m \in \mathbb{N}_0,$$

with p_k, $k \in \mathbb{N}_0$, being a polynomial in two variables of degree k. Note that the coefficients of p_{n+m} contain different powers of c, c', σ' and γ. An explicit formula is proven in Lemma C.3 below.

Lemma C.3. *Let $\sigma, \sigma' > 0$ and $\gamma \in \mathbb{R}$ be given. Define the function*

$$I_h : \mathbb{R}^2 \longrightarrow \mathbb{R},\ (\alpha, \beta) \mapsto 2\pi\sqrt{c\sigma'}\exp(c\alpha^2 + c'\beta^2 - c\sigma'\gamma\alpha\beta).$$

Then the partial derivatives of I_h are given by the formula

$$\frac{\partial^{n+m} I_h}{\partial\alpha^n \partial\beta^m}(\alpha, \beta) = I_h(\alpha, \beta) p_{n+m}(\alpha, \beta), \qquad n, m \in \mathbb{N}_0,$$

with

$$\begin{aligned} &p_{n+m}(\alpha, \beta) \\ &= \sum_{i=0}^{m}\sum_{k_1=0}^{i}\sum_{\substack{k_2=0 \\ i-k_1-k_2 \text{ even}}}^{i-k_1}\sum_{j=m-i}^{n}\sum_{\substack{l=0 \\ n-j-l \text{ even}}}^{n-j}\binom{m}{i}\frac{j!}{(j-m+i)!}R\left(i, \frac{i-k_1-k_2}{2}, k_2\right) \\ &\times R\left(n, \frac{n-j-l}{2}, l\right)c^{n+k_1-\frac{n-j-l}{2}}c'^{k_2+\frac{i-k_1-k_2}{2}}(\sigma'\gamma)^{k_1+j}\alpha^{k_1+l}\beta^{k_2+j-m+i} \end{aligned}$$

where we set

$$R(i, j, k) := \frac{i!}{j!(i-2j)!}\binom{i-2j}{k}(-1)^{i-2j-k}2^k, \qquad i, j, k \in \mathbb{N}_0.$$

Lemma C.4. *For any $0 \le \tau_i \le 1$, $i = 1, 2$, and $\alpha, \beta, \gamma, \sigma, \sigma'$ as in Eq. (33), the following bound holds*

$$I_h(\alpha, \beta) = 2\pi\sqrt{c\sigma'}\exp(c\alpha^2 + c'\beta^2 - c\sigma'\gamma\alpha\beta) \le \frac{\pi}{\sqrt{1-\varepsilon^2}}\exp\left(\varepsilon^2\frac{x^2}{t^{2H}}\right). \tag{C.2}$$

Lemma C.5. *Let $0 < \varepsilon \le 1$ be given. The polynomial $q : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by*

$$q(x, y) = \varepsilon^2x^4 + 2\varepsilon^2x^2 + \varepsilon^2y^4 + 2\varepsilon^2y^2 - 2\varepsilon^2x^3y - 2\varepsilon^2xy^3 - 2\varepsilon^2x^2y^2 + 4xy$$

is bounded by 4 on $[0, 1]^2$.

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